From SU(2) to SU(3) and Beyond

We've seen that to understand irreps of $SU(2)$ it's good to use $SL(2,\mathbb{C}) \supseteq SU(2)$. Complex irreps of $SU(2)$ correspond to complex-linear irreps of $\mathfrak{sl}(2,\mathbb{C})$, and $\mathfrak{sl}(2,\mathbb{C})$ is nice because it has "raising and lowering operators". $\mathfrak{sl}(2,\mathbb{C})$ has basis

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{raising operator}$$

$$F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{lowering operator}$$

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and the spin-j irrep of $\mathfrak{sl}(2,\mathbb{C})$ works like this:

$$\begin{tikzcd}
U^3 \arrow[r, bend right, "d\alpha_j(F)"'] & U^2V \arrow[l, bend right, "d\alpha_j(E)"'] \arrow[r, bend right, "d\alpha_j(F)"'] & UV^2 \arrow[l, bend right, "d\alpha_j(E)"'] \arrow[r, bend right, "d\alpha_j(F)"'] & V^3 \arrow[l, bend right, "d\alpha_j(E)"']
\end{tikzcd}$$

raising, lowering

when $j = 3/2$.

since

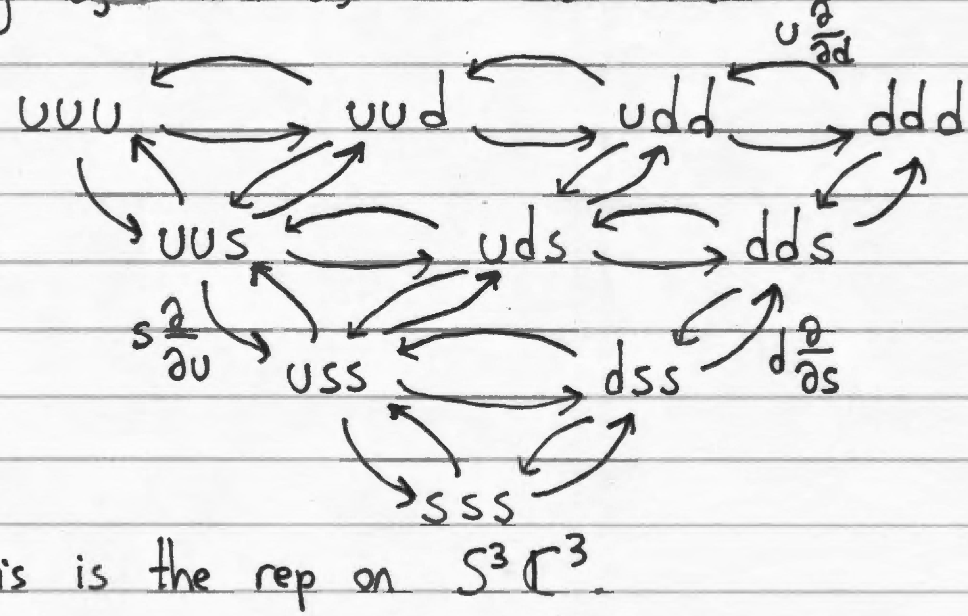
$$d\alpha_j : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{gl}(S^n\mathbb{C}^2)$$

has

$$d\alpha_j(E) = u\frac{\partial}{\partial v} \qquad \text{(raises power of } u\text{)}$$

$$d\alpha_j(F) = v\frac{\partial}{\partial u} \qquad \text{(lowers power of } u\text{)}$$

The same idea works for $SU(3) \subseteq SL(3,\mathbb{C})$, which has irreps on $S^n\mathbb{C}^3$ (the space of degree-n homogeneous polynomials in 3 variables, say u, d and s) but also others:



This is the rep on $S^3\mathbb{C}^3$.

This rep

$$d\rho : \mathfrak{sl}(3, \mathbb{C}) \to \mathfrak{gl}(S^3\mathbb{C}^3)$$

has 3 raising operators that work like this:

$$d\rho\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = u\frac{\partial}{\partial d}$$

$$d\rho\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = u\frac{\partial}{\partial s}$$

$$d\rho\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = d\frac{\partial}{\partial s}$$

and 3 lowering operators coming from lower triangular elementary matrices.

And so on for $SU(n) \leq SL(n, \mathbb{C})$.

But why am I calling the 3 variables u, d and s? The answer lies in physics.

After physicists used $SU(2)$ to study spin, Heisenberg used it to study "isospin" in 1932. The proton and neutron have only the same mass, so it's natural to try considering them as 2 states in $\mathbb{C}^2$: two states of a single particle, the "nucleon".

$$p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \uparrow \qquad \text{"isospin up"}$$

$$n = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \downarrow \qquad \text{"isospin down"}$$

In 1947 people discovered "pions", 3 particles, and described them as 3 states in the isospin-1 rep of $SU(2)$, $S^2\mathbb{C}^2 \cong \mathbb{C}^3$. In the 1960s they discovered other particles including the "Δ baryons":

$$\left.\begin{aligned}\Delta^{++} &= \uparrow\uparrow\uparrow \\ \Delta^{+} &= \uparrow\uparrow\downarrow \\ \Delta^{0} &= \uparrow\downarrow\downarrow \\ \Delta^{-} &= \downarrow\downarrow\downarrow\end{aligned}\right\} \in S^3\mathbb{C}^2 \cong \mathbb{C}^4$$

Then particles were discovered that didn't fit into this scheme, and in 1964 Gell-Mann introduced $SU(3)$. This has a tautologous rep on $\mathbb{C}^3$ with basis

$$\begin{pmatrix}1\\0\\0\end{pmatrix} = u \quad \text{"up quark"}$$

$$\begin{pmatrix}0\\1\\0\end{pmatrix} = d \quad \text{"down quark"}$$

$$\begin{pmatrix}0\\0\\1\end{pmatrix} = s \quad \text{"strange quark"}$$

The proton & neutron were revealed to be made of quarks, and so were pions, Δ baryons & many other particles! Particles could be grouped into irreps of $SU(3)$. For example, $SU(3)$ has an irrep on $S^3\mathbb{C}^3$ which includes these particles (with funny names):

Δ^{++}	Δ^{+}	Δ^{0}	Δ^{-}
uuu	uud	udd	ddd
Σ^{*+}	Σ^{*0}	Σ^{*-}	
uus	uds	dds	
Ξ^{*0}	Ξ^{*-}		
uss	dss		
Ω^{-}			
sss			

The Ω^- was not yet observed when Gell-Mann predicted its existence & its properties. It was soon seen, and he won the Nobel Prize for this in 1969.

Gell-Mann's $SU(3)$ was only an approximate symmetry and far from the last word in particle physics. The "Standard Model", our current best theory, has symmetry group $SU(3) \times SU(2) \times U(1)$ describing three aspects of nature:

$SU(3)$: the strong nuclear force

$SU(2)$: the weak nuclear force

$U(1)$: the electromagnetic force

The $SU(3)$ here is used in a completely different way than Gell-Mann's $SU(3)$!