Lie Groups Rule The World

In quantum physics, every physical system has a Hilbert space $\mathcal{H}$ whose unit vectors describe states of the system. If a Lie group G acts as symmetries, we have a unitary representation

$$\rho : G \to U(H)$$

This gives a Lie algebra representation

$$d\rho : \mathfrak{g} \to \mathfrak{u}(H)$$

so each $v \in \mathfrak{g}$ gives a self-adjoint operator

$$i\, d\rho(v) : H \to H$$

– that is, an observable! So:

symmetries give observables!

Lets look at examples:

1) Time translation gives energy. The passage of time is described by the Lie group $\mathbb{R}$ where the group operation is addition. Given a unitary rep

$$\rho : \mathbb{R} \to U(H)$$

the unitary operator

$$\rho(t) : H \to H \qquad t \in \mathbb{R}$$

means "wait for a time t". We get a Lie algebra rep

$$d\rho : \mathbb{R} \to \mathfrak{u}(H)$$

where now $\mathbb{R} = T_0\mathbb{R}$ is the Lie algebra with $[\cdot,\cdot] = 0$. The vector $1 \in \mathbb{R}$ gives a self-adjoint operator

$$i\,d\rho(1) : H \to H$$

and this is called <u>energy</u> or the <u>Hamiltonian</u>.

2) Spatial translations give momentum.

Translations in 3d Euclidean space are described by the Lie group $\mathbb{R}^3$ where the group operation is addition. Given a unitary rep

$$\rho : \mathbb{R}^3 \to U(H)$$

the unitary operator

$$\rho(\vec{v}) : H \to H \qquad \vec{v} \in \mathbb{R}^3$$

means "translate the system by the amount $\vec{v} \in \mathbb{R}^3$". We get a Lie algebra rep

$$d\rho : \mathbb{R}^3 \to \mathfrak{u}(H)$$

where now $\mathbb{R}^3 = T_{\vec{0}}\mathbb{R}^3$ is the Lie algebra with $[\cdot,\cdot] = 0$. Any vector $\vec{v} \in \mathbb{R}^3$ gives a self-adjoint operator

$$i\, d\rho(\vec{v}) : H \to H$$

called <u>momentum in the $\vec{v}$ direction</u> (at least when $\vec{v}$ is a unit vector).

3) Rotations give angular momentum.

Rotations in 3d Euclidean space are described by the Lie group $SO(3)$ or its double cover $\mathrm{Spin}(3) \cong \mathrm{Sp}(1) \cong SU(2)$.

Given a unitary rep

$$\rho : SU(2) \to U(H)$$

the unitary operator

$$\rho(g) : H \to H \qquad g \in SU(2)$$

means roughly "rotate the system by the amount $g \in SU(2)$" – but notice two different elements of $SU(2)$ correspond to each element of $SO(3)$. We get a Lie algebra rep

$$d\rho : \mathfrak{su}(2) \to \mathfrak{u}(H)$$

and any vector $\vec{v} \in (\mathbb{R}^3, \times) \cong \mathfrak{su}(2)$ gives a self-adjoint operator

$$i\, d\rho(\vec{v}) : H \to H$$

called <u>angular momentum in the $\vec{v}$ direction</u> (at least when $\vec{v}$ is a unit vector).

We can combine spatial translations and rotations into a single group, the <u>Euclidean group</u> $SO(3) \ltimes \mathbb{R}^3$, where

$$(g, \vec{v})(h, \vec{w}) = (gh, g\vec{w} + \vec{v})$$

This acts on Euclidean space, $\mathbb{R}^3$, by

$$\alpha(g, \vec{v})(\vec{x}) = g\vec{x} + \vec{v}$$

Check that it's an action:

$$\begin{aligned} \alpha(g, \vec{v})\, \alpha(h, \vec{w})(\vec{x}) &= \alpha(g, \vec{v})(h\vec{x} + \vec{w}) \\ &= g(h\vec{x} + \vec{w}) + \vec{v} \\ &= gh\vec{x} + (g\vec{w} + \vec{v}) \\ &= \alpha(gh, g\vec{w} + \vec{v})(\vec{x}) \\ &= \alpha((g, \vec{v}), (h, \vec{w}))(\vec{x}) \end{aligned}$$

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We can also include time translations and get a Lie group

$$\underbrace{\mathbb{R}}_{\text{time translations}} \times \underbrace{(SO(3) \ltimes \mathbb{R}^3)}_{\text{Euclidean group}}$$

This is a 7-dimensional Lie group which acts on <u>spacetime</u>, that is,

$$\mathbb{R}^4 \ni (t, x, y, z) = (t, \vec{x})$$

It acts as follows:

$$\beta(s, g, \vec{v})(t, \vec{x}) = (s+t, \; g\vec{x} + \vec{v})$$

This Lie group is a subgroup of an even bigger group, the 10-dimensional <u>Galilei group</u>, which includes extra symmetries called <u>boosts</u>, that act as follows on spacetime:

$$(t, \vec{x}) \mapsto (t, \vec{x} + t\vec{u}) \qquad \vec{u} \in \mathbb{R}^3$$

where $\vec{v}$ is called the <u>velocity</u>: if the system was stationary before, after this boost it is moving at a velocity $\vec{v}$. If we have a unitary representation of the Galilei group, the boost symmetries give extra observables, called <u>position</u>: our system will have a position!

The Galilei group was our best guess about the symmetry group of the world until Einstein came along. Thanks to his "special relativity", we now use a *different* 10-dimensional Lie group called the "Poincaré group". This is actually simpler than the Galilei group!

In the "Standard Model" of particle physics, our best theory of particles & forces other than gravity, elementary particles are certain unitary irreps of

$$\underbrace{P}_{\text{Poincaré group}} \times \underbrace{SU(3)}_{\text{strong force symmetries}} \times \underbrace{SU(2)}_{\text{weak force symmetries}} \times \underbrace{U(1)}_{\text{electromagnetic force symmetries}}$$

But the Standard Model does not describe

- gravity
- dark matter
- dark energy
- other mysteries?

So, this is not the last word!

THE END