Irreducible representations

One dream of physics is to find elementary building blocks from which everything can be built. For systems described by representations of compact Lie groups this is true: every such rep is a direct sum of "irreducible" reps, which contain no smaller reps (except the 0-dimensional rep).

Definition: Given a representation $\rho: G \to GL(V)$ of a Lie group G on a finite-dimensional vector space, an <u>invariant subspace</u> $W \subseteq V$ is a linear subspace W such that

$$w \in W \Rightarrow \rho(g) w \in W \qquad \forall g \in G$$

Given an invariant subspace $W \subseteq V$, we've seen that we can define a representation $\tilde{\rho} : G \to GL(W)$ by

$$\tilde{\rho}(g)(w) = \rho(g)(w) \qquad \forall w \in W,\ g \in G$$

i.e.

$$\tilde{\rho}(g) = \rho(g)|_W : W \to W$$

We call $\tilde{\rho}$ a <u>subrepresentation</u> of ρ.

Every representation $\rho : G \to GL(V)$ has $\{0\}$ and V as invariant subspaces, so we call these invariant subspaces & the resulting subrepresentations <u>trivial</u>.

Henceforth let G be a Lie group and V a finite-dimensional vector space.

Definition: A representation $\rho: G \to GL(V)$ is <u>reducible</u> if V has a nontrivial invariant subspace. Otherwise it is <u>irreducible</u>.

Definition: A representation $\rho: G \to GL(V)$ is <u>decomposable</u> if $V \cong W \oplus W'$ where W and W' are nontrivial invariant subspaces. Otherwise it is <u>indecomposable</u>.

Lemma - If a rep is decomposable, it is reducible.

Proof - Obvious. $\blacksquare$

The converse is not always true, but it is for compact Lie groups G.

Earlier we defined the concept of "subrepresentation" and "direct sum of representations". It follows straight from these definitions that:

Lemma – A rep $\rho: G \to GL(V)$ is irreducible iff ρ has no nontrivial subreps (i.e., subreps not isomorphic to $\{0\}$ or ρ).

Lemma – A rep $\rho: G \to GL(V)$ is indecomposable iff ρ is not a direct sum of two nontrivial subreps.

We will prove that if G is compact,

every reducible rep of G is decomposable,

so

$$\text{reducible} \iff \text{decomposable}$$

in this case. But note:

Not every reducible rep of $\mathbb{R}$ is decomposable!

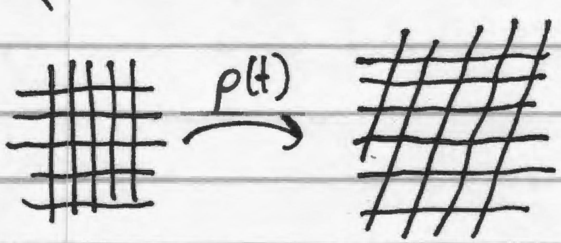
Consider $\rho: \mathbb{R} \to GL(\mathbb{R}^2)$ given by

$$\rho(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

Note it's a rep:

$$\rho(s)\rho(t) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & s+t \\ 0 & 1 \end{pmatrix} = \rho(s+t)$$

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+ty \\ y \end{pmatrix}$$

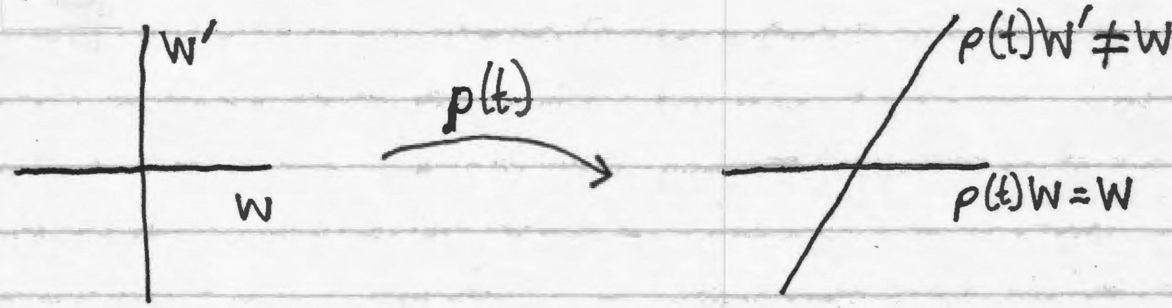


The x axis is a nontrivial invariant subspace, so this rep is reducible, but we cannot write

$$\mathbb{R}^2 = \underbrace{W}_{\text{the } x \text{ axis}} \oplus W'$$

where W' is some other invariant subspace, so this rep is not decomposable!

The problem is that no complementary subspace W' is mapped to itself by all $\rho(t): \mathbb{R}^2 \to \mathbb{R}^2$:



For an unitary rep this problem can't happen: we can take W' to be the <u>orthogonal complement</u>:

$$W^{\perp} = \{ v : \langle v, w \rangle = 0 \quad \forall w \in W \}$$

Now let H be a finite-dimensional Hilbert space.

Lemma - If $\rho: G \to U(H)$ is a unitary rep of G and $W \subseteq H$ is an invariant subspace, then $W^\perp$ is an invariant subspace.

