

Irreducible representations of $SU(2)$

We've looked at the spin-1 rep of $SU(2)$ and claimed it's isomorphic to the rep of $SU(2)$ on

$$S^2\mathbb{C}^2 = \{a\uparrow\otimes\uparrow + b(\uparrow\otimes\downarrow + \downarrow\otimes\uparrow) + c\downarrow\otimes\downarrow : a, b, c \in \mathbb{C}\}$$

where

$$\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{C}^2$$

We can also think of $S^2\mathbb{C}^2$ as the space of homogeneous quadratic polynomials in 2 variables:

$$S^2\mathbb{C}^2 \cong \{au^2 + buv + cv^2 : a, b, c \in \mathbb{C}\}$$

There is an isomorphism

$$\uparrow\otimes\uparrow \mapsto u^2$$

$$\frac{1}{2}(\uparrow\otimes\downarrow + \downarrow\otimes\uparrow) \mapsto uv = vu$$

$$\downarrow\otimes\downarrow \mapsto v^2$$

We can get higher-spin reps by generalizing this idea!

Theorem - $SU(2)$ has an irreducible rep

on the space of homogeneous polynomials of

degree n in 2 variables:

$$S^n \mathbb{C}^2 \cong \{ a_n U^n + a_{n-1} U^{n-1} V + \dots + a_1 U V^{n-1} + a_0 V^n : a_i \in \mathbb{C} \}.$$

(Note: this theorem is still true if we replace

both appearances of "2" by any other natural number.)

This space $S^n \mathbb{C}^2$ has dimension $n+1 = 2j+1$

where $j = 0, \frac{1}{2}, 1, \dots$ is called the spin. This

representation of $SU(2)$ is also called the spin- j

representation, and it's isomorphic to the spin- j

rep we've already seen when $j = 0, \frac{1}{2}$, or 1.

Theorem - Every irrep of $SU(2)$ is isomorphic to a spin- j rep.

To begin, note $GL(2, \mathbb{C})$ has a tautologous rep on $\mathbb{C}^2$. We call the standard basis of $\mathbb{C}^2$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = u = \uparrow, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} = v = \downarrow$$

Then $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL(2, \mathbb{C})$ acts as follows:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha \\ \gamma \end{pmatrix}$$

$\uparrow_u$ $\uparrow_{\alpha u + \gamma v}$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \beta \\ \delta \end{pmatrix}$$

$\uparrow_v$ $\uparrow_{\beta u + \delta v}$

or:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} u = \alpha u + \gamma v \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} v = \beta u + \delta v$$

This gives a rep

$$\rho: GL(2, \mathbb{C}) \rightarrow GL(S^n \mathbb{C}^2)$$

with

$$\rho \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} (u^j v^k) = (\alpha u + \gamma v)^j (\beta u + \delta v)^k \quad \star$$

The rep $\rho: GL(2, \mathbb{C}) \rightarrow GL(S^d \mathbb{C}^2)$

gives a Lie algebra rep

$$d\rho: \mathfrak{gl}(2, \mathbb{C}) \rightarrow \mathfrak{gl}(S^d \mathbb{C}^2)$$

We'll want to see this explicitly too.

Note $\mathfrak{gl}(2, \mathbb{C})$ has a basis

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Theorem - $d\rho$ acts on degree- n homogeneous polynomials as follows:

$$d\rho \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = v \frac{\partial}{\partial u} \quad d\rho \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = v \frac{\partial}{\partial v}$$

$$d\rho \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = v \frac{\partial}{\partial u} \quad d\rho \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = v \frac{\partial}{\partial v}$$

Proof - We'll show one; they all work the same.

