MATHEMATICS IN THE 21ST CENTURY

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MARCH 25, 2021

Mathematics will continue to change dramatically in the 21st century, both for

- “endogenous” reasons — its internal logic

and

- “exogenous” reasons — coming from the outside world.

I’ll talk about both — but especially how mathematics may be affected by, and affect, the Anthropocene.

In the early 21st century, mathematicians are trying to do some exciting things. We could all make our own list. Here’s mine.

1. **Embrace homotopical / higher-categorical thinking.**

   I am pretty strongly convinced that there is an ongoing reversal in the collective consciousness of mathematicians: the right hemispherical and homotopical picture of the world becomes the basic intuition, and if you want to get a discrete set, then you pass to the set of connected components of a space defined only up to homotopy. — Yuri Manin, 2009

$(\infty, 1)$-categories: doing math where everything holds "up to homotopy".

homotopy type theory: doing math with new foundations based on homotopy types rather than sets.

full-fledged ∞-categories: Grothendieck's dream.

2. **Understand quantum field theory & string theory.**

Only the loftiest peaks, which reach above the clouds, are seen in the mathematical theories of today, and these splendid peaks are studied in isolation, because above the clouds they are isolated from one another. Still lost in the mist is the body of the range, with its quantum field theory bedrock and the great bulk of the mathematical treasures. So there is one rather safe, though perhaps seemingly provocative, prediction about twenty-first century mathematics: trying to come to grips with quantum field theory will be one of the main themes. — Edward Witten, 1998

3. Broaden our view of algebraic geometry.

The struggle to understand the "field with one element", tropical algebra, etc., is pushing us to generalize the duality

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that underlies algebraic geometry.

What sort of "space" has $\mathbb{Z}$ as its ring of functions, really? Can understanding this help us prove the Riemann Hypothesis?

What is the subject of commutative algebra really about?

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Commutative monoid objects "up to homotopy"?

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Items 1–3 are all related, and they interact in exciting ways!

We can't ascend these peaks as solitary explorers! We also need to do some other things:

4. **Make mathematics more computer-friendly.**

Ultimately all results should be computer-verified, easy to access world-wide, and annotated in many ways.

More importantly:

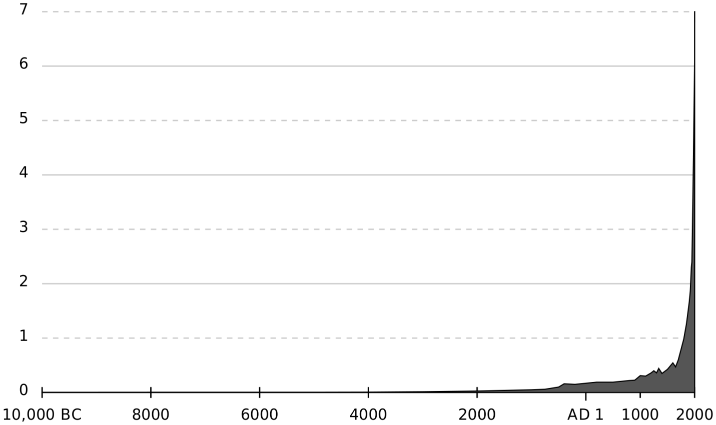
5. **Make mathematics more human-friendly.**

Math is harder to understand than it needs to be. Khan Academy, Wikipedia, nLab, etc. are great — but we could do much more. The challenge starts in kindergarten.

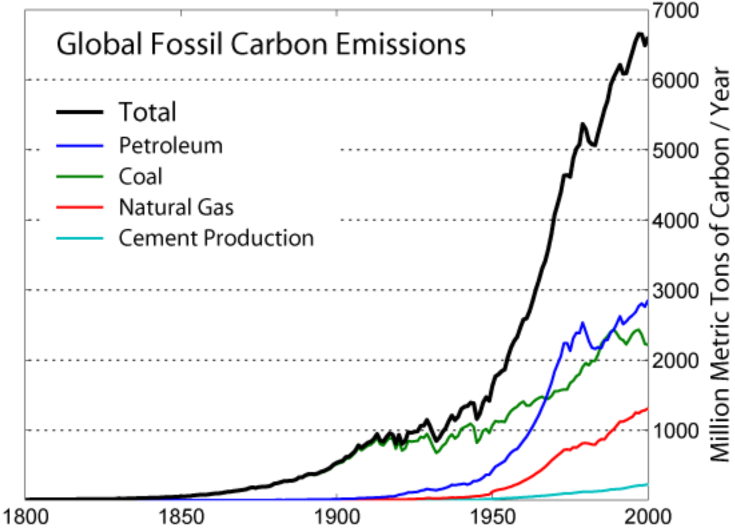
Items 4&5 are connected.

