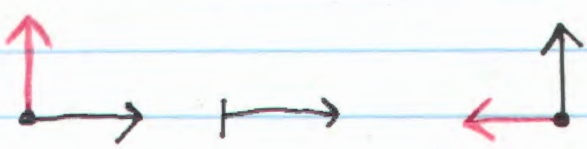
Some more very important examples of matrix Lie groups:

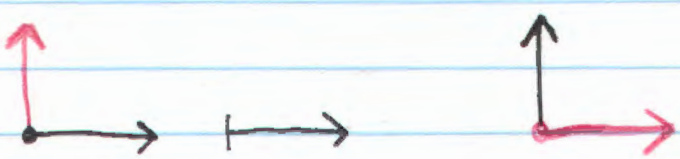
$$\begin{tikzcd}
GL(2,\mathbb{R}) \arrow[rr, hook] & & GL(2,\mathbb{C}) & \\
& O(2) \arrow[ul, hook] \arrow[rr, hook] & & U(2) \arrow[ul, hook] \\
SL(2,\mathbb{R}) \arrow[uu, hook] \arrow[rr, hook] & & SL(2,\mathbb{C}) \arrow[uu, hook] & \\
& SO(2) \arrow[ul, hook] \arrow[uu, hook] \arrow[rr, hook] & & SU(2) \arrow[ul, hook] \arrow[uu, hook]
\end{tikzcd}$$

$GL(2,\mathbb{R})$ is linear transformations of the plane:

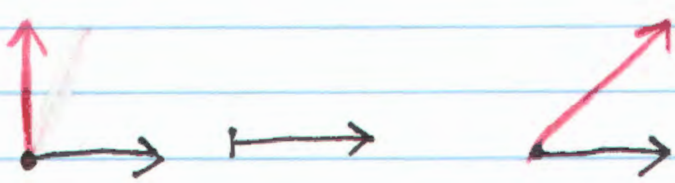
$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$:



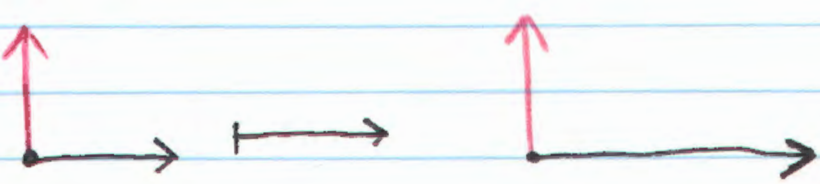
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$:



$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$:



$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$:



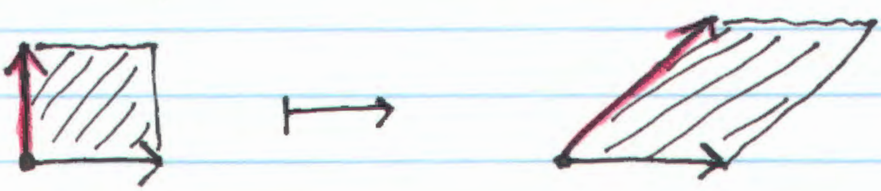
Are these in $O(2)$, $SL(2,\mathbb{R})$, both, neither?

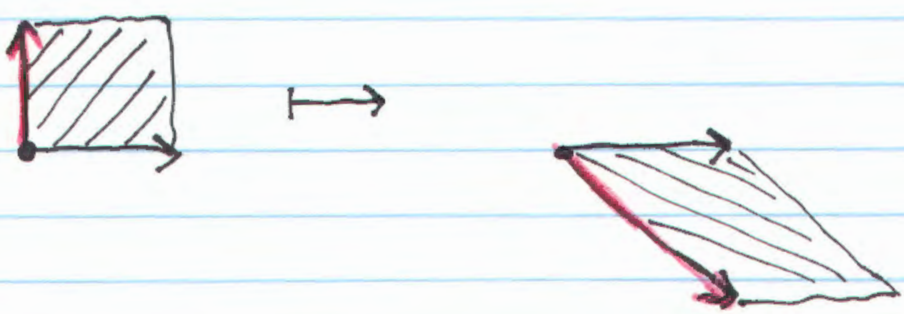
$$O(2) = \{ g \in GL(2,\mathbb{R}) : gg^* = 1 \}$$

consists of all linear transformations of the plane that preserve inner products (dot products), hence unoriented angles & lengths.

$$SL(2,\mathbb{R}) = \{ g \in GL(2,\mathbb{R}) : \det g = 1 \}$$

consists of all linear transformations of the plane that preserve oriented areas:

good: 

bad: 

$$SO(2) = O(2) \cap SL(2,\mathbb{R})$$

$$= \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R} \right\}$$

consists of rotations.

So as Lie groups,

$$SO(2) \cong S^1 = \mathbb{R}/2\pi\mathbb{Z}$$
$$\cong U(1) = \{z \in \mathbb{C} : |z| = 1\}$$

What about $O(2)$?

Lemma – If $g \in U(n)$, $|\det g| = 1$.

Proof – $\det(gg^*) = \det(g)\det(g^*)$
$$= \det(g)\overline{\det(g)}$$

so $gg^* = 1 \Rightarrow |\det(g)|^2 = 1$.

Corollary – If $g \in O(n)$, $\det g = \pm 1$.

So if $g \in O(n)$, either $g \in SO(n)$ or $rg \in SO(n)$ where $r = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$ is a reflection, so $g = rh$ for some $h \in SO(n)$.

So as a <u>manifold</u> $O(n) \cong SO(n) \sqcup SO(n)$.

Q: Is $O(2) \cong \mathbb{Z}_2 \times SO(2)$ as a group?

Q: Is $O(3) \cong \mathbb{Z}_2 \times SO(3)$ as a group?

To better understand $SL(2,\mathbb{R})$, write any 2×2 real matrix this way:

$$g = \begin{pmatrix} x+z & w-y \\ w+y & x-z \end{pmatrix}$$

with $x, y, z, w \in \mathbb{R}$. Then

$$\det g = (x+z)(x-z) - (w+y)(w-y)$$
$$= x^2 + y^2 - z^2 - w^2$$

Thus as a manifold

$$SL(2,\mathbb{R}) \cong \{(x,y,z,w) \in \mathbb{R}^4 : x^2+y^2 = z^2+w^2+1\}$$

The slice where $w=0$ is a hyperboloid with one sheet:

$w=0$:

$x^2+y^2 = z^2+1$

diffeomorphic to $\mathbb{R}\times S^1$: a circle for each $z \in \mathbb{R}$

So,

$SL(2,\mathbb{R}) \cong \{(x,y,z,w) \in \mathbb{R}^4 : x^2+y^2 = z^2+w^2+1\}$ is diffeomorphic to $\mathbb{R}^2 \times S^1$, since we get a circle $x^2+y^2=z^2+w^2+1$ for each choice of $(z,w) \in \mathbb{R}^2$.

The circle with $z=w=0$ is just $SO(2) \leq SL(2,\mathbb{R})$!

$$\left\{ \begin{pmatrix} x+z & w-y \\ w+y & x-z \end{pmatrix} : \begin{matrix} x^2+y^2 = z^2+w^2+1, \\ z=w=0 \end{matrix} \right\}$$

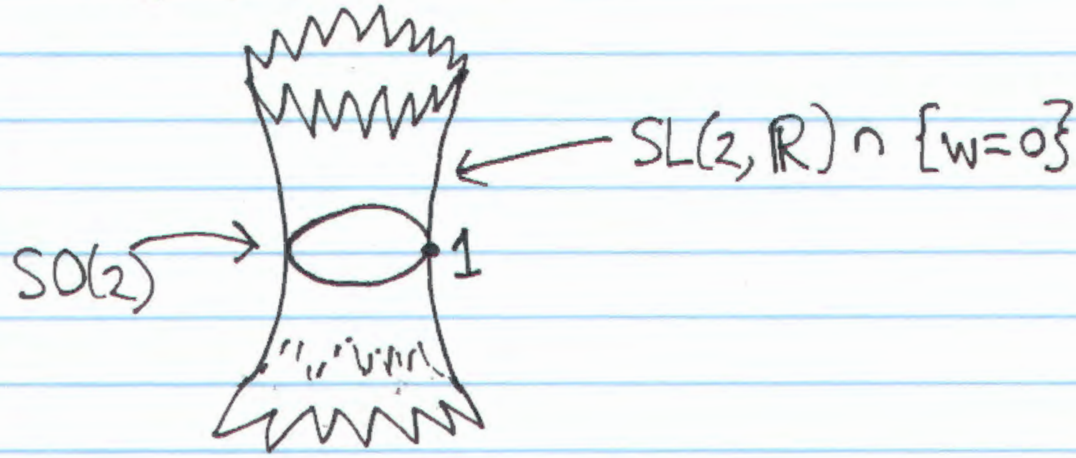
$$\|$$

$$\left\{ \begin{pmatrix} x & -y \\ y & x \end{pmatrix} : x^2+y^2=1 \right\}$$

$$\|$$

$$\left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R} \right\}$$

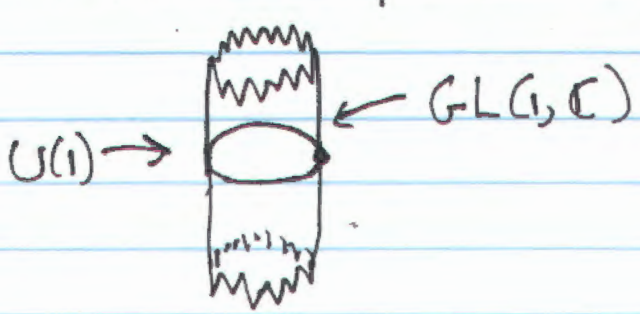
So, we can picture a slice of $SL(2,\mathbb{R})$ like this:



$SL(2,\mathbb{R})$ is diffeomorphic to $SO(2)\times\mathbb{R}^2$ but not isomorphic to it as a Lie group — it's nonabelian!

Thm. – Any connected Lie group G is diffeomorphic to $K \times \mathbb{R}^n$ where $K \subseteq G$ is a <u>maximal</u> <u>compact</u> <u>subgroup</u> of G – a compact subgroup not contained in a bigger compact subgroup. All maximal compact subgroups are conjugate, hence isomorphic as Lie groups.

Another example we've seen:



$$GL(1,\mathbb{C}) \cong U(1)\times\mathbb{R}$$

Thm. —

$U(n)$ is a maximal compact subgroup of $GL(n, \mathbb{C})$
$SU(n)$ " " $SL(n, \mathbb{C})$
$O(n)$ " " $GL(n, \mathbb{R})$
$SO(n)$ " " $SL(n, \mathbb{R})$

$$
\begin{tikzcd}
GL(n,\mathbb{R}) \arrow[rrr, hook] & & & GL(n,\mathbb{C}) & \\
& O(n) \arrow[ul, hook] \arrow[rrr, hook] & & & U(n) \arrow[ul, hook] \\
SL(n,\mathbb{R}) \arrow[uu, hook] \arrow[rrr, hook] & & & SL(n,\mathbb{C}) \arrow[uu, hook] & \\
& SO(n) \arrow[ul, hook] \arrow[uu, hook] \arrow[rrr, hook] & & & SU(n) \arrow[ul, hook] \arrow[uu, hook]
\end{tikzcd}
$$

In fact any inner product $(\cdot,\cdot)$ on $\mathbb{C}^n$ picks out a maximal compact subgroup of $GL(n, \mathbb{C})$, namely

$$K = \{ g \in GL(n, \mathbb{C}) : (gv, gw) = (v, w) \ \forall v, w \}$$

Similarly any inner product on $\mathbb{R}^n$ picks out a maximal compact subgroup of $GL(n, \mathbb{R})$.