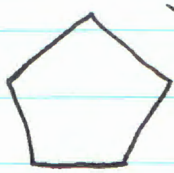
Lie group actions

Groups act as symmetries of things:



$\mathbb{Z}_5$ acts on the regular pentagon

Lie groups act as symmetries of manifolds:

$U(1)$ acts on the circle

Definition – An <u>action</u> A of a Lie group G on a manifold X is a smooth map

$$A : G \times X \to X$$

such that $\forall g, h \in G$ and $\forall x \in X$

$$A(1, x) = x$$

$$A(gh, x) = A(g, A(h, x)).$$

We say G <u>acts</u> on X via A.

Given an action $A: G \times X \to X$, for each $g \in G$ we define

$$\alpha(g): X \to X$$

by

$$\alpha(g)(x) = A(g,x)$$

Then $\alpha(g)$ is smooth and

$$\alpha(1) = 1_X$$

$$\alpha(gh) = \alpha(g)\alpha(h) \quad \forall g,h \in G$$

so therefore

$$\alpha(g^{-1}) = \alpha(g)^{-1}$$

Thus $\alpha(g): X \to X$ is a diffeomorphism of X

The diffeomorphisms of X form a group $\mathrm{Diff}(X)$ and the equations above imply

$$\alpha: G \to \mathrm{Diff}(X)$$

is a group homomorphism.

Lemma - If $\alpha : G \to \mathrm{Diff}(X)$ is a group homomorphism and the function

$$A: G \times X \to X$$

defined by

$$A(g,x) = \alpha(g)(x)$$

is smooth, then A is an action of G on X.

Example: $O(n)$ acts on the sphere

$$\{x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i^2 = 1\} = S^{n-1}$$

via $A(g,x) = gx$. Similarly $U(n)$ acts on the sphere

$$\{x \in \mathbb{C}^n : \sum |x_i|^2 = 1\} \underset{\text{diffeo}}{\cong} S^{2n-1}$$

and $Sp(n)$ acts on the sphere

$$\{x \in \mathbb{H}^n : \sum |x_i|^2 = 1\} \underset{\text{diffeo}}{\cong} S^{4n-1}$$

We can <u>transport</u> an action along a diffeomorphism:

Lemma - If $\alpha : G \to \mathrm{Diff}(X)$ gives an action of G on X and $f : X \to Y$ is a diffeomorphism then $\beta : G \to \mathrm{Diff}(Y)$ gives an action of G on Y where

$$\beta(g) = f \circ \alpha(g) \circ f^{-1} \qquad \forall g \in G.$$

Example : $U(n)$ acts on

$$\{ x \in \mathbb{C}^n : \sum |x_i|^2 = 1 \} \underset{\text{diffeo}}{\cong} S^{2n-1}$$

so it acts on S^{2n-1}.

We can <u>restrict</u> an action to a closed subgroup:

Lemma – If $H \leq G$ is a closed subgroup of a Lie group G and $\alpha : G \to \mathrm{Diff}(X)$ gives an action of G on the manifold X, then H is a Lie group and $\beta : H \to \mathrm{Diff}(X)$ gives an action of H on X, where

$$\beta(h) = \alpha(h) \qquad \forall h \in H$$

Example: Since $O(n)$ acts on S^{n-1}, so does $SO(n) \leq O(n)$. Since $U(n)$ acts on S^{2n-1}, so does $SU(n) \leq U(n)$.