Def. – A homomorphism from a Lie group G to a Lie group H is a group homomorphism $f: G \to H$ that is also a smooth map.

There is a category LieGrp with Lie groups as objects & Lie group homomorphisms as morphisms.

Example: A closed subgroup $H \leq G$ of a Lie group is a submanifold of G and a Lie group in its own right. The inclusion map

$$i: H \hookrightarrow G$$

is a Lie group homomorphism.

Recall a matrix Lie group is a closed subgroup of the general linear group

$$GL(n, \mathbb{C}) = \{ g \in M_n(\mathbb{C}) : \det g \neq 0 \}$$

So, if H is a matrix Lie group we have a 1-1 Lie group homomorphism

$$i: H \hookrightarrow GL(n, \mathbb{C})$$

Examples of matrix Lie groups

1) The <u>general linear group</u>

$$GL(n,\mathbb{R}) = \{ g \in GL(n,\mathbb{C}) : g_{ij} \in \mathbb{R} \}$$

is a closed subgroup of $GL(n,\mathbb{C})$
since

$$g_\alpha \to g \quad \text{iff} \quad (g_\alpha)_{ij} \to g_{ij} \quad \forall i, j$$

so

$$g_\alpha \to g \quad \& \quad g_\alpha \in GL(n,\mathbb{R})$$

$$\Downarrow$$

$$(g_\alpha)_{ij} \to g_{ij} \quad \& \quad (g_\alpha)_{ij} \in \mathbb{R}$$

$$\Downarrow$$

$$g_{ij} \in \mathbb{R}$$

$$\Downarrow$$

$$g \in GL(n,\mathbb{R})$$

2) The special linear group

$$SL(n,\mathbb{C}) = \{ g \in GL(n,\mathbb{C}) : \det g = 1 \}$$

is closed in $GL(n,\mathbb{C})$ since det is continuous.

3) The special linear group

$$SL(n,\mathbb{R}) = \{ g \in GL(n,\mathbb{R}) : \det g = 1 \}$$
$$= GL(n,\mathbb{R}) \cap SL(n,\mathbb{C})$$

is a closed subgroup, being the intersection of two closed subgroups.

4) The unitary group

$$U(n) = \{ g \in GL(n,\mathbb{C}) : gg^* = 1 \}$$

where

$$(g^*)_{ij} = \overline{g_{ji}}$$

defines the adjoint (conjugate transpose) of g.

$U(n)$ is closed because gg^* depends in a continuous way on $g \in GL(n,\mathbb{C})$.

Lemma - For any $g \in GL(n, \mathbb{C})$ the following are equivalent:

1) $gg^* = 1$

2) $g^*g = 1$

3) g preserves inner products:

$\langle gv, gw \rangle = \langle v, w \rangle \quad \forall v, w \in \mathbb{C}^n$

where $\langle v, w \rangle = \sum_{i=1}^{n} \overline{v_i} \, w_i$.

4) g preserves lengths:

$\|gv\| = \|v\| \qquad \forall v \in \mathbb{C}^n$

where $\|v\| = \sqrt{\langle v, v \rangle}$.

1) $\Leftrightarrow$ 2) : since g has an inverse, any right inverse is a left inverse & vice versa.

2) $\Leftrightarrow$ 3) : $\langle gv, gw \rangle = \langle v, g^* g w \rangle$

3) $\Rightarrow$ 4) : easy

4) $\Rightarrow$ 3) : polarization identity

So, $U(n)$ is a subgroup of $GL(n,\mathbb{C})$:

$$g, h \in U(n)$$

$$\Downarrow$$

$$\|ghv\| = \|hv\| = \|v\| \quad \forall v \in \mathbb{C}^n$$

$$\Downarrow$$

$$gh \in U(n)$$

Thus $U(n)$ is a matrix Lie group.

5) The orthogonal group

$$O(n) = \{g \in GL(n,\mathbb{R}) : gg^* = 1\}$$
$$= U(n) \cap GL(n,\mathbb{R})$$

is a matrix Lie group, being an intersection of closed Lie subgroups of $GL(n,\mathbb{C})$.