By definition

$$d\rho\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)(u^j v^k) = \left. \frac{d}{dt} \rho\left(\exp\left(t\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\right)(u^j v^k) \right|_{t=0}$$

and

$$\begin{aligned} \exp\left(t\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) &= 1 + t\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} + \frac{t^2}{2!}\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}^2 + \dots \\ &= \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \end{aligned}$$

$\nwarrow$ vanishes

so

$$d\rho\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)(u^j v^k) = \left. \frac{d}{dt} \rho\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right)(u^j v^k) \right|_{t=0}$$

Remember $\star$:

$$\rho\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)(u^j v^k) = (\alpha u + \gamma v)^j (\beta u + \delta v)^k$$

so

$$\begin{aligned} d\rho\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)(u^j v^k) &= \left. \frac{d}{dt} (1u + 0v)^j (tu + 1v)^k \right|_{t=0} \\ &= \left. \frac{d}{dt} u^j (tu + v)^k \right|_{t=0} \end{aligned}$$

Continuing:

$$\begin{aligned} d\rho \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} (u^j v^k) &= \frac{d}{dt} u^j (t u + v)^k \Big|_{t=0} \\ &= u^j \cdot k u (t u + v)^{k-1} \Big|_{t=0} \\ &= k u^{j+1} v^{k-1} \\ &= u \frac{\partial}{\partial v} u^j v^k \end{aligned}$$

so

$$d\rho \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = u \frac{\partial}{\partial v}$$

as desired. $\square$

There are more conceptual ways to see this:

$GL(n, \mathbb{C})$ acts as linear transformations of $\mathbb{C}^n$ so $\mathfrak{gl}(n, \mathbb{C})$ acts to give linear

vector fields on $\mathbb{C}^n$, i.e. vector fields that

depend linearly on the coordinates, like $u \frac{\partial}{\partial v}$.

We can now restrict the rep

$$\rho: GL(2, \mathbb{C}) \rightarrow GL(S^n \mathbb{C}^2)$$

to $SU(2)$ and get a rep

$$\alpha_j: SU(2) \rightarrow GL(S^n \mathbb{C}^2)$$

called the spin-j rep of $SU(2)$, where

$j = n/2$. We can also restrict the rep

$$d\rho: \mathfrak{gl}(2, \mathbb{C}) \rightarrow \mathfrak{gl}(S^n \mathbb{C}^2)$$

to $\mathfrak{su}(2)$ and get a rep

$$d\alpha_j: \mathfrak{su}(2) \rightarrow \mathfrak{gl}(S^n \mathbb{C}^2).$$

Now we're ready to prove this:

Theorem- The spin-j rep of $SU(2)$ is irreducible.

Proof - We need to show any invariant subspace $W \subseteq S^n \mathbb{C}^2$ is trivial: either $\{0\}$ or all of $S^n \mathbb{C}^2$. So, assume it's not $\{0\}$, and show it's all of $S^n \mathbb{C}^2$.

Lemma 1 - If the invariant subspace $W \subseteq S^n \mathbb{C}^2$ is nonzero, it contains a monomial.

Proof - Let

$$\psi = \sum_{j=0}^n a_j u^j v^{n-j} \in W$$

be a vector in W with the lowest possible number, say m , of nonzero coefficients a_j .

We know $m > 0$ since $W \neq \{0\}$. We'll

suppose $m > 1$ and get a contradiction. So

$m=1$ and ψ must be a monomial.

If $\alpha \in \mathbb{C}$ has $|\alpha| = 1$ then

$$g = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \in SU(2)$$

Thus by $\star$

$$\begin{aligned} \alpha_j(g) \psi &= \sum a_j (\alpha u)^j (\alpha^{-1} v)^{n-j} \\ &= \sum a_j \alpha^{2j-n} u^j v^{n-j} \end{aligned}$$

We can choose α such that each power α^{2j-n} ($0 \leq j \leq n$) is distinct. Suppose

a_ℓ is one of the nonzero coefficients. Then $\alpha_j(g) \psi - \alpha^{2\ell-n} \psi = \sum a_j (\alpha^{2j-n} - \alpha^{2\ell-n}) u^j v^{n-j}$

is W (because W is invariant) and now it's missing the $u^\ell v^{n-\ell}$ term only, so it has one fewer nonzero coefficients than ψ did -

a contradiction! 

Lemma 2- If the invariant subspace $W \subseteq S^n \mathbb{C}^2$ contains a monomial, it is all of $S^n \mathbb{C}^2$.

Proof- Assume the invariant subspace W contains a monomial. We'll show W contains all monomials $u^j v^{n-j}$ ($0 \leq j \leq n$), so it is all of $S^n \mathbb{C}^2$.

