Lie Group Representations

The simplest manifolds are finite-dimensional vector spaces, and the simplest smooth maps are linear maps. So:

Definition — If G is a Lie group, an action

$$A : G \times X \to X$$

is called a <u>representation</u> of G or <u>rep</u> if X is a finite-dimensional (real or complex) vector space and each map

$$\begin{aligned} \alpha(g) : X &\to X \\ x &\mapsto A(g,x) \end{aligned}$$

is linear.

We say α is <u>real</u> or <u>complex</u> depending on which case we're considering. We often try to handle both cases simultaneously.

Definition - If V is a vector space, let $GL(V)$ be the group of invertible linear maps $T: V \to V$

We have $GL(V) \cong GL(n, \mathbb{R})$ if $V \cong \mathbb{R}^n$ is real and $GL(V) \cong GL(n, \mathbb{C})$ if $V \cong \mathbb{C}^n$, so in either case $GL(V)$ is a Lie group.

Henceforth let G be a Lie group and V be a finite-dimensional real or complex vector space.

Lemma - $R: G \times V \to V$ is a representation if and only if $\rho: G \to GL(V)$ is a Lie group homomorphism, where

$$\rho(g): V \to V$$
$$v \mapsto R(g, v)$$

There is a category $\mathrm{Act}(G)$ where:

- objects are actions of G: $A: G \times X \to X$
- a morphism from $A: G \times X \to X$ to

$B: G \times Y \to Y$ is a smooth map $f: X \to Y$

such that

$$\begin{tikzcd} G \times X \arrow[r, "A"] \arrow[d, "1 \times f"'] & X \arrow[d, "f"] \\ G \times Y \arrow[r, "B"'] & Y \end{tikzcd}$$

commutes, i.e.

$$f(A(g,x)) = B(g, f(x)) \quad \forall g \in G, x \in X$$

or

$$f(\alpha(g)(x)) = \beta(g)(f(x)) \quad \forall g \in G, x \in X$$

where as usual

$$\alpha(g)(x) = A(g,x)$$

$$\beta(g)(y) = B(g,y)$$

Act(G) has a subcategory Rep(G) where:

- objects are representations of G:

$$R: G\times V \to V$$

- a morphism from $R: G\times V \to V$ to $S: G\times W \to W$ is a linear map $f: V \to W$ such that

$$\begin{tikzcd} G\times V \arrow[r, "R"] \arrow[d, "1\times f"'] & V \arrow[d, "f"] \\ G\times W \arrow[r, "S"'] & W \end{tikzcd}$$

commutes, i.e.

$$f(R(g,v)) = S(g, f(v)) \quad \forall g\in G, v\in V$$

or

$$f(\rho(g)v) = \sigma(g)(f(v)) \quad \forall g\in G, v\in V$$

where as usual

$$\rho(g)(v) = R(g,v)$$

$$\sigma(g)(w) = S(g,w)$$

An object of $\mathrm{Act}(G)$ is sometimes called a *G-space*, and a morphism is often called a *G-equivariant map* or *equivariant map*. A morphism in $\mathrm{Rep}(G)$ is often called an *intertwining operator* or *intertwiner*.

Example: The group $GL(n,\mathbb{R})$ has a *tautological representation* on $\mathbb{R}^n$, where any $g \in GL(n,\mathbb{R})$ acts on any $x \in \mathbb{R}^n$ by matrix multiplication. This corresponds to the Lie group homomorphism

$$1 : GL(n,\mathbb{R}) \to GL(\mathbb{R}^n) = GL(n,\mathbb{R})$$

Any closed subgroup of $GL(n,\mathbb{R})$ also has a tautological representation, e.g.

$$i : SL(n,\mathbb{R}) \to GL(\mathbb{R}^n)$$

$$i' : O(n) \to GL(\mathbb{R}^n)$$

$$i'' : SO(n) \to GL(\mathbb{R}^n)$$

Similarly $GL(n, \mathbb{C})$ or any closed subgroup of $GL(n, \mathbb{C})$ has a tautological rep on $\mathbb{C}^n$.

Example: Given two reps of G, say $\rho : G \to GL(V)$ and $\sigma : G \to GL(W)$, their <u>direct</u> <u>sum</u> is a rep $\rho \oplus \sigma : G \to GL(V \oplus W)$ given by

$$(\rho \oplus \sigma)(g)(v, w) = (\rho(g)v, \sigma(g)w)$$

There are morphisms of reps

$$\begin{tikzcd} \rho \arrow[dr, "i_1"'] & & \sigma \arrow[dl, "i_2"] \\ & \rho \oplus \sigma \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \rho & & \sigma \end{tikzcd}$$

$$i_1(v) = (v, 0)$$
$$i_2(w) = (0, w)$$

$$p_1(v, w) = v$$
$$p_2(v, w) = w$$

Example: Given two reps $\rho: G \to GL(V)$, $\sigma: G \to GL(W)$, their <u>tensor product</u> is a rep $\rho \otimes \sigma : G \to GL(V \otimes W)$ given by

$$(\rho \otimes \sigma)(g)(v \otimes w) = \rho(g)(v) \otimes \sigma(g)(w)$$

Example: For any rep $\rho: G \to GL(V)$

$$\rho \otimes \rho \cong S^2\rho \oplus \Lambda^2\rho$$

in $\mathrm{Rep}(G)$, where $S^2\rho$ & $\Lambda^2\rho$ are reps we'll define. First, note that for any vector space V we have a vector space

$$S^2 V = \left\{ \sum_i v_i \otimes v_i' + v_i' \otimes v_i : v_i, v_i' \in V \right\} \subseteq V \otimes V$$

of <u>symmetric</u> tensors & a space

$$\Lambda^2 V = \left\{ \sum_i v_i \otimes v_i' - v_i' \otimes v_i : v_i, v_i' \in V \right\} \subseteq V \otimes V$$

of <u>skew-symmetric</u> or <u>antisymmetric</u> tensors, Lemma – $V \otimes V \cong S^2 V \oplus \Lambda^2 V$.

Proof – Define a linear map by

$$\sigma : V \otimes V \longrightarrow V \otimes V$$
$$v \otimes v' \longmapsto v' \otimes v$$

Note $\sigma^2 = 1$ and

$$S^2 V = \{ x \in V \otimes V : \sigma(x) = x \}$$

$$\Lambda^2 V = \{ x \in V \otimes V : \sigma(x) = -x \}$$

It now suffices to show if $T : L \to L$ is any linear map with $T^2 = 1$ then

$$L \cong \{ x \in L : Tx = x \} \oplus \{ x \in L : Tx = -x \}$$

To see this, first note if $Tx = x$ & $Tx = -x$ then $x = 0$ since $x = Tx = -x$. Then note any $x \in L$ is a sum of vectors in the two subspaces:

$$x = \frac{x + Tx}{2} + \frac{x - Tx}{2}$$

where $T\left(\frac{x \pm Tx}{2}\right) = \pm\left(\frac{x \pm Tx}{2}\right)$. $\boxed{/\!/}$

Next, if $\rho : G \to GL(V)$ is a rep, note that $\forall g \in G$ the linear map

$$(\rho \otimes \rho)(g) : V \otimes V \to V \otimes V$$

maps the subspace S^2V into itself, and similarly for $\Lambda^2 V$:

$$x \in S^2V \Rightarrow x = \sum_i v_i \otimes v_i' + v_i' \otimes v_i$$

$$\Rightarrow (\rho\otimes\rho)(g)(x) = \sum_i \rho(g)(v_i) \otimes \rho(g)(v_i') + \rho(g)(v_i') \otimes \rho(g)(v_i)$$

$$\Rightarrow (\rho\otimes\rho)(g)(v) \in S^2V$$

and similarly for $\Lambda^2 V$. Next use:

Lemma - If $\rho : G \to GL(X)$ is a rep & $Y \leq X$ is a vector subspace with

$$y \in Y \Rightarrow \rho(g) y \in Y \qquad \forall g \in G$$

then ρ restricts to a rep $\tilde{\rho} : G \to GL(Y)$ with

$$\tilde{\rho}(g) = \rho(g)|_Y : Y \to Y \qquad \forall g \in G$$

So, given a rep $\rho : G \to GL(V)$, we get a rep $\rho \otimes \rho : G \to GL(V \otimes V)$, and we can define reps

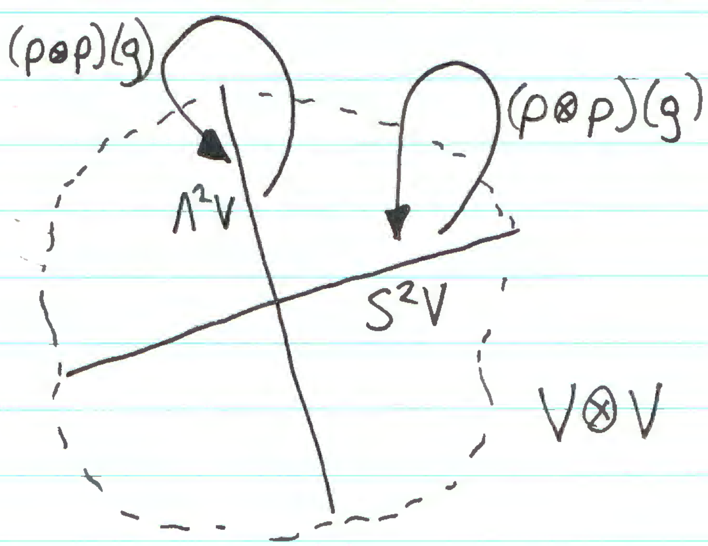
$$S^2\rho(g) := (\rho \otimes \rho)\big|_{S^2V} : S^2V \to S^2V$$

and

$$\Lambda^2\rho(g) := (\rho \otimes \rho)\big|_{\Lambda^2V} : \Lambda^2V \to \Lambda^2V$$

and since $V \otimes V \cong S^2V \oplus \Lambda^2 V$ we get

$$\rho \otimes \rho \cong S^2\rho \oplus \Lambda^2\rho$$



Example: $GL(2,\mathbb{C})$ has a tautological rep on $\mathbb{C}^2$. Following the physicists, write the standard basis of $\mathbb{C}^2$ as $\uparrow, \downarrow$.

Then

$$\mathbb{C}^2 = \langle \uparrow, \downarrow \rangle \leftarrow \text{span : set of linear combinations}$$

$$\mathbb{C}^2 \otimes \mathbb{C}^2 = \langle \uparrow\otimes\uparrow, \uparrow\otimes\downarrow, \downarrow\otimes\uparrow, \downarrow\otimes\downarrow \rangle$$

$$S^2\mathbb{C}^2 = \langle \uparrow\otimes\uparrow, \downarrow\otimes\downarrow, \uparrow\otimes\downarrow + \downarrow\otimes\uparrow \rangle$$

$$\Lambda^2\mathbb{C}^2 = \langle \uparrow\otimes\downarrow - \downarrow\otimes\uparrow \rangle$$

and the tautological rep tensored with itself is the direct sum of a 3d rep on $S^2\mathbb{C}^2$ and a 1d rep on $\Lambda^2\mathbb{C}^2$.

Same for $SL(2,\mathbb{C})$, $U(2)$ & $SU(2)$.

Some important jargon : if $\rho : G \to GL(X)$ is a rep, $Y \leq X$ and

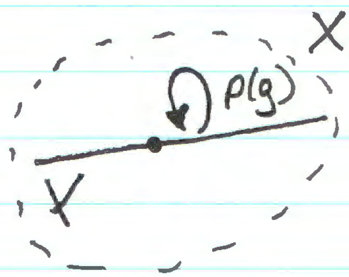
$$y \in Y \Rightarrow \rho(g)(y) \qquad \forall g \in G$$

then we call Y an <u>invariant</u> <u>subspace</u> of X.

If we then define a rep $\tilde{\rho} : G \to GL(Y)$ as in the lemma:

$$\tilde{\rho}(g) = \rho(g)\big|_Y : Y \to Y$$

we call $\tilde{\rho}$ a <u>subrepresentation</u> or <u>subrep</u> of ρ.



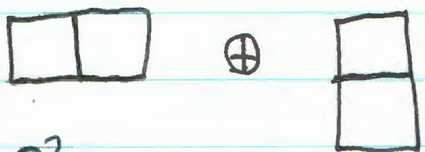
Note the inclusion $i : Y \to X$ gives a morphism of reps $i : \tilde{\rho} \to \rho$ since

$$\underset{\rho(g)(y)}{\underset{\shortparallel}{\rho(g)(i(y))}} = \underset{\tilde{\rho}(g)(y)}{\underset{\shortparallel}{i(\tilde{\rho}(g)(y))}} \quad \forall g \in G, y \in Y$$

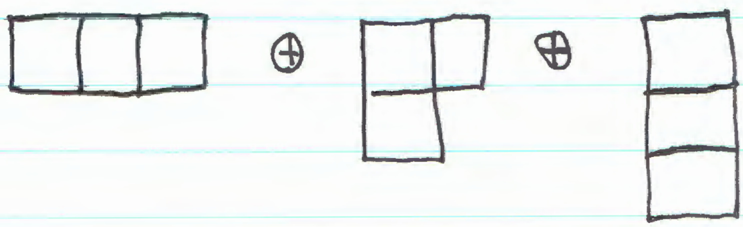
Example: $\rho\otimes\rho$ has $S^2\rho$ and $\Lambda^2\rho$ as subreps and

$$\rho\otimes\rho \cong S^2\rho \oplus \Lambda^2\rho$$

We can generalize this to $\rho\otimes\rho\otimes\rho$, $\rho\otimes\rho\otimes\rho\otimes\rho$, etc. using "Young diagrams":

$\rho\otimes\rho \cong$ 

$S^2\rho$ $\Lambda^2\rho$

$\rho\otimes\rho\otimes\rho \cong$ 

etc.

But why bother? In quantum mechanics, if ρ describes a particle of some kind, $\rho\otimes\rho$ describes 2 particles of this kind, etc. So we need this math for physics!