Lie Algebras in Quantum Physics

In quantum physics any system has a Hilbert space H whose unit vectors describe states.

Usually there is a unitary Lie group representation $\rho: G \to U(H)$ describing symmetries of the system.

This in turn gives a Lie algebra representation

$$d\rho: \mathfrak{g} \to \mathfrak{u}(H)$$

The linear maps $d\rho(v)$ for $v \in \mathfrak{g}$ are skew-adjoint, so the linear maps $i\, d\rho(v)$ are "self-adjoint", where a linear map $A: H \to H$ from a finite-dimensional Hilbert space to itself is <u>self-adjoint</u> if

$$A^* = A$$

or in other words

$$\langle \psi, A\phi \rangle = \langle A\psi, \phi \rangle \quad \forall \psi, \phi \in H$$

Lemma - $iA: H \to H$ is self-adjoint iff
$A: H \to H$ is skew-adjoint.

Proof -

$$A = -A^* \iff iA = -iA^* \iff (iA) = (iA)^*. \quad \blacksquare$$

If H is a finite-dimensional Hilbert space, a linear map $A: H \to H$ is called an <u>operator</u>, and A is self-adjoint iff H has an orthonormal basis of eigenvectors of A, $e_i \in H$, with real eigenvalues:

$$\langle e_i, e_j \rangle = \delta_{ij}$$

$$A e_i = \lambda_i e_i \qquad \lambda_i \in \mathbb{R}$$

In quantum physics a self-adjoint operator on the Hilbert space H of a system is an <u>observable</u>: something you can measure about your system, which takes real values. Examples include:

- energy
- momentum
- angular momentum

and these come from unitary representations of various Lie groups on H:

$$\rho : G \to U(H)$$

and $v \in \mathfrak{g}$ give the observable $i\,d\rho(v)$.

How does a self-adjoint operator describe an observable in quantum physics? Its eigenvectors are states where the observable has a well-defined value. The eigenvalues are those values!

Axiom 2: Suppose $A: H \to H$ is self-adjoint with $Ae_i = \lambda e_i$ for some orthonormal basis e_i. Measuring the corresponding observable when the system is in the state e_i, we always get the number λ_i.

But what if we measure the observable in a general state $\psi \in H$?

Suppose the eigenvalues λ_i are distinct.

Suppose the system is in the state $\psi \in H$.

If we measure the observable A, we get the answer λ_i with probability

$$|\langle e_i, \psi \rangle|^2$$

This "sort of follows" from our axioms.

By Axiom 1, the system will be found to be in the state e_i with probability

$$|\langle e_i, \psi \rangle|^2$$

By Axiom 2, the value of the observable is λ_i in the state e_i (and no other state e_j, since the eigenvalues are distinct).

Suppose $\psi = \sum_i c_i e_i$ for some $c_i \in \mathbb{C}$.

$$c_i = \langle e_i, \sum_j c_j e_j \rangle = \langle e_i, \psi \rangle$$

The <u>expected</u> or <u>mean</u> or <u>average</u> value of the observable A in the state ψ is

$$\sum_i \lambda_i \, |\langle e_i, \psi \rangle|^2 = \sum_i \lambda_i |c_i|^2$$

value of A when the system is in the state e_i

probability of finding the system in the state e_i

$$= \sum_{i,j} \bar{c}_i c_j \lambda_j \langle e_i, e_j \rangle$$

$$= \sum_{i,j} \langle c_i e_i, \lambda_j c_j e_j \rangle$$

$$= \langle \sum_i c_i e_i, \sum_j \lambda_j c_j e_j \rangle$$

$$= \langle \sum_i c_i e_i, A \sum_j c_j e_j \rangle$$

$$= \langle \psi, A\psi \rangle$$

Example: If $\mathcal{H} = \mathbb{C}^2$ is the Hilbert space of the spin-$\frac{1}{2}$ particle, we'll see later that the observable

$$A = \begin{pmatrix} 1/2 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

is the "z component of the spin" of this particle.

$$A\uparrow = \frac{1}{2}\uparrow \quad \text{where} \quad \uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$A\downarrow = -\frac{1}{2}\downarrow \quad \text{where} \quad \downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

so the z component of the spin is $\frac{1}{2}$ in the <u>spin-up</u> <u>state</u> $\uparrow$, and $-\frac{1}{2}$ in the <u>spin-down</u> state. In the state

$$\psi = c_1\uparrow + c_2\downarrow \qquad |c_1|^2 + |c_2|^2 = 1$$

the spin is $\frac{1}{2}$ with probability $|c_1|^2 = |\langle\uparrow,\psi\rangle|^2$, and $-\frac{1}{2}$ with probability $|c_2|^2$.

But where does the observable

$$A = \begin{pmatrix} 1/2 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

come from? It comes from the spin-$\frac{1}{2}$ representation of $SU(2)$ on $\mathbb{C}^2$, the tautologous representation we were calling

$$\alpha_{1/2} : SU(2) \to U(\mathbb{C}^2)$$

There is a Lie algebra element $v \in \mathfrak{su}(2)$ such that $\exp(tv) \in SU(2)$ is the symmetry "rotating the particle an angle t counterclockwise around the z axis". $d\alpha_{1/2}(v)$ is skew-adjoint, and

$$i\, d\alpha_{1/2}(v) = \begin{pmatrix} 1/2 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

is our observable A!