Since W is invariant it becomes a subrep of the rep of $SU(2)$ on $S^n \mathbb{C}^2$, and thus a subrep of the rep of $\mathfrak{su}(2)$. But in fact it becomes a rep of the larger Lie algebra $\mathfrak{sl}(2, \mathbb{C})$! For this we use

$$\mathfrak{sl}(n, \mathbb{C}) \cong \mathfrak{su}(n) \otimes \mathbb{C}$$

Or in simpler terms:

Lemma 3- Every $v \in \mathfrak{sl}(n, \mathbb{C})$ is of the form $a+ib$ for unique $a, b \in \mathfrak{su}(n)$, and given any rep

$$\gamma: \mathfrak{su}(n) \rightarrow \mathfrak{gl}(V)$$

on a complex vector space V , it extends uniquely to a complex-linear rep

$$\tilde{\gamma}: \mathfrak{sl}(n, \mathbb{C}) \rightarrow \mathfrak{gl}(V)$$

given by

$$\tilde{\gamma}(a+ib) = \gamma(a) + i\gamma(b)$$

Proof - Note $v \in \mathfrak{sl}(n, \mathbb{C})$ has $v^* \in \mathfrak{sl}(n, \mathbb{C})$, and $v = a+ib$ where

$$a = \frac{v-v^*}{2}, \quad b = \frac{v+v^*}{2i}$$

are not only in $\mathfrak{sl}(n, \mathbb{C})$ but also skew-adjoint, hence in $\mathfrak{su}(n)$. Check that is the only

way to write $v = a + ib$ with $a, b \in \mathfrak{su}(n)$;

check that $\tilde{\gamma}$ as defined above is a rep:

$$\tilde{\gamma}([v, w]) = [\tilde{\gamma}(v), \tilde{\gamma}(w)].$$



Proof of Lemma 2, cont. - Recall we have

an invariant subspace $W \subseteq S^n \mathbb{C}^2$ that contains

a monomial $u^l v^{n-l}$, and we are trying to

show it contains all monomials $u^k v^{n-k}$ ($0 \leq k \leq n$).

W is an invariant subspace for the rep

$$d\alpha_j : \mathfrak{su}(2) \rightarrow \mathfrak{gl}(S^n \mathbb{C}^2)$$

and thus for the rep

$$\tilde{d}\alpha_j : \mathfrak{sl}(2, \mathbb{C}) \rightarrow \mathfrak{gl}(S^n \mathbb{C}^2)$$

since $\tilde{d}\alpha_j(a + ib) = d\alpha_j(a) + i d\alpha_j(b) : W \rightarrow W$.

But recall $d\alpha_j$ was defined as a restriction of

$$d\rho: \mathfrak{gl}(2, \mathbb{C}) \rightarrow \mathfrak{gl}(S^n \mathbb{C}^2)$$

which is complex-linear, so $\tilde{d\alpha_j}$, being

the complex-linear extension of $d\alpha_j$ from

$\mathfrak{su}(2)$ to $\mathfrak{sl}(2, \mathbb{C}) \subseteq \mathfrak{gl}(2, \mathbb{C})$, is also

a restriction of $d\rho$! (Think about it.)

Now

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{C}) \subseteq \mathfrak{gl}(2, \mathbb{C})$$

and we saw that

$$d\rho \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = u \frac{\partial}{\partial v}, \quad d\rho \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = v \frac{\partial}{\partial u}$$

So, if $u^l v^{n-l}$ is in W , so is the

result of applying $u \frac{\partial}{\partial v}$ or $v \frac{\partial}{\partial u}$ to $u^l v^{n-l}$

any number of times. This gives all

monomials $u^k v^{n-k}$, since $u \frac{\partial}{\partial v}$ boosts the power of u by 1 and decreases the power of v by 1, while $v \frac{\partial}{\partial u}$ does the reverse: it boosts the power of v and lowers the power of u . 

In math, the monomials $u^k v^{n-k}$ are called "weight vectors" for the rep of $\mathfrak{sl}(2, \mathbb{C})$ on $S^n \mathbb{C}^2$, while $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ are called "E" and "F", respectively.

In physics, $d_p \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $d_p \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ are called "ladder operators" or "raising and lowering operators".

All these ideas generalize to many other Lie groups.