Proof - Assume W is invariant.

$$v \in W^\perp \Rightarrow \langle v, w \rangle = 0 \quad \forall w \in W$$

$$\Rightarrow \langle v, \rho(g) w \rangle = 0 \quad \forall g \in G, w \in W$$

since W invariant

$$\Rightarrow \langle \rho(g)^* v, w \rangle = 0 \quad \forall g \in G, w \in W$$

$$\Rightarrow \langle \rho(g)^{-1} v, w \rangle = 0 \quad \forall g \in G, w \in W$$

since $\rho(g)$ unitary

$$\Rightarrow \langle \rho(g^{-1}) v, w \rangle = 0 \quad \forall g \in G, w \in W$$

Since g was arbitrary so is g^{-1}, so

$$\Rightarrow \langle \rho(h) v, w \rangle = 0 \quad \forall h \in G, w \in W$$

$$\Rightarrow \rho(h) v \in W^\perp \quad \forall h \in G$$

so $W^\perp$ is invariant. $\blacksquare$

Lemma – If $\rho : G \to U(H)$ is a unitary rep,

ρ is decomposable $\Leftrightarrow$ ρ is reducible.

Proof –

$\Rightarrow$: We've shown this in an earlier lemma; this is the obvious direction.

$\Leftarrow$: Suppose ρ is reducible; then there is a nontrivial invariant subspace $W \subseteq H$, i.e. an invariant subspace with $W \neq \{0\}, H$. By the previous lemma $W^\perp$ is also invariant, and nontrivial: $W^\perp \neq H, \{0\}$. So, $H = W \oplus W^\perp$ is decomposable. $\blacksquare$

Lemma - Any rep $\rho: G \to GL(V)$ is a direct sum of finitely many indecomposable reps.

Proof - If ρ is indecomposable we are done. Otherwise $V = W \oplus W'$ for nontrivial invariant subspaces $W, W' \subseteq V$, so $\rho \cong \tilde{\rho} \oplus \tilde{\rho}'$ where

$\tilde{\rho} : G \to GL(W)$ has $\tilde{\rho}(g) = \rho(g)\big|_W : W \to W$

$\tilde{\rho}' : G \to GL(W')$ has $\tilde{\rho}'(g) = \rho(g)\big|_{W'} : W' \to W'$.

If $\tilde{\rho}$ and $\tilde{\rho}'$ are indecomposable we are done; otherwise repeat this procedure to break up the decomposable one(s). Since $\dim(W), \dim(W') < \dim V$ this process must terminate after finitely many steps, writing ρ as a finite direct sum of indecomposable reps. $\blacksquare$

Lemma - Any unitary rep $\rho: G \to U(H)$ is a direct sum of finitely many irreducible reps.

Proof - Apply the previous lemma and note that at each step of the process $W, W' \subseteq H$ inherit inner products from H that are preserved by $\tilde{\rho}(g)$, $\tilde{\rho}'(g)$ respectively $\forall g \in G$. Thus $\tilde{\rho}$ and $\tilde{\rho}'$ are unitary too. When we finally reach indecomposable unitary reps, these are irreducible by an earlier lemma: all indecomposable unitary reps are irreducible. $\blacksquare$

Now, the finishing touch-

Theorem 1 - If G is compact, for any rep $\rho: G \to GL(V)$ there is an inner product on V preserved by all $\rho(g): V \to V$, thus making ρ into a unitary rep.

Proof - Later. $\blacksquare$

And the payoff for all this work:

Theorem 2 - If G is compact, every rep $\rho: G \to GL(V)$ is a direct sum of irreducible reps.

Proof - There is an inner product on V such that ρ is a unitary rep, so by an earlier lemma ρ is a direct sum of irreducible unitary reps. $\blacksquare$

Proof of Theorem 1 – We use "Haar measure".

Lemma – If G is a compact Lie group, there is a unique Borel measure μ on G, called Haar measure, such that:

1) μ is a probability measure: $\int_{g \in G} 1 \, d\mu(g) = 1$

2) μ is left-invariant: $\int_{g \in G} f(g) \, d\mu(g) = \int_{g \in G} f(hg) \, d\mu(g)$ for all $h \in G$ and all integrable $f: G \to \mathbb{C}$.

3) μ is right-invariant: $\int_{g \in G} f(g) \, d\mu(g) = \int_{g \in G} f(gh) \, d\mu(g)$ for all $h \in G$ and all integrable $f: G \to \mathbb{C}$

Proof – See a book on measure theory, e.g. Folland's Real Analysis. Note: either 1) & 2) or 1) & 3) is enough for uniqueness. ▨

Now, given a rep $\rho: G \to GL(V)$ where G is compact, take any inner product $\langle\langle \cdot , \cdot \rangle\rangle$ on V and "average it over the group" to get a new inner product

$$\langle v, w \rangle = \int_{h \in G} \langle\langle \rho(h) v, \rho(h) w \rangle\rangle \, d\mu(h) \quad \forall v, w \in V$$

where μ is Haar measure. Check that $\langle \cdot , \cdot \rangle$ is an inner product. Note that it is preserved by $\rho(g): V \to V$ for all $g \in G$:

$$\begin{aligned}
\langle \rho(g) v, \rho(g) w \rangle &= \int_{h \in H} \langle\langle \rho(h)\rho(g) v, \rho(h)\rho(g) w \rangle\rangle \, d\mu(h) \\
&= \int_{h \in H} \langle\langle \rho(hg) v, \rho(hg) w \rangle\rangle \, d\mu(h) \\
&= \int_{h \in H} \langle\langle \rho(h) v, \rho(h) w \rangle\rangle \, d\mu(h) \\
&= \langle v, w \rangle
\end{aligned}$$

Since $\rho(g)$ is invertible and preserves the inner product, it is unitary. Thus $\rho: G \to U(V)$ is a unitary rep when we make V into a Hilbert space with $\langle \cdot, \cdot \rangle$ as its inner product.

$\blacksquare$

Our next challenge is to take a compact Lie group G and determine all its irreducible reps. This has been done for all compact Lie groups! This is part of how physicists classify elementary particles.

Let's try it for our favorite, $G = SU(2)$. For any $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ it has a "spin-j" rep, and these are all its irreps!

irreducible reps