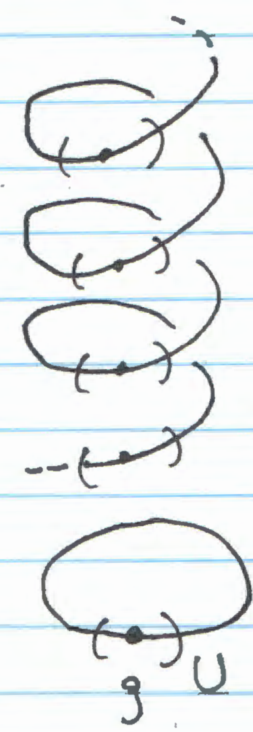
Universal Covers

A homomorphism of Lie groups $f: H \to G$ is a <u>cover</u> if every $g \in G$ has an open neighborhood U such that $f^{-1}(U)$ is a disjoint union of open sets U_α such that

$$f\big|_{U_\alpha} : U_\alpha \longrightarrow U$$

is a homeomorphism (& thus a diffeomorphism).



$$\begin{array}{c} \mathbb{R} \\ \Big\downarrow f \\ U(1) \cong SO(2) \cong S^1 \end{array} \qquad f(x) = e^{ix}$$

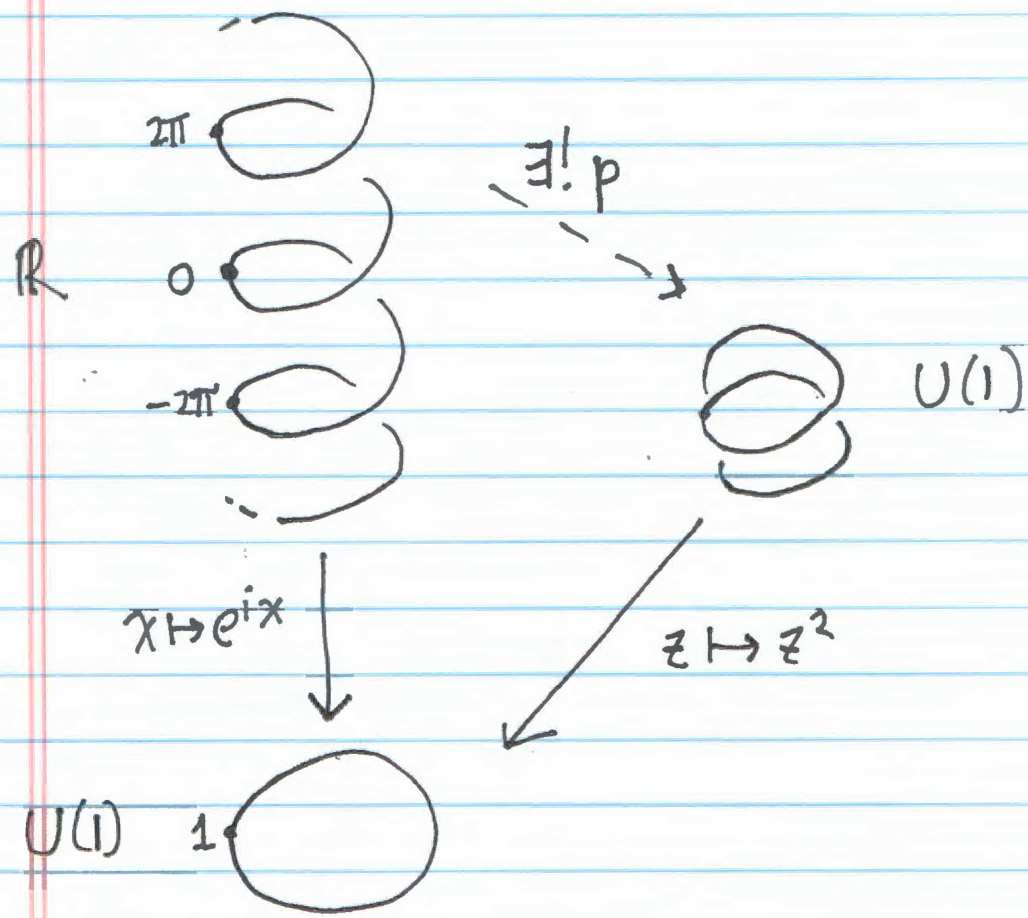
Theorem - Every connected Lie group G has a <u>universal</u> <u>cover</u> $\pi : \tilde{G} \to G$: a cover such that for any cover $p: H \to G$ there exists a unique Lie group homomorphism $p: \tilde{G} \to H$ such that this diagram commutes:

$$\begin{tikzcd}
\tilde{G} \arrow[dr, "\exists! p"] \arrow[dd, "\pi"'] & \\
& H \arrow[dl, "f"] \\
G &
\end{tikzcd}$$

In fact p is a cover.

