

Lie Groups Rule The World

In quantum physics, every physical system has a Hilbert space H whose unit vectors describe states of the system. If a Lie group G acts as symmetries, we have a unitary representation

$$\rho: G \rightarrow U(H)$$

This gives a Lie algebra representation

$$d\rho: \mathfrak{g} \rightarrow \mathfrak{u}(H)$$

so each $v \in \mathfrak{g}$ gives a self-adjoint operator

$$i d\rho(v): H \rightarrow H$$

— that is, an observable! So:

symmetries give observables!

Lets look at examples:

1) Time translation gives energy. The passage of time is described by the Lie group $\mathbb{R}$ where the group operation is addition.

Given a unitary rep

$$\rho: \mathbb{R} \rightarrow U(H)$$

the unitary operator

$$\rho(t): H \rightarrow H \quad t \in \mathbb{R}$$

means "wait for a time t ". We get a

Lie algebra rep

$$d\rho: \mathbb{R} \rightarrow \mathfrak{u}(H)$$

where now $\mathbb{R} = T_0 \mathbb{R}$ is the Lie algebra

with $[\cdot, \cdot] = 0$. The vector $1 \in \mathbb{R}$

gives a self-adjoint operator

$$i d\rho(1) : H \rightarrow H$$

and this is called energy or the Hamiltonian.

2) Spatial translations give momentum.

Translations in 3d Euclidean space are described by the Lie group $\mathbb{R}^3$ where the group operation is addition. Given a unitary rep

$$\rho : \mathbb{R}^3 \rightarrow U(H)$$

the unitary operator

$$\rho(\vec{v}) : H \rightarrow H \quad \vec{v} \in \mathbb{R}^3$$

means "translate the system by the amount $\vec{v} \in \mathbb{R}^3$ ". We get a Lie algebra rep

$$d\rho : \mathbb{R}^3 \rightarrow \mathfrak{m}(H)$$

where now $\mathbb{R}^3 = T_0 \mathbb{R}^3$ is the Lie algebra

with $[\cdot, \cdot] = 0$. Any vector $\vec{v} \in \mathbb{R}^3$ gives a self-adjoint operator

$$i d\rho(\vec{v}) : H \rightarrow H$$

called momentum in the $\vec{v}$ direction (at least when $\vec{v}$ is a unit vector).

3) Rotations give angular momentum.

Rotations in 3d Euclidean space are

described by the Lie group $SO(3)$ or its

double cover $Spin(3) \cong Sp(1) \cong SU(2)$.

Given a unitary rep

$$\rho : SU(2) \rightarrow U(H)$$

the unitary operator

$$\rho(g) : H \rightarrow H \quad g \in SU(2)$$

means roughly "rotate the system by the amount $g \in SU(2)$ " - but notice two different elements of $SU(2)$ correspond to each element of $SO(3)$. We get a Lie algebra rep

$$d\rho: \mathfrak{su}(2) \rightarrow \mathfrak{m}(H)$$

and any vector $\vec{v} \in (\mathbb{R}^3, \times) \cong \mathfrak{su}(2)$

gives a self-adjoint operator

$$i d\rho(\vec{v}) : H \rightarrow H$$

called angular momentum in the $\vec{v}$ direction

(at least when $\vec{v}$ is a unit vector).

We can combine spatial translations and rotations into a single group, the Euclidean group $SO(3) \ltimes \mathbb{R}^3$, where

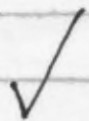
$$(g, \vec{v})(h, \vec{w}) = (gh, g\vec{w} + \vec{v})$$

This acts on Euclidean space, $\mathbb{R}^3$, by

$$\alpha(g, \vec{v})(\vec{x}) = g\vec{x} + \vec{v}$$

Check that it's an action:

$$\begin{aligned}\alpha(g, \vec{v})\alpha(h, \vec{w})(\vec{x}) &= \alpha(g, \vec{v})(h\vec{x} + \vec{w}) \\ &= g(h\vec{x} + \vec{w}) + \vec{v} \\ &= gh\vec{x} + (g\vec{w} + \vec{v}) \\ &= \alpha(gh, g\vec{w} + \vec{v})(\vec{x}) \\ &= \alpha((g, \vec{v})(h, \vec{w}))(\vec{x})\end{aligned}$$



We can also include time translations and get a Lie group

$$\mathbb{R} \times (SO(3) \times \mathbb{R}^3)$$

(time translations (Euclidean group

This is a 7-dimensional Lie group which acts on spacetime, that is,

$$\mathbb{R}^4 \ni (t, x, y, z) = (t, \vec{x})$$

It acts as follows:

$$\beta(s, g, \vec{V})(t, \vec{x}) = (s+t, g\vec{x} + \vec{V})$$

This Lie group is a subgroup of an even bigger group, the 10-dimensional Galilei group, which includes extra symmetries called boosts, that act as follows on spacetime:

$$(t, \vec{x}) \mapsto (t, \vec{x} + t\vec{U}) \quad \vec{U} \in \mathbb{R}^3$$

where $\vec{U}$ is called the velocity: if the system was stationary before, after this boost it is moving at a velocity $\vec{U}$. If we have a unitary representation of the Galilei group, the boost symmetries give extra observables, called position: our system will have a position!

The Galilei group was our best guess about the symmetry group of the world until Einstein came along. Thanks to his "special relativity", we now use a different 10-dimensional Lie group called the "Poincaré group". This is actually simpler than the Galilei group!

In the "Standard Model" of particle physics, our best theory of particles & forces other than gravity, elementary particles are certain unitary irreps of

$$\begin{array}{cccc} P & \times & SU(3) & \times & SU(2) & \times & U(1) \\ \swarrow & & \swarrow & & \swarrow & & \swarrow \\ \text{Poincaré} & & \text{strong} & & \text{weak} & & \text{electromagnetic} \\ \text{group} & & \text{force} & & \text{force} & & \text{force} \\ & & \text{symmetries} & & \text{symmetries} & & \text{symmetries} \end{array}$$

But the Standard Model does not describe

- gravity
- dark matter
- dark energy
- other mysteries?

So, this is not the last word!

THE END