Klein's ideas on geometry

In his "Erlangen program" of 1872, Felix Klein argued geometry – Euclidean geometry and different kinds of non-Euclidean geometry – is really the study of groups and their actions! We'll look at examples with Lie groups acting on manifolds.

Example: <u>Euclidean geometry</u>. The Lie group $O(n)$ acts on $\mathbb{R}^n$ by

$$\alpha(g)\,x = gx \qquad g \in O(n),\ x \in \mathbb{R}^n$$

The Lie group $\mathbb{R}^n$ acts on $\mathbb{R}^n$ by

$$\beta(v)\,x = x + v \qquad x, v \in \mathbb{R}^n$$

Both these actions preserve distances & angles.

Can we combine these actions into a single action of a bigger Lie group?

Yes! – the Euclidean group $E(n)$.

As a manifold

$$E(n) = O(n) \times \mathbb{R}^n$$

but it is not the product of these Lie groups – the multiplication is different.

We can figure it out by trying to define an action $\gamma : E(n) \to \mathrm{Diff}(\mathbb{R}^n)$ by

$$\gamma(g, v)(x) = gx + v \qquad \begin{matrix} g \in O(n) \\ x, v \in \mathbb{R}^n \end{matrix}$$

$$\gamma(1, 0)(x) = 1x + 0 = x$$

so $(1, 0) \in E(n)$ will be the identity.

But what's the multiplication in $E(n)$?

Given $(g,v), (h,w) \in E(n)$ and $x \in \mathbb{R}^n$,

$$\begin{aligned}\gamma(g,v)\,\gamma(h,w)\,x &= \gamma(g,v)\,(hx+w) \\ &= g(hx+w)+v \\ &= ghx + gw + v\end{aligned}$$

This will be $\gamma((g,v)(h,w))(x)$ iff

$$(g,v)(h,w) = (gh, gw+v)$$

so this how we define multiplication in $E(n)$.

Check that it's associative &

$$(g,v)^{-1} = (g^{-1}, -g^{-1}v)$$

The multiplication & inverses are smooth so $E(n)$ is a Lie group.

Theorem – every function $f: \mathbb{R}^n \to \mathbb{R}^n$ preserving distances ($\|fv - fw\| = \|v-w\|$) equals $\gamma(g,v)$ for some $(g,v) \in E(n)$.

There is not only an action γ of $E(n)$ on the manifold $P = \mathbb{R}^n$ of <u>points</u> in $\mathbb{R}^n$; there is also a manifold L of <u>lines</u> in $\mathbb{R}^n$ and an action of $E(n)$ on that:

$$\delta : E(n) \to \mathrm{Diff}(L)$$

Puzzle: prove that L is a manifold.

There is an important relation: "the point $p \in P$ lies on the line $\ell \in L$". This defines a submanifold

$$I = \{ (p, \ell) \in P \times L : p \text{ lies on } \ell \} \subseteq P \times L$$

and

$$(p, \ell) \in I \implies (\gamma(g) p, \gamma(g) \ell) \in I$$

so in fact $E(n)$ acts on I.

Here we rely on some general facts:

Lemma - If G acts on X and Y it acts on $X \times Y$. If $\alpha : G \to \mathrm{Diff}(X)$ & $\beta : G \to \mathrm{Diff}(Y)$ are actions there is an action

$$(\alpha, \beta) : G \to \mathrm{Diff}(X \times Y)$$

given by

$$(\alpha,\beta)(g)(x,y) = (\alpha(g)x, \beta(g)y).$$

Lemma - If $\alpha : G \to \mathrm{Diff}(X)$ is an action and $Y \subseteq X$ is a submanifold such that

$$x \in Y \Rightarrow \alpha(g)x \in Y \quad \forall g \in G$$

then there is an action $\beta : G \to \mathrm{Diff}(Y)$ given by

$$\beta(g)(y) = \alpha(g)(y) \quad \forall g \in G,\ y \in Y$$

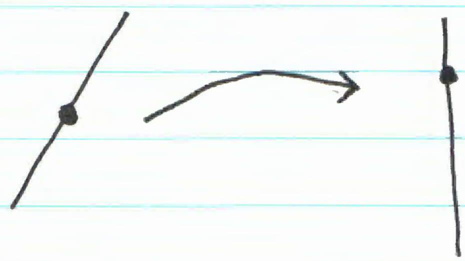
Definition – An action $\alpha : G \to \mathrm{Diff}(X)$ is <u>transitive</u> if

$$\forall x, y \in X \quad \exists g \in G \quad \alpha(g)(x) = y$$

Example: $E(n)$ acts transitively on P (points) L (lines) and

$$I = \{ (p, \ell) : p \text{ lies on } \ell \}$$

(so-called <u>point-line flags</u>).



This says that all points "look alike" in the geometry governed by the Euclidean group $E(n)$, and similarly for lines and point-line flags.