Theorem - A cover $p: H \to G$ is universal iff H is <u>simply connected</u>: it's connected & every loop in H is contractible.

Theorem - The kernel of $\pi : \tilde{G} \to G$ is isomorphic to the fundamental group of G, $\pi_1(G)$.



$\mathbb{R}$ is the universal cover of $U(1)$, so $\exists!\ p : \mathbb{R} \to U(1)$ such that the triangle commutes :

$$e^{ix} = p(x)^2 \qquad \forall x \in \mathbb{R}$$

What is p?

Note the kernel of

$$\begin{aligned} \mathbb{R} &\to U(1) \\ x &\mapsto e^{ix} \end{aligned}$$

is $2\pi\mathbb{Z}$, isomorphic to $\pi_1(U(1)) \cong \mathbb{Z}$.

$SO(2) \cong U(1)$, so we've just seen

$$\widetilde{SO(2)} \cong \mathbb{R} \quad \& \quad \pi_1(SO(2)) \cong \mathbb{Z}.$$

Next we'll see

$$\widetilde{SO(3)} \cong SU(2) \quad \& \quad \pi_1(SO(3)) \cong \mathbb{Z}_2.$$

Some harder facts:

$$\widetilde{SO(4)} \cong SU(2) \times SU(2) \quad \& \quad \pi_1(SO(4)) \cong \mathbb{Z}_2$$

$$\widetilde{SO(5)} \cong Sp(2) \quad \& \quad \pi_1(SO(5)) \cong \mathbb{Z}_2$$

$$\widetilde{SO(6)} \cong SU(4) \quad \& \quad \pi_1(SO(6)) \cong \mathbb{Z}_2$$

No higher $\widetilde{SO(n)}$'s have cute descriptions like this, but we define the <u>spin groups</u>

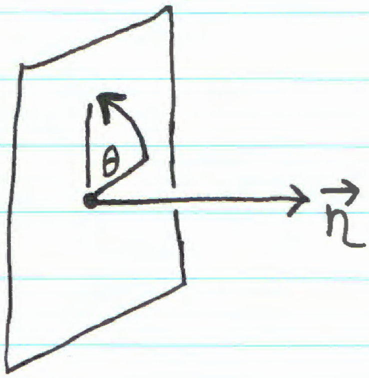
$$\mathrm{Spin}(n) = \widetilde{SO(n)} \qquad (n > 2)$$

& we have

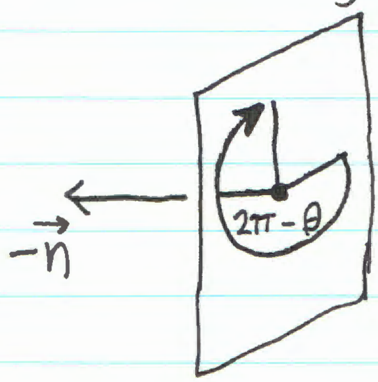
$$\pi_1(SO(n)) \cong \mathbb{Z}_2 \qquad (n > 2)$$

so $\pi : \mathrm{Spin}(n) \to SO(n)$ is a 2-1 Lie group homomorphism.

