A many-faceted Lie algebra

Theorem – The following Lie algebras are isomorphic:

1) $\mathfrak{o}(3) = \{a \in M_3(\mathbb{R}) : a^* = -a\}$

2) $\mathfrak{so}(3) = \{a \in M_3(\mathbb{R}) : a^* = -a,\ tr(a) = 0\}$

3) $\mathfrak{su}(2) = \{a \in M_2(\mathbb{C}) : a^* = -a,\ tr(a) = 0\}$

4) $\mathfrak{sp}(1) = \{a \in \mathbb{H} : a^* = -a\}$

5) $\mathbb{R}^3$ with its cross product as Lie bracket.

Proof –

1) = 2) : $\mathfrak{o}(3) = \mathfrak{so}(3)$. We've seen

$a^* = -a \Rightarrow tr(a) = 0$ when $a \in M_n(\mathbb{R})$,

and this does the job. Also: $SO(3) \leq O(3)$

is the connected component of $O(3)$ containing 1,

so it has the same Lie algebra.

2) $\cong$ 4) : $\mathfrak{so}(3) \cong \mathfrak{sp}(1)$. We've seen there is a double cover

$$\rho : Sp(1) \to SO(3)$$

By the Lemma below, whenever $\pi : H \to G$ is a cover $d\pi : \mathfrak{h} \to \mathfrak{g}$ is a Lie algebra isomorphism. So

$$d\rho : \mathfrak{sp}(1) \to \mathfrak{so}(3)$$

is a Lie algebra isomorphism.

Lemma – If $f : H \to G$ is a Lie group homomorphism and $f|_U : U \to f(U)$ is a diffeomorphism for some open neighborhood U of 1, $df : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra isomorphism.

Proof – The conditions imply $df : \mathfrak{g} \to \mathfrak{h}$

is an isomorphism of vector spaces, and it's a Lie algebra homomorphism, so its inverse will be a Lie algebra homomorphism too (check this with a calculation). $\blacksquare$

By the way, recall that

$$\rho : Sp(1) \to SO(3)$$

maps any element $g \in Sp(1)$ – that is, any quaternion $g \in \mathbb{H}$ with $|g| = 1$, to $\rho(g) \in SO(3)$ given by

$$\begin{aligned} \rho(g) : \mathbb{R}^3 &\to \mathbb{R}^3 \\ \vec{v} &\mapsto g \vec{v} g^{-1} \end{aligned}$$

where we think of $\mathbb{R}^3$ as the imaginary quaternions $\{\vec{v} = v_1 i + v_2 j + v_3 k\} \subseteq \mathbb{H}$.

$3) \cong 4)$: $\mathfrak{su}(2) \cong \mathfrak{sp}(1)$. We have seen there is an algebra homomorphism

$$\alpha : \mathbb{H} \hookrightarrow M_2(\mathbb{C})$$

which restricts to give a Lie group isomorphism

$$\alpha : Sp(1) \to SU(2)$$

Thus,

$$d\alpha : \mathfrak{sp}(1) \to \mathfrak{su}(2)$$

is a Lie algebra isomorphism.

By the way, recall that

$$\alpha : i \mapsto I = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

$$j \mapsto J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$k \mapsto K = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$1 \mapsto 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$I^2 = J^2 = K^2 = IJK = -1$$

4) $\cong$ 5) : $sp(1) \cong (\mathbb{R}^3, \times)$. Note

$$sp(1) = \{ai + bj + ck : a,b,c \in \mathbb{R}\} \subseteq \mathbb{H}$$

$$\mathbb{R}^3 = \{a\vec{i} + b\vec{j} + c\vec{k} : a,b,c \in \mathbb{R}\}$$

so the sure look isomorphic! But the obvious isomorphism of vector spaces does not preserve the Lie bracket! In $(\mathbb{R}^3, \times)$ we have $\vec{i} \times \vec{j} = \vec{k}$, but in $sp(1)$ we have $[i,j] = ij - ji = k - (-k) = 2k$.

So this is a Lie algebra isomorphism:

$$\beta : sp(1) \longrightarrow (\mathbb{R}^3, \times)$$

$$ai + bj + ck \longmapsto 2(a\vec{i} + b\vec{j} + c\vec{k})$$

For example:

$$\beta([i,j]) = \beta(ij - ji) = \beta(2k) = 4\vec{k}$$

$$\beta(i) \times \beta(j) = 2\vec{i} \times 2\vec{j} = 4\vec{k}. \quad \boxtimes$$

$$
\begin{tikzcd}
\mathfrak{sp}(1) \arrow[r, "d\rho", "\sim"'] \arrow[rr, bend left, "d\beta", "\sim"'] \arrow[rrr, bend left=50, "d\alpha", "\sim"'] & \mathfrak{su}(2) & (\mathbb{R}^3, \times) & \mathfrak{so}(3)
\end{tikzcd}
$$

$$
\begin{array}{cccc}
i & I = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} & 2\vec{i} & 2\vec{i} \times - \\[2em]
j & J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & 2\vec{j} & 2\vec{j} \times - \\[2em]
k & K = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} & 2\vec{k} & 2\vec{k} \times -
\end{array}
$$

Physicists turn I, J, K into self-adjoint matrices called <u>Pauli matrices</u> by multiplying them by i. They write

$$\sigma_1 = iI = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_2 = iJ = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_3 = iK = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$