6) So is the special unitary group

$$SU(n) = \{g \in GL(n,\mathbb{C}) : gg^* = 1, \det g = 1\}$$
$$= U(n) \cap SL(n,\mathbb{C})$$

7) So is the <u>special</u> <u>orthogonal</u> <u>group</u>

$$
\begin{aligned}
SO(n) &= \{ g \in GL(n,\mathbb{R}) : gg^* = 1, \det g = 1 \} \\
&= O(n) \cap SL(n,\mathbb{R})
\end{aligned}
$$

For example, $SO(3)$ is the group of 3×3 matrices describing rotations in $\mathbb{R}^3$.

Putting it all together:

$$
\begin{tikzcd}
 & GL(n,\mathbb{R}) \arrow[rr, hook] & & GL(n,\mathbb{C}) & \\
 & & O(n) \arrow[ul, hook] \arrow[rr, hook] & & U(n) \arrow[ul, hook] \\
 SL(n,\mathbb{R}) \arrow[uur, hook] \arrow[rrr, hook] & & & SL(n,\mathbb{C}) \arrow[uu, hook] & \\
 & SO(n) \arrow[ul, hook] \arrow[uur, hook] \arrow[rr, hook] & & SU(n) \arrow[u, hook] \arrow[uur, hook] &
\end{tikzcd}
$$

Let's look at some low-dimensional examples!

$$
\begin{tikzcd}
 & GL(1,\mathbb{R}) \arrow[rr, hook] & & GL(1,\mathbb{C}) & \\
SL(1,\mathbb{R}) \arrow[ur, hook] & & O(1) \arrow[ul, hook] & SL(1,\mathbb{C}) \arrow[u, hook] & U(1) \arrow[ul, hook] \\
 & SO(1) \arrow[ul, hook] \arrow[ur, hook] \arrow[rr, hook] & & SU(1) \arrow[u, hook] \arrow[ur, hook] &
\end{tikzcd}
$$

$GL(1,\mathbb{C}) \cong \mathbb{C}^* := \mathbb{C} - \{0\}$,
a group under multiplication.

$GL(1,\mathbb{R}) \cong \mathbb{R}^* := \mathbb{R} - \{0\}$

$U(1) \cong \{z \in \mathbb{C} : |z| = 1\}$,

$\cong S^1$, the circle group

$O(1) \cong \{x \in \mathbb{R} : |x| = 1\}$

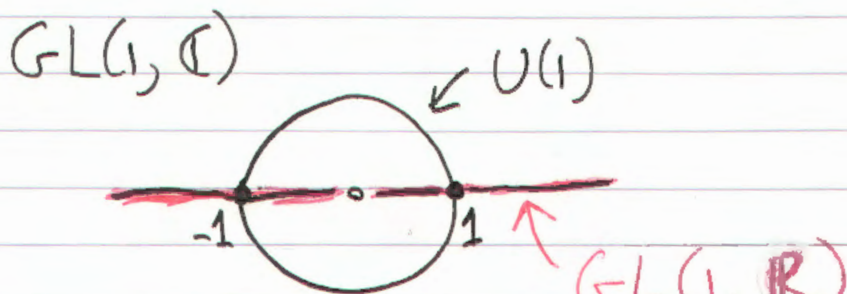
$\cong \mathbb{Z}_2$

$\cong S^0$

Note :

$$\begin{aligned} GL(1,\mathbb{C}) &\cong \mathbb{C}^* \\ &\cong \{ e^r e^{i\theta} : r, \theta \in \mathbb{R} \} \\ &\cong \mathbb{R} \times S^1 \end{aligned}$$

$$\begin{aligned} GL(1,\mathbb{R}) &\cong \mathbb{R}^* \\ &\cong \{ e^r x : r \in \mathbb{R}, x = \pm 1 \} \\ &\cong \mathbb{R} \times S^0 \end{aligned}$$



$\cong$