Compact Lie groups

Theorem - The Lie groups $O(n)$, $U(n)$ & $Sp(n)$ are compact.

Proof - We do $U(n)$ because they're all similar.

$$U(n) = \{ g \in M_n(\mathbb{C}) : gg^* = g^*g = 1 \}$$

This is clearly a <u>closed</u> subset of the finite-dimensional real vector space $M_n(\mathbb{C}) \cong \mathbb{R}^{2n^2}$, since $g \mapsto gg^*$ and $g \mapsto g^*g$ are continuous and $\{1\}$ is closed in $M_n(\mathbb{C})$.

Thus to show $U(n)$ is compact we show it is also <u>bounded</u>:

$$gg^* = 1 \Rightarrow \sum_j g_{ij} (g^*)_{jk} = \delta_{ik}$$

$$\Rightarrow \sum_j g_{ij} \overline{g_{kj}} = \delta_{ik}$$

$$\overset{i=k}{\Rightarrow} \sum_j |g_{ij}|^2 = 1$$

so $|g_{ij}|^2 \leq 1$ for all $i,j = 1, \dots, n$. $\blacksquare$

Cor. – The groups $SO(n)$, $SU(n)$ & $Spin(n) = \widetilde{SO(n)}$ $(n > 2)$ are compact.

Proof – $SO(n)$ is a closed subset of $O(n)$ since det is continuous; a closed subset of a compact set is compact. Similarly $SU(n)$ is a closed subset of $O(n)$.

For $Spin(n)$ we need the fact that $\widetilde{SO(n)}$ is a double cover of $SO(n)$ when $n > 2$, since $\pi_1(SO(n)) = \mathbb{Z}_2$ for $n > 2$. Then use:

Lemma – A finite-to-one cover of a compact topological space is compact.

$\square$

Note $\widetilde{SO(2)} \cong \mathbb{R}$ is not compact, and $\pi_1(SO(2)) \cong \mathbb{Z}$ is not finite, so $\widetilde{SO(2)}$ is not a finite-to-one cover.

By the way, $GL(n,\mathbb{R})$, $GL(n,\mathbb{C})$, $GL(n,\mathbb{H})$, $SL(n,\mathbb{R})$, $SL(n,\mathbb{C})$, $SL(n,\mathbb{H})$ are not compact except for some trivial cases like

$$SL(1,\mathbb{R}) \cong SL(1,\mathbb{C}) \cong SL(1,\mathbb{H}) \cong 1.$$

It's impossible to classify compact Lie groups, since a 0-dimensional compact Lie group is the same as a finite group, and these are a huge mess:

Theorem (Besche, Eick & O'Brien, 2001): Up to isomorphism there are

$$49{,}910{,}529{,}484$$

groups of order ≤ 2000, ~99.15% of which have order 1024.

However, amazingly, we *can* classify compact *connected* Lie groups!

Some examples we've seen: $SO(n)$, $SU(n)$, $Sp(n)$, $U(n)$, $\mathrm{Spin}(n)$ but not

$$O(n) \overset{\text{diffeo}}{\cong} SO(n) \sqcup SO(n)$$

Theorem - Every compact connected Lie group K has a finite cover L that is a product of:

- a compact connected abelian Lie group
- a compact simply connected Lie group

Every compact connected abelian Lie group is isomorphic to a <u>torus</u>

$$\mathbb{T}^n = \underbrace{U(1) \times \cdots \times U(1)}_{n}$$

for some n.

Every compact simply connected Lie group is isomorphic to a product of finitely many copies of:

- $Spin(n) = \widetilde{SO}(n)$ $(n > 2)$
- $SU(n)$ $(n \geq 1)$
- $Sp(n)$ $(n \geq 1)$

and five more, called

- F_4, G_2, E_6, E_7, E_8.

Note: the theorem doesn't quite classify all compact connected Lie groups K, but only certain finite covers L of these, which are products of a torus and a simply connected compact Lie group. The covering map

$$p: L \to K$$

has a kernel $\ker p$ that can be any finite normal subgroup of L. We can classify all the possibilities & thus all compact Lie groups... but not today!

$SO(n)$, $SU(n)$, $Sp(n)$ & $Spin(n)$ are called <u>classical</u> compact Lie groups. F_4, G_2, E_6, E_7 & E_8 are called <u>exceptional</u> & they're built using "octonions."

Example: How about $U(n)$? The theorem says it's covered by a product of $SO(n)$'s, $SU(n)$'s, $Sp(n)$'s, $U(1)$'s, etc.

Every $g \in U(n)$ has $gg^* = 1$ so $|\det g|^2 = 1$ so $\det g \in U(1)$. $\det(gh) = \det(g)\det(h)$ & $\det 1 = 1$ so

$$\det : U(n) \to U(1)$$

is a Lie group homomorphism with kernel $SU(n)$.

Is $U(n) \cong U(1) \times SU(n)$?

We have a Lie group homomorphism

$$p: U(1) \times SU(n) \longrightarrow U(n)$$
$$(\alpha, h) \longmapsto \begin{pmatrix} \alpha & & 0 \\ & \alpha & \\ 0 & & \ddots \\ & & & \alpha \end{pmatrix} h$$

p is onto: given $g \in U(n)$,

$$h = \begin{pmatrix} \alpha^{-1} & & 0 \\ & \alpha^{-1} & \\ 0 & & \ddots \; \alpha^{-1} \end{pmatrix} g$$

has

$$\det h = \alpha^{-n} \det g$$

so $h \in SU(n)$ if $\alpha^n = \det g$, and then

$$g = \begin{pmatrix} \alpha & & 0 \\ & \alpha & \\ 0 & & \ddots \; \alpha \end{pmatrix} h \in \operatorname{im} p$$

But p is not 1-1!

$$p(\alpha, h) = \begin{pmatrix} \alpha & & 0 \\ & \ddots & \\ 0 & & \alpha \end{pmatrix} h = 1$$

if $h = \begin{pmatrix} \alpha^{-1} & & 0 \\ 0 & \ddots & \alpha^{-1} \end{pmatrix} \in SU(n)$, which is true iff

$\alpha^n = 1$. So $\ker p \cong \mathbb{Z}_n$, the nth

roots of unity.

In short

$$p : U(1) \times SU(n) \to U(n)$$

is n-to-1 and onto!

Theorem – If a Lie group homomorphism $p: H \to G$ is n-to-1 and onto, it is a cover.

Corollary – $p: U(1) \times SU(n) \to U(n)$ is a cover.