In Klein's approach to geometry, a Lie group G gives a *geometry*. An action $\alpha : G \to \mathrm{Diff}(X)$ gives a *type of figure*: the elements of X are *figures* of this type (e.g. points, lines, etc.) All figures of this type "look alike" iff α is transitive.

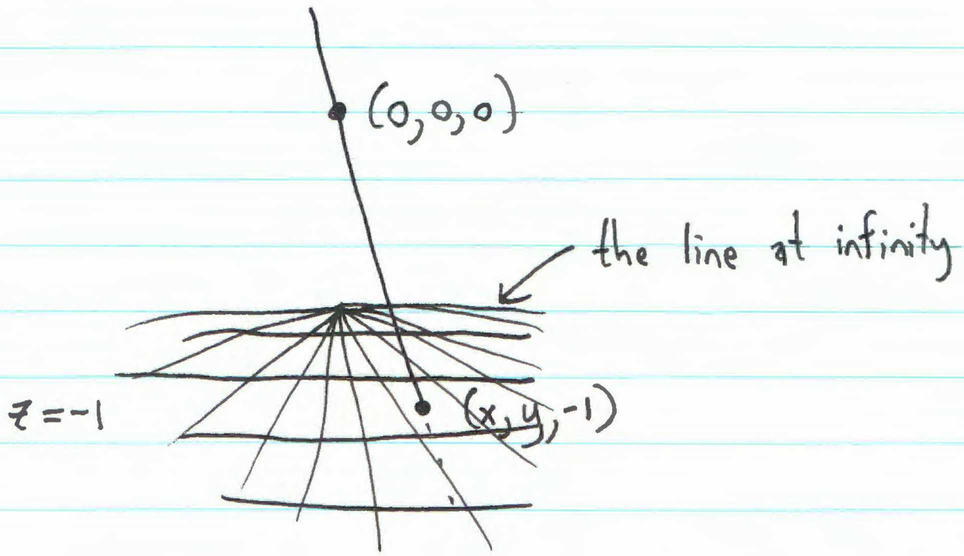
Famous groups give famous geometries:

$E(n)$ — Euclidean geometry

$O(n)$ — spherical geometry

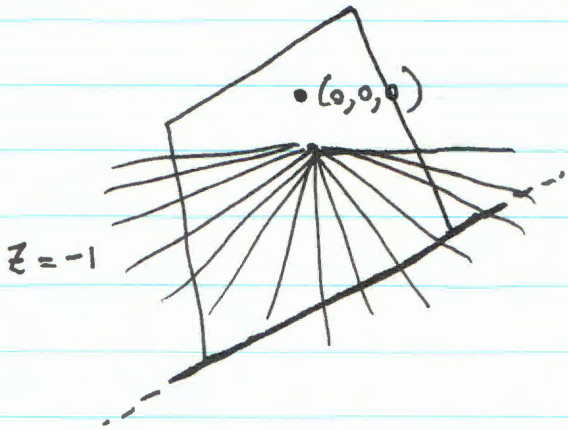
$$\left.\begin{array}{l} GL(n,\mathbb{R}) - \text{real projective geometry} \\ GL(n,\mathbb{C}) - \text{complex projective geometry} \\ GL(n,\mathbb{H}) - \text{quaternionic projective geometry} \end{array}\right\} \text{approximately}$$

The group $GL(3,\mathbb{R})$ acts on the set P of lines through the origin in $\mathbb{R}^3$, which is diffeomorphic to the <u>real projective plane</u> $\mathbb{R}P^2$:

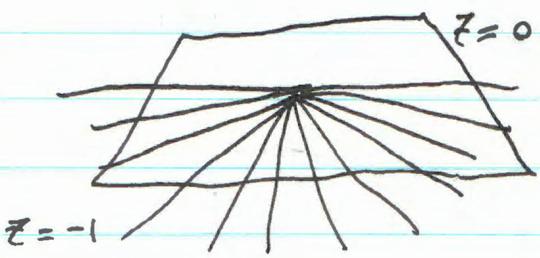


$\mathbb{R}P^2$ can be seen as the union of $\mathbb{R}^2$ and the "line at infinity", $\mathbb{R}P^1$, consisting of lines through the origin in $\mathbb{R}^3$ that don't intersect the plane $\{(x,y,-1) : x,y \in \mathbb{R}^2\}$

$GL(3,\mathbb{R})$ also acts on the set L of planes through the origin in $\mathbb{R}^3$, which give lines in $\mathbb{RP}^2$:

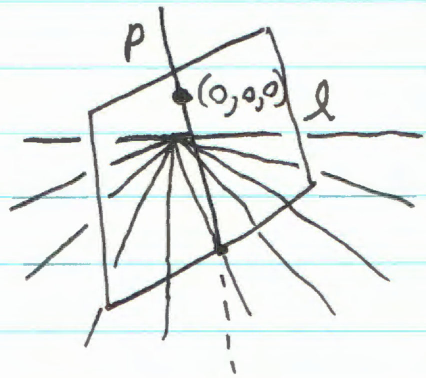


All but one plane through the origin in $\mathbb{R}^3$ intersects the plane $z=-1$ in an ordinary line as above. The plane $z=0$ does not intersect $z=-1$. This gives the <u>line at infinity</u> in $\mathbb{RP}^2$.