One reason better math education is urgent: we're in a slow-burning crisis.

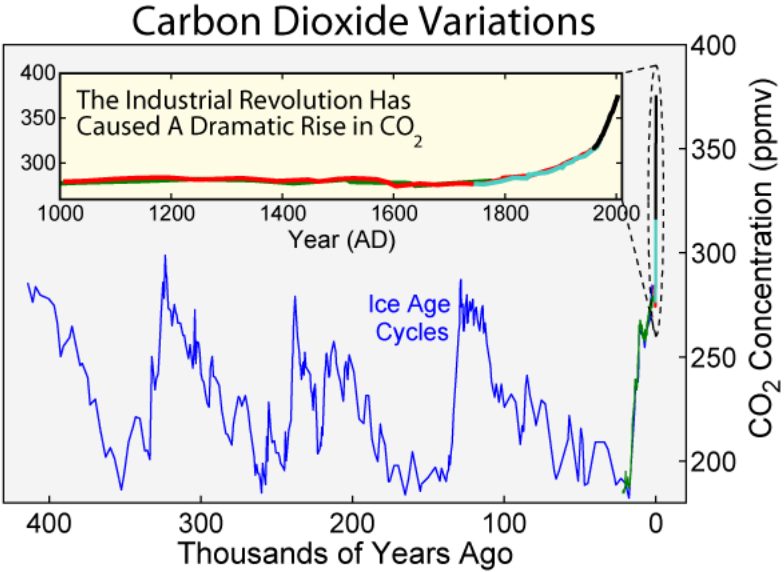
We have left the Holocene and entered a new epoch, the Anthropocene, when the biosphere is rapidly changing due to human activities. Global warming is just *part* of this process.



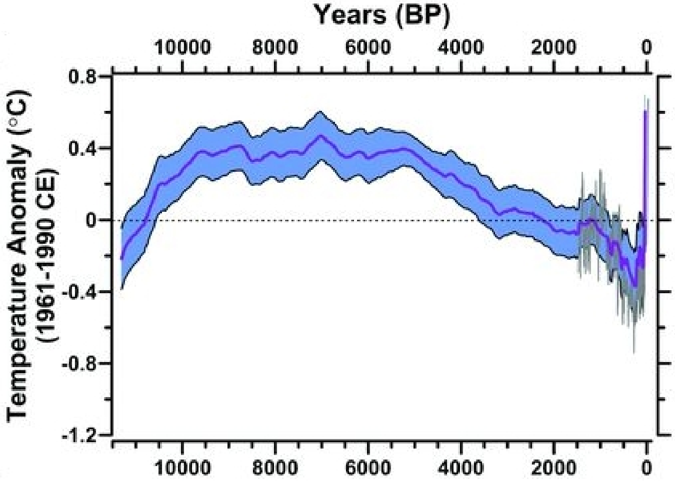
HUMAN POPULATION, IN BILLIONS



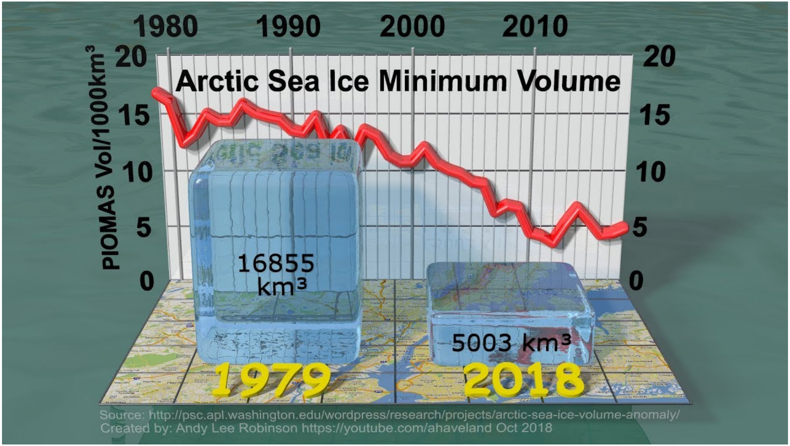
Carbon Dioxide Information Analysis Center



Antarctic ice cores and other data — Global Warming Art



Reconstruction of temperature from 73 different records — Marcott *et al.*



PIOMAS data, art by Andy Lee Robinson

San Francisco — September 9, 2020

WE DOMINATE THE BIOSPHERE

- About 1/4 of all chemical energy produced by plants is now used by humans.
- The biomass of farmed birds is 2.5 times that of wild birds.
- The biomass of farmed mammals is 14 times that of wild mammals.
- Populations of large ocean fish have declined 90% since 1950.
- The rate of species going extinct is 100-1000 times the usual background rate.
- Humans now take more nitrogen from the atmosphere and convert it into nitrates than all other processes combined.
- 8-9 times as much phosphorus is flowing into oceans than the natural background rate.

