What about $SU(2)$? It's a 3-sphere!

To understand this it's best to think about the quaternions, a 4-dimensional algebra called $\mathbb{H}$ after their discoverer, Hamilton.

There are just 3 spheres that are Lie groups:

$$S^0 \cong \{x \in \mathbb{R} : |x| = 1\}$$

$$S^1 \cong \{x \in \mathbb{C} : |x| = 1\}$$

$$S^3 \cong \{x \in \mathbb{H} : |x| = 1\}$$

These come from the 3 associative normed division algebras: finite-dimensional associative algebras over $\mathbb{R}$ with a norm $|\cdot|$ such that

$$|ab| = |a|\,|b|$$

There are just 3 associative normed division algebras: $\mathbb{R}$, $\mathbb{C}$ & $\mathbb{H}$. $\mathbb{R}$ & $\mathbb{C}$ are fields but $\mathbb{H}$ is noncommutative!

$\mathbb{H}$ consists of expressions

$$a = \underbrace{a_0 1}_{\substack{\text{scalar}\\ \text{part}}} + \underbrace{a_1 i + a_2 j + a_3 k}_{\substack{\text{vector}\\ \text{part}}}$$

$$= a_0 + \vec{a}$$

We multiply them using the rules

$$1i = i = i1 \qquad 1j = j = j1 \qquad 1k = k = k1$$

$$i^2 = j^2 = k^2 = -1$$

$$ij = k = -ji$$

$$jk = i = -kj$$

$$ki = j = -ik$$

so

$$ab = (a + \vec{a})(b + \vec{b})$$

$$= a_0 b_0 - \vec{a} \cdot \vec{b} + a_0 \vec{b} + b_0 \vec{a} + \vec{a} \times \vec{b}$$

We define conjugation for quaternions by:

$$a = a_0 + \vec{a} = a_0 + a_1 i + a_2 j + a_3 k$$

$$\Downarrow$$

$$\bar{a} = a_0 - \vec{a} = a_0 - a_1 i - a_2 j - a_3 k$$

Note:

$$a\bar{a} = a_0 a_0 + \vec{a}\cdot\vec{a} - a_0\vec{a} + a_0\vec{a} + \vec{a}\times\vec{a}$$

$$= a_0^2 + a_1^2 + a_2^2 + a_3^2$$

& similarly $\bar{a}a = a_0^2 + a_1^2 + a_2^2 + a_3^2$.

So, we can define a norm on $\mathbb{H}$ by

$$|a| = \sqrt{a\bar{a}} = \sqrt{\bar{a}a}$$

Lemma - for all $a, b \in \mathbb{H}$,

$$|ab| = |a|\,|b|$$

Proof - compute! You'll use lots of identities, especially

$$\|\vec{a}\|^2 \|\vec{b}\|^2 = |\vec{a}\cdot\vec{b}|^2 + \|\vec{a}\times\vec{b}\|^2$$

Lemma - Quaternion multiplication is associative.

Proof - Compute!

Theorem -

$$S^3 = \{ a \in \mathbb{H} : |a| = 1 \}$$

is a group under multiplication.