The topology of SO(3)



Every rotation $g \in SO(3)$ fixes a unit vector $\vec{n} \in \mathbb{R}^3$ and rotates the plane orthogonal to $\vec{n}$ counterclockwise by some angle θ. But it also fixes $-\vec{n}$ and rotates counterclockwise by $-\theta$ about $-\vec{n}$, or counterclockwise by $2\pi - \theta$:

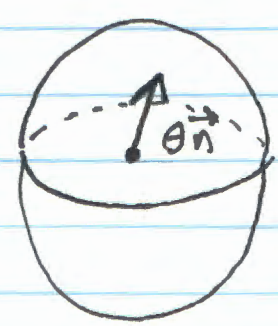


So any rotation $g \in SO(3)$ is rotation counterclockwise about some unit vector $\vec{n}$ by an angle $0 \leq \theta \leq \pi$, and this $\vec{n}$ is unique unless:

1) $\theta = 0$: then every $\vec{n}$ gives the same rotation, namely $g = 1 \in SO(3)$

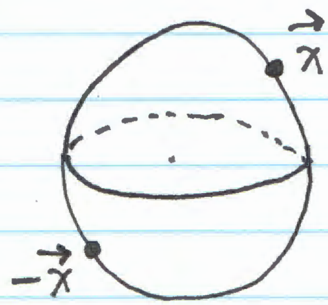
2) $\theta = \pi$: then $\vec{n}$ & $-\vec{n}$ give the same rotation.

So, we can describe $g \in SO(3)$ by a point $\theta\vec{n}$ in the ball of radius π, where the points $\pi\vec{n}$ & $-\pi\vec{n}$ are identified.



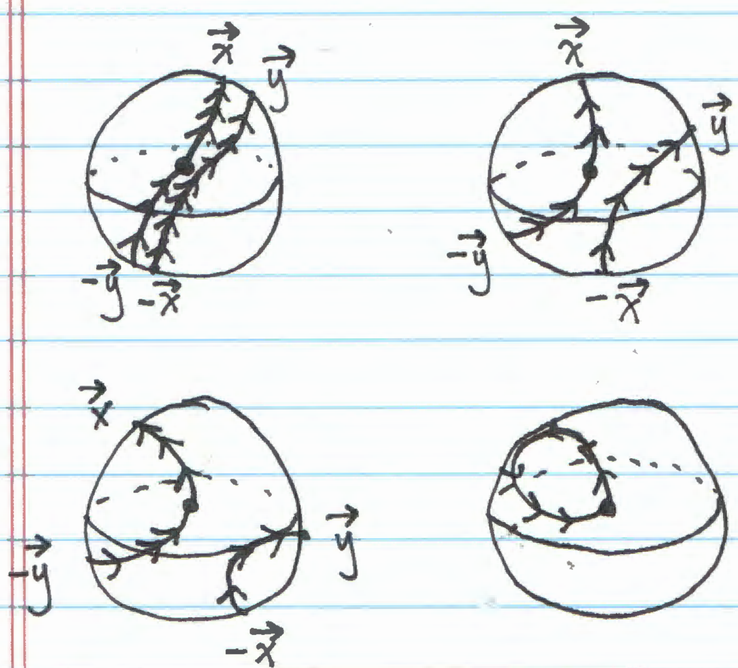
$0 \leq \theta \leq \pi$

So, $SO(3)$ is homeomorphic to a 3d ball, B^3, with opposite points on its boundary identified:



rotating π counterclockwise around $\vec{x}$ = rotating π counterclockwise around $-\vec{x}$

This space is called $\mathbb{R}P^3$, <u>real projective 3-space</u>. It has $\pi_1(\mathbb{R}P^3) \cong \mathbb{Z}_2$:



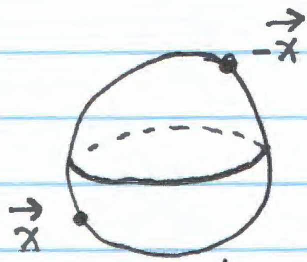
The loop in $SO(3)$ where we rotate two full turns ($4\pi = 720°$) about an axis is contractible!

We can think of $\mathbb{RP}^n$ in two ways:

1) It's the ball B^n with opposite points on its boundary S^{n-1} identified:

$n=2$

2) It's the sphere S^n with opposite points identified:



$n=2$

The second way makes it easier to prove that $\mathbb{RP}^n$ is a smooth n-dimensional manifold with a cover

$$\begin{array}{c} S^n \\ \pi\big\downarrow \\ \mathbb{RP}^n \end{array}$$

with π being 2-1. When $n=3$ we get

$$\begin{array}{ccc} S^3 & \underset{\text{diffeo}}{\cong} & SU(2) \\ \pi\big\downarrow & & \\ \mathbb{RP}^3 & \underset{\text{diffeo}}{\cong} & SO(3) \end{array}$$

This suggests:

Theorem – There is a 2-1 Lie group homomorphism $\pi : SU(2) \to SO(3)$ making $SU(2)$ into a universal cover of $SO(3)$.

