Social contagion modeled on random networks

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no category theory

No category theory is discussed.

These are ideas that I'd like to lift into category theory.

climate change

Controlling climate change requires action along many fronts

- science
- engineering & tech
- political & economical
- $\cdots$
- **social**

the social dimension of climate change

Controlling climate change $\cdots$

requires collective action

$\Downarrow$

convincing loads of people to act

$\Downarrow$

maximizing people exposed to the message

Hence, understanding dynamics of social networks can be leveraged to control climate change

In this talk we overview a simple model of social contagion.

what's social contagion?

(def) Social contagion is the spread of ideas, opinions, beliefs, behaviors, etc through social networks.

social contagion $\approx$ biological contagion

difference between social & biological

(social)

∗ force correlates with exposure

(e.g.) having 100 friends see Black Panther is a far stronger influence on you than if only 1 friend did.

∗ spreads many to one

(e.g.) influence is distributed over friends

(biological)

∗ force "is" independent of exposure

(e.g.) chance of catching a flu is marginally more likely after interacting with 100 infected friends instead of 1

∗ spread one to one

(e.g.) flue comes from a single source

Watts model

Duncan Watts – " A simple model of global cascades on random networks" (2002)

Communicated by Murray Gell-Mann and currently has 2169 citations on Google scholar

The basic ingredients is a simple, undirected graph, which we call a **network**

Nodes represent a person

Edges represent friendship

Watts model updated to weighted, directed, and temporal networks.

Watts motivation

Consider a network subjected to external forces. (e.g. a Facebook marketing campaign)

Why can a small external force cause global cascades while much larger forces fail?

(e.g.) stock market volatility, viral video, power grid failures

Two aims for Watts:

- explain triggering of cascades in terms of network connectivity
- address two qualitative observations
  - global cascades can be triggered by external events that are small relative to network size
  - global cascades are rare relative to the number of shocks the system receives

Watts motivation

A starting point: binary decisions with externalities

binary decision

Should I do the thing or not ?

Formal description: a map

$$\texttt{decide}\colon A \to \{0, 1\}$$

from a set A of *agents*

Watts model motivation

externalities

To make a decision, each agent $x \in A$ is incentivized to collect information about distinct agents $x' \in A$.

(ex1) agent x has limited information to make a decision, so relies on the actions of other $x' \in A$. *Should I go to that restaurant?*

(ex2) agent x has difficulty making sense of lots of information, so relies on the actions of other $x' \in A$. *Should I buy that stock?*

(ex3) the value of a purchase increases the more purchases are made. *Fax machines*

Economist call this class of decision making **binary decisions with externalities**.

a formalism for BDwE

We use random networks (networks = simple undirected graphs).

These do not accurately reflect real-life networks[1], but are well studied and provide a tractable starting place.

(1. Strogatz, S.H. (2001). "Exploring complex networks". Nature. 410 (6825): 268–276.)

(Gilbert random network)

Fix a probability $p \in [0, 1]$ and number of agents $N \in \mathbb{N}$.

The random network $G(N, p)$ has N nodes, $\binom{N}{2}$ possible edges each appearing independently with probability p.

(Erdos-Renyi random network)

The random network $G(N, E)$ places a uniform probability distribution on all edges with N nodes and E edges.

(Watts random network)

Important: control the neighborhoods

Fix a probability distribution $p_{(-)}$ on $\mathbb{N}$. Construct a random network $G(N, p_{(-)})$ on N nodes by choosing node x to have k neighbors with probability p_k.

the Watts model

Fix a Watt's random network $G(N, p_{(-)})$.

To account for variation in knowledge, preference, observational capacity across agents fix a probability distribution

$$f\colon [0,1] \to [0,1]$$

Define a threshold function

$$\theta\colon N \to [0,1]$$

using f.

The Watts model is the pair $(G(N, p_{(-)}), \theta)$

Watts model dynