Lie Algebra Representations

Remember:

Definition – A <u>representation</u> of a Lie group G is a Lie group homomorphism $\rho: G \to GL(V)$ for some (finite-dimensional) vector space V. A <u>unitary</u> representation of G is a Lie group homomorphism $\rho: G \to U(H)$ for some (finite-dimensional) Hilbert space H.

Here

$$GL(V) = \{ f: V \to V : f \text{ linear \& invertible} \}$$

and

$$U(H) = \{ f: H \to H : f \text{ is unitary} \}$$

are Lie groups.

Similarly:

Definition – A <u>representation</u> of a Lie algebra L is a Lie algebra homomorphism $\varphi: L \to \mathfrak{gl}(V)$ for some (finite-dimensional) vector space V.

A <u>unitary</u> representation of L is a Lie algebra homomorphism $\varphi: L \to \mathfrak{u}(H)$ for some (finite-dimensional) Hilbert space H.

Here $\mathfrak{gl}(V)$ is the Lie algebra of $GL(V)$, and

$$\mathfrak{gl}(V) = \{f: V \to V : f \text{ linear}\}$$

since we showed this for $\mathfrak{gl}(\mathbb{C}^n) \cong \mathfrak{gl}(n, \mathbb{C})$.

$\mathfrak{u}(H)$ is the Lie algebra of $U(H)$, and

$$\mathfrak{u}(H) = \{f: H \to H : f \text{ linear } \& \ f^* = -f\}$$

"skew-adjoint"

since we showed this for $\mathfrak{u}(\mathbb{C}^n) \cong \mathfrak{u}(n)$

Theorem – Any Lie group representation $\rho : G \to GL(V)$ gives a Lie algebra representation $d\rho : \mathfrak{g} \to \mathfrak{gl}(V)$. Any unitary Lie group representation $\rho : G \to U(H)$ gives a unitary Lie algebra representation $d\rho : \mathfrak{g} \to \mathfrak{u}(H)$.

Proof – We've seen any Lie group homomorphism $f : G \to H$ gives a Lie algebra homomorphism $df : \mathfrak{g} \to \mathfrak{h}$. $\blacksquare$

Example – If $G \leq GL(n, \mathbb{C})$ is any matrix Lie group, its tautologous representation $i : G \to GL(n, \mathbb{C}) \cong GL(\mathbb{C}^n)$ gives a <u>tautologous</u> <u>representation</u>

$$di : \mathfrak{g} \to \mathfrak{gl}(n, \mathbb{C}) \cong \mathfrak{gl}(\mathbb{C}^n)$$