The spin-1 particle

Rotating by an angle t counterclockwise around the z axis gives an element

$$R(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SO(3)$$

and $R(s+t) = R(s)R(t)$ so we get a Lie group homomorphism

$$R : \mathbb{R} \to SO(3)$$

Thus by a theorem we proved,

$$R(t) = \exp(tv) \qquad \forall t \in \mathbb{R}$$

for some unique $v \in \mathfrak{so}(3)$. What is v?

$$\frac{d}{dt} R(t)\Big|_{t=0} = \frac{d}{dt} \exp(tv)\Big|_{t=0}$$

$$= v\exp(tv)\Big|_{t=0}$$

$$= v.$$

So:

$$v = \frac{d}{dt} R(t)\Big|_{t=0} = \frac{d}{dt}\begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}\Bigg|_{t=0}$$

$$= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{so}(3)$$

Note

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{=\vec{i}} = \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{=\vec{j}}$$

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{=\vec{j}} = \underbrace{\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}}_{=-\vec{i}}$$

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{=\vec{k}} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}_{=\vec{0}}$$

So

$$v(\vec{w}) = \vec{k} \times \vec{w} \qquad \forall \vec{w} \in \mathbb{R}^3$$

More generally rotations about any unit vector $\vec{v} \in \mathbb{R}^3$ are $\exp(tv) \in SO(3)$ where

$$v(\vec{w}) = \vec{v} \times \vec{w}.$$

Remember: the spin-1 rep

$$\alpha_1 : SO(3) \to U(\mathbb{C}^3)$$

is the tautologous rep

$$SO(3) \hookrightarrow U(3) \cong U(\mathbb{C}^3)$$

so $\alpha_1(g)$ is just $g \in SO(3)$ thought of as a complex matrix. Similarly

$$d\alpha_1 : \mathfrak{so}(3) \to \mathfrak{u}(\mathbb{C}^3)$$

sends any real 3x3 skew-adjoint matrix to the corresponding complex one. So if

$$v = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{so}(3)$$

then $d\alpha_1(v) \in \mathfrak{u}(\mathbb{C}^3)$ looks just the same, and the corresponding observable is the self-adjoint matrix

$$i\, d\alpha_1(v) = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This observable is called <u>angular momentum</u> around the z axis.

Angular momentum around the z axis has an orthonormal basis of eigenvectors:

$$\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \text{eigenvalue is } 0$$

$$\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} /\sqrt{2} = \begin{pmatrix} -i \cdot i \\ i \cdot 1 \\ 0 \end{pmatrix} /\sqrt{2} = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} /\sqrt{2}$$

eigenvalue is 1

$$\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} /\sqrt{2} = \begin{pmatrix} -i \cdot -i \\ i \cdot 1 \\ 0 \end{pmatrix} /\sqrt{2} = \begin{pmatrix} -1 \\ i \\ 0 \end{pmatrix} /\sqrt{2}$$

eigenvalue is -1

So: the angular momentum of a spin-1 particle around the z axis can only be 1, 0, or -1!

The same is true for any other axis: we used the z axis just to make the calculation easy.

More generally, the angular momentum of a "spin-j particle" around any axis can only be $j, j-1, j-2, \ldots, -j$. Here $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ The dimension of the spin-j rep is the number of these eigenvalues $j, j-1, j-2, \ldots, -j$, namely $2j+1$.

The spin-$\frac{1}{2}$ particle, revisited

For any unitary rep $\rho: SO(3) \to U(H)$, "angular momentum around the z axis" is $i d\rho(v)$ for $v \in \mathfrak{so}(3)$ such that $\exp(tv) \in SO(3)$ is rotation by an angle t counterclockwise around the z axis.

The spin-$\frac{1}{2}$ particle is a unitary rep not of $SO(3)$ but $SU(2)$, which double covers $SO(3)$:

$$\pi: SU(2) \to SO(3)$$

For any unitary rep $\rho: SU(2) \to U(H)$, angular momentum around the z axis is defined to be $i d\rho(\tilde{v})$ where $\tilde{v} \in \mathfrak{su}(2)$ has $d\pi(\tilde{v}) = v$. A calculation shows that for the spin-$\frac{1}{2}$ rep

$$i d\rho(\tilde{v}) = \begin{pmatrix} 1/2 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

which has eigenvalues $\frac{1}{2}$ and $-\frac{1}{2}$. So, the spin-$\frac{1}{2}$ particle can only have angular momentum $\frac{1}{2}$ or $-\frac{1}{2}$ around the z axis (or any axis).