Proof – Write

$$\underset{\substack{\text{scalars} \\ a_0}}{\mathbb{H} = \mathbb{R}} \oplus \underset{\substack{\text{vectors} \\ a_1 i + a_2 j + a_3 k = \vec{a}}}{\mathbb{R}^3}$$

In a lemma we'll show that if

$g \in \{ x \in \mathbb{H} : |x| = 1 \} = Sp(1) \cong SU(2)$

and $\vec{v} \in \mathbb{R}^3 \subseteq \mathbb{H}$ then $g \vec{v} g^{-1} \in \mathbb{R}^3$.

Moreover $|g \vec{v} g^{-1}| = |g| \, |\vec{v}| \, |g^{-1}| = |\vec{v}|$

so the map

$$\vec{v} \mapsto g \vec{v} g^{-1}$$

preserves lengths.

$$\rho(g): \begin{array}{rcl} \mathbb{R}^3 & \to & \mathbb{R}^3 \\ \vec{v} & \mapsto & g\vec{v}g^{-1} \end{array}$$

preserves lengths and it's linear so $\rho(g) \in O(3)$.

$$\rho : Sp(1) \to O(3)$$

is a group homomorphism:

$$\begin{aligned} \rho(gh)(\vec{v}) &= gh\vec{v}(gh)^{-1} \\ &= gh\vec{v}h^{-1}g^{-1} \\ &= \rho(g)\rho(h)\vec{v} \qquad \forall g,h \in Sp(1) \end{aligned}$$

ρ is smooth so it's a Lie group homomorphism. $Sp(1) \overset{\text{diffeo}}{\cong} S^3$ is connected so ρ maps $Sp(1)$ into one connected component of $O(3) \overset{\text{diffeo}}{\cong} SO(3) \sqcup SO(3)$, the component containing $\rho(1) = 1$.

So,

$$\rho : Sp(1) \to SO(3)$$

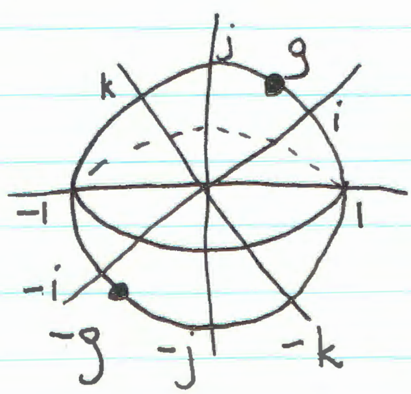
is a Lie group homomorphism. It's at least 2-1 since $g \in Sp(1) \Rightarrow -g \in Sp(1)$ and

$$\rho(-g)\vec{v} = -g\vec{v}(-g)^{-1} = g\vec{v}g^{-1} = \rho(g)\vec{v}$$

With more work one can show ρ is 2-1, onto, and a cover. It's just our friend

$$\pi : S^3 \to \mathbb{R}P^3$$

in disguise! Since S^3 is simply connected ρ is a universal cover.



$g, -g \in Sp(1)$ map to same element of $SO(3)$

In short,

$$\rho : \underset{\substack{\wr\| \\ SU(2)}}{Sp(1)} \longrightarrow SO(3)$$

is a universal cover as desired. $\blacksquare$

Lemma – If $g \in Sp(1)$ & $\vec{v} \in \mathbb{R}^3 \subseteq \mathbb{H}$ then $g\vec{v}g^{-1} \in \mathbb{R}^3 \subseteq \mathbb{H}$.

Proof – A quaternion $a \in \mathbb{H}$ has $a \in \mathbb{R}^3$ iff it's "imaginary": $a^* = -a$. So we have $\vec{v}^* = -\vec{v}$ and want to show $(g\vec{v}g^{-1})^* = -(g\vec{v}g^{-1})^*$. Since $|g| = 1$ we have $g^{-1} = g^*$, and in general $(ab)^* = b^*a^*$ & $a^{**} = a$, so

$$(g\vec{v}g^{-1})^* = (g\vec{v}g^*)^* = g^{**}\vec{v}^*g^* = -g\vec{v}g^{-1}.$$

$\blacksquare$