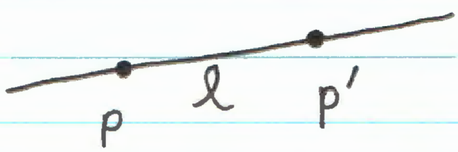


Note a "point" $p \in P$ is a 1d subspace of $\mathbb{R}^3$;
a "line" $\ell \in L$ is a 2d subspace of $\mathbb{R}^3$.

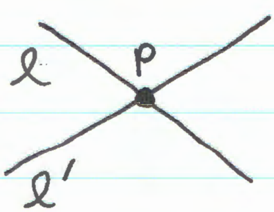
We say a point $p \in P$ <u>lies</u> <u>on</u> a line $\ell \in L$ if $p \subseteq \ell$:



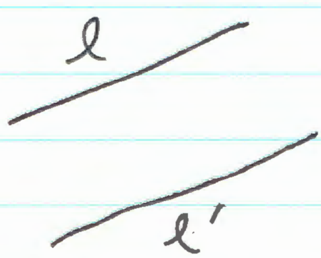
Projective geometry fixes a "flaw" in Euclidean plane geometry, where every pair of distinct points lies on a unique line:



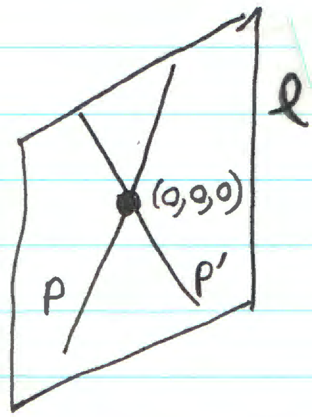
but not every pair of distinct lines has a unique point lying on them:



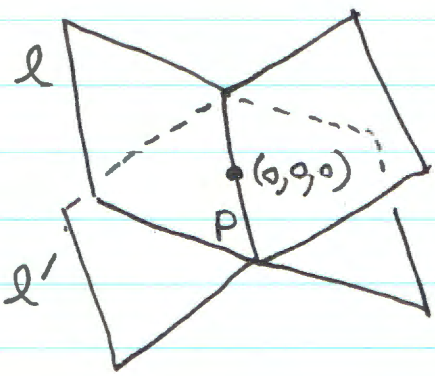
but



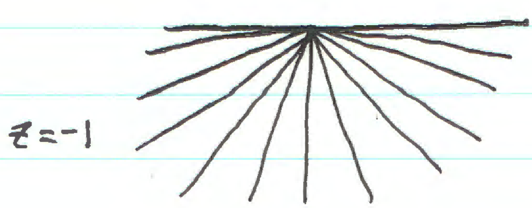
In projective plane geometry every pair of distinct points lies on a unique line:



and every pair of distinct lines has a unique point lying on them:



Parallel lines in $\{z=-1\}$ intersect in a point on the line at infinity:



The sets of points:

$$P = \{ \text{1d subspaces of } \mathbb{R}^3 \}$$

lines:

$$L = \{ \text{2d subspaces of } \mathbb{R}^3 \}$$

and point-line flags

$$I = \{ (p, \ell) \in P \times L : p \subseteq \ell \}$$

are manifolds, and $GL(3, \mathbb{R})$ acts transitively on all three.

All this generalizes to n dimensions: there's a manifold called a <u>Grassmannian</u>

$$Gr(k, \mathbb{R}^n) = \{ k\text{-dimensional subspaces of } \mathbb{R}^n \}$$

and various "flag manifolds" like

$$\{ (U, V, W) \in Gr(2, \mathbb{R}^6) \times Gr(3, \mathbb{R}^6) \times Gr(5, \mathbb{R}^6) : U \subseteq V \subseteq W \}$$

and $GL(n, \mathbb{R})$ acts transitively on all these!

All this stuff – projective geometry, Grassmannians, flag manifolds – works the same way for $\mathbb{C}^n$ & $\mathbb{H}^n$ as well, giving us interesting manifolds on which the Lie groups $GL(n, \mathbb{C})$ and $GL(n, \mathbb{H})$ act transitively.

For a deep introduction try:

- Hans Freudenthal, Lie groups in the foundations of geometry, Advances in Mathematics 1 (1964), 145-190.

Available free online!