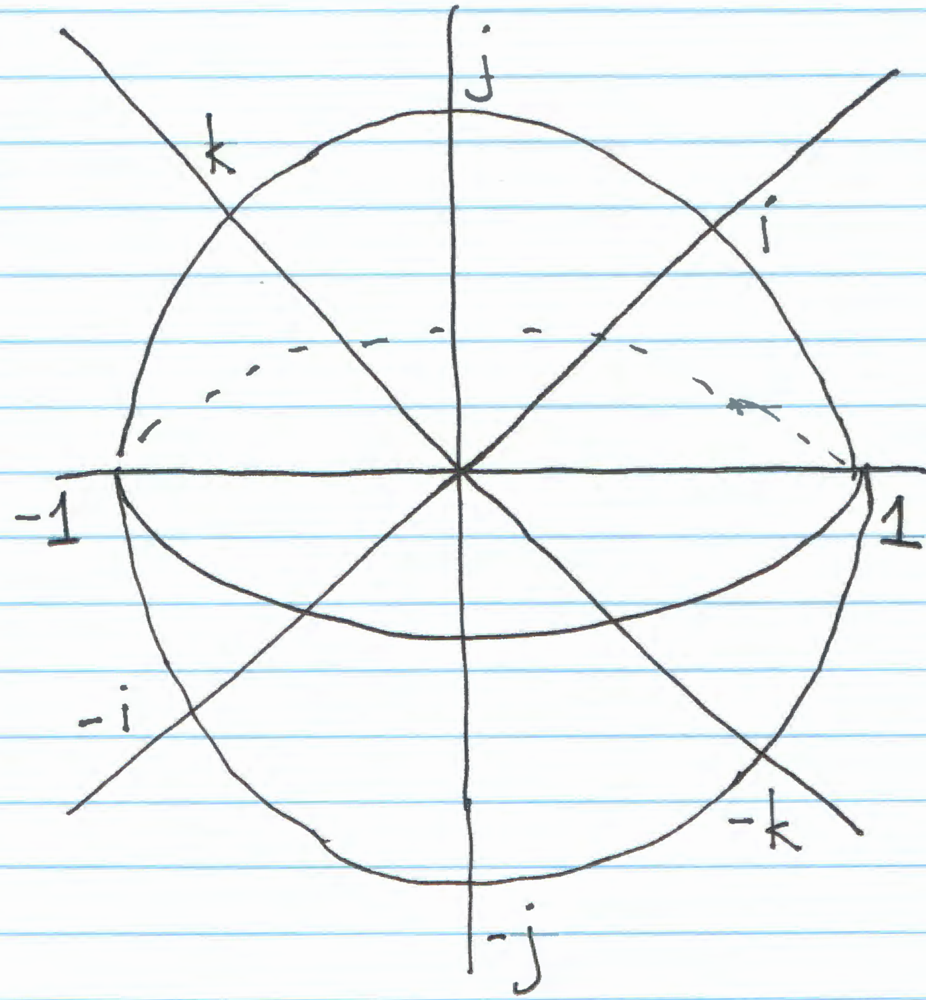
Proof - It's closed under multiplication because

$$|a| = 1, \; |b| = 1 \implies |ab| = |a||b| = 1,$$

multiplication is associative, and each element a has a 2-sided inverse $\bar{a}$:

$$a\bar{a} = \bar{a}a = |a|^2 = 1.$$

$\blacksquare$



j
k
i
-1
1
-i
-k
-j

Theorem - As Lie groups, $SU(2)$ is isomorphic to

$$\{a \in \mathbb{H} : |a| = 1\}$$

so $SU(2)$ is diffeomorphic to S^3.

Proof - To see this, we construct a 1-1 algebra homomorphism

$$\varphi : \mathbb{H} \hookrightarrow M_2(\mathbb{C})$$

and then show

$$\varphi : \{a \in \mathbb{H} : |a| = 1\} \xrightarrow{\sim} SU(2)$$

The key : let

$$I = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad K = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

& check

$$I^2 = J^2 = K^2 = -1$$

$$IJ = K = -JI$$
$$JK = I = -KJ$$
$$KI = J = -IK$$

So, if we define

$$\varphi(a_0 + a_1 i + a_2 j + a_3 k) = a_0 1 + a_1 I + a_2 J + a_3 K$$

then φ is an <u>algebra</u> <u>homomorphism</u>: it's linear and

$$\varphi(ab) = \varphi(a)\varphi(b)$$

since I, J, K obey the same multiplication table as i, j, k. Also:

Lemma - $\varphi(\bar{a}) = \varphi(a)^*$

so φ maps

$$\{a \in \mathbb{H} : |a| = 1\} = \{a \in \mathbb{H} : a\bar{a} = \bar{a}a = 1\}$$

<u>into</u>

$$\{g \in M_2(\mathbb{C}) : gg^* = g^*g = 1\} = U(2)$$

in a 1-1 way (since φ is 1-1).

Why does φ map $\{a \in \mathbb{H} : |a| = 1\}$ into $SU(2)$?

Note if $a \in \mathbb{H}$ then

$$\varphi(a) = a_0 + a_1 I + a_2 J + a_3 K$$

$$= \begin{pmatrix} a_0 - i a_3 & -a_2 - i a_1 \\ a_2 - i a_1 & a_0 + i a_3 \end{pmatrix}$$

so if $|a| = 1$ then

$$\det(\varphi(a)) = (a_0 + i a_3)(a_0 - i a_3) + (a_2 + i a_1)(a_2 - i a_1)$$

$$= a_0^2 + a_3^2 + a_2^2 + a_1^2$$

$$= |a|^2 = 1$$

so $\varphi(a) \in SU(2)$

Why does φ map onto $SU(2)$?
Note $|a| = 1$ implies

$$\varphi(a) = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}$$

with $|\alpha|^2 + |\beta|^2 = 1$.

So, to show

$$\varphi : \{ a \in \mathbb{H} : |a| = 1 \} \to SU(2)$$

is onto, we need to show every $g \in SU(2)$ has

$$g = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \text{ with } |\alpha|^2 + |\beta|^2 = 1$$

Now, if $g \in SU(2)$ then clearly

$$g = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \text{ with } gg^* = 1$$

or $\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ \bar{\gamma} & \bar{\delta} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

so $|\alpha|^2 + |\gamma|^2 = 1, \quad |\beta|^2 + |\delta|^2 = 1, \quad \alpha\bar{\beta} + \gamma\bar{\delta} = 0$

$$\Downarrow$$

so $\qquad \gamma\bar{\delta} = -\alpha\bar{\beta}$

$$\Downarrow$$

$1 = \det g = \alpha\delta - \beta\gamma \qquad \gamma = \dfrac{-\alpha\bar{\beta}}{\bar{\delta}}$

$\qquad = \alpha\delta + \alpha\beta\bar{\beta}/\bar{\delta}$

$$\Downarrow$$

or $\bar{\delta} = \alpha\delta\bar{\delta} + \alpha\beta\bar{\beta} = \alpha \Rightarrow \gamma = -\bar{\beta}$

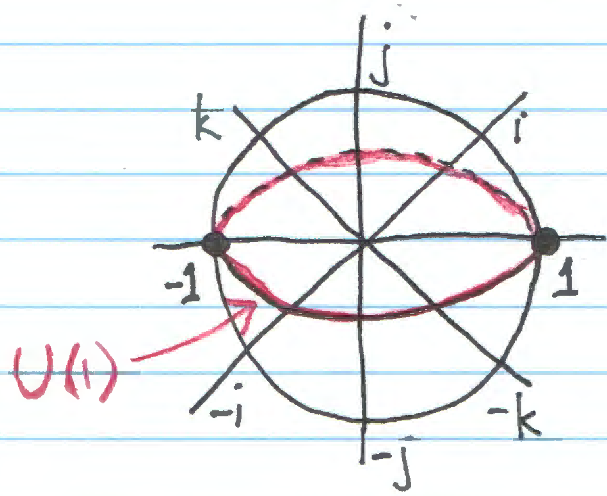
& $g = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}$ with $|\alpha|^2 + |\beta|^2 = 1 \quad \blacksquare$

We have seen

$$S^0 \cong \{x \in \mathbb{R} : |x| = 1\} = O(1) \cong \mathbb{Z}_2$$

$$S^1 \cong \{x \in \mathbb{C} : |x| = 1\} = U(1) \cong SO(2)$$

$$S^3 \cong \{x \in \mathbb{H} : |x| = 1\} = Sp(1) \cong SU(2)$$



We have subgroups

$$O(1) \subseteq U(1) \subseteq Sp(1)$$

or in other words

$$S^0 \subseteq S^1 \subseteq S^3$$

We can define quaternionic analogues of our favorite Lie groups:

$$
\begin{tikzcd}
GL(n,\mathbb{R}) \arrow[rr, hook] & & GL(n,\mathbb{C}) \arrow[rr, hook] & & GL(n,\mathbb{H}) & \\
& O(n) \arrow[ul, hook] \arrow[rr, hook] & & U(n) \arrow[ul, hook] \arrow[rr, hook] & & Sp(n) \arrow[ul, hook] \\
SL(n,\mathbb{R}) \arrow[uu, hook] \arrow[rr, hook] & & SL(n,\mathbb{C}) \arrow[uu, hook] \arrow[rr, hook] & & SL(n,\mathbb{H}) \arrow[uu, hook] & \\
& SO(n) \arrow[ul, hook] \arrow[uu, hook] \arrow[rr, hook] & & SU(n) \arrow[ul, hook] \arrow[uu, hook] \arrow[rr, hook] & & Sp(n) \arrow[ul, hook] \arrow[uu, hook]
\end{tikzcd}
$$

$$GL(n,\mathbb{H}) = \{ g \in M_n(\mathbb{H}) : g \text{ invertible} \}$$

$$Sp(n) = \{ g \in GL(n,\mathbb{H}) : gg^* = 1 \}$$

$$SL(n,\mathbb{H}) = \{ g \in GL(n,\mathbb{H}) : \det(g) = 1 \}$$

However quaternionic determinants are subtle — they are <u>complex</u>-valued, and

$$gg^* = 1 \implies \det(g) = 1$$

so $Sp(n) \subseteq SL(n,\mathbb{H})$ & there are just 3 quaternionic groups in the chart above, not 4!