In view of all this, what should mathematicians do?

Like everyone else, we should do our part to curb global warming.

US citizens emit 16 tonnes of carbon dioxide per year on average.

- **Stop flying to conferences.** By not flying from Riverside to San Francisco I saved ~1 tonne of carbon dioxide.
- **Eat less meat.** Switching from a high-meat diet to a vegan diet can cut your CO_2 emissions about 1.5 tonnes per year.
- **Consider having fewer children, or adopting.** In the US each child adds ~9400 tonnes of CO_2 to the air.

But what what special things can *mathematicians* do?

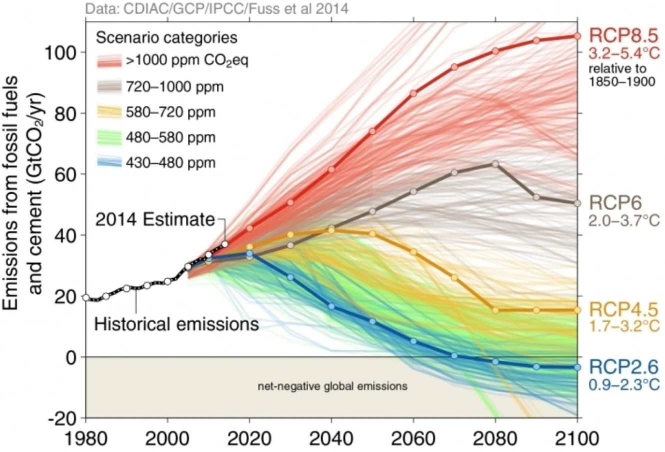
Pure mathematicians tend to feel helpless. “Is there any way I can help solve global warming while continuing to work on algebraic K-theory?”

But in fact there is a lot that even pure mathematicians can do.

1. **Learn more about the problems and solutions — and talk to people about them!**

If you’re watching this talk, you are probably better than average at dealing with quantitative data. For global warming, *the numbers matter.* Let me give you some.

To keep global warming below 2°C by 2100, it's likely that we will need *negative carbon emissions*:



But how much?

We are now putting 38 gigatonnes of CO_2 into the air each year.

To keep global warming below 2°C by 2100, we will probably need to remove about 10 gigatonnes of CO_2 from the air each year by 2050, and double that by 2100.

How could we do this? Some proposed solutions just aren't big enough: *the numbers matter!*

For example: make plastics from CO_2 in the air? We make about 360 megatonnes of plastic per year — not nearly enough to make a difference!

We could pull down over 10 gigatonnes of CO_2 per year this way:

- 1 gigatonne by planting trees,
- 1.5 gigatonnes by better forest management,
- 3 gigatonnes by better agricultural practices,
- 5.2 gigatonnes by biofuels with carbon capture.

This is not enough to cancel the 38 gigatonnes we're dumping into the air each year now, but *together with reducing emissions* it could be enough!

2. **Work on math connected to real-world problems.**

If you're doing applied math: shift toward math that helps the world, instead of making the rich and powerful even more so.

If you're doing pure math: shift your research *a little bit* toward real-world problems. Start talking to people in these subjects:

- applied algebraic topology
- applied algebraic geometry
- applied category theory
- etc...

The Topos Institute promises to be a great place for this.

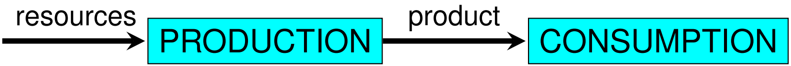
As an example: I'm using category theory to study networks. Why?

In general, we have been pretending that:

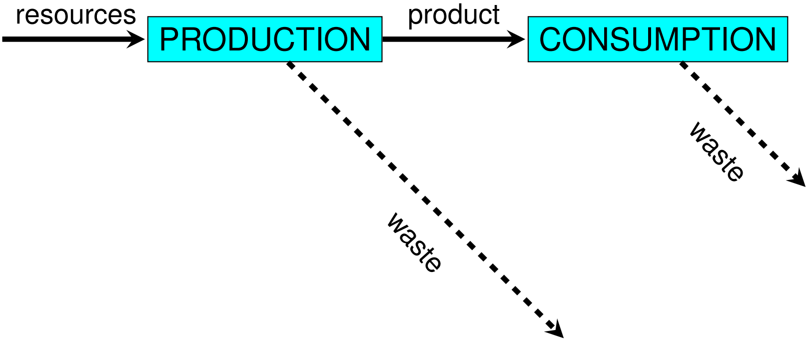
- the Earth is essentially infinite
- our effect on the biosphere is negligible
- exponential growth is a normal condition.

Acting as if these are true inevitably brings us to a point where they *stop* being true.

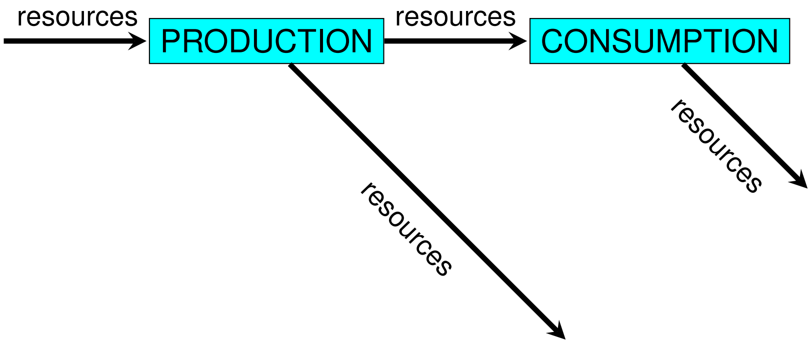
Our simplified picture:



Our simplified picture neglects “waste”, until it causes trouble:



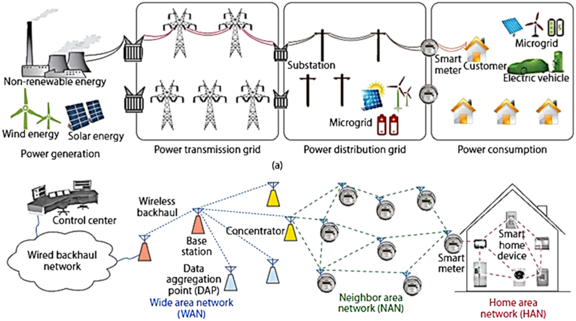
In nature there is no waste, only more resources:



We need to bring nature into our economics:

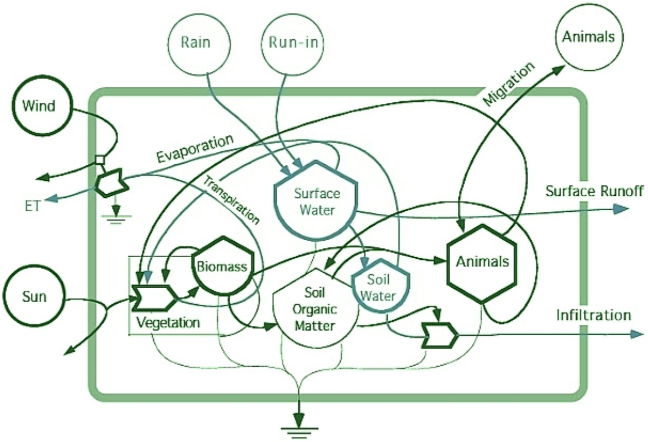
- *The Economics of Biodiversity: The Dasgupta Review.*

But we also need to **develop a nervous system for the planet**. The “smart grid” is a start.

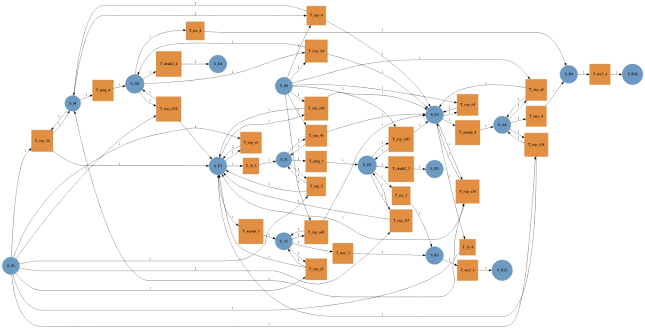


But we need a general, mathematical theory of networks.

To understand ecosystems, ultimately will be to understand networks. — B. C. Patten and M. Witkamp



The math of structured cospans led quickly to compositional models for COVID-19.



But the potential of category theory to help us design flexible networks has barely begun to be tapped.

The 21st century will be a *very* bumpy ride.
But mathematicians can help build a saner and happier civilization, if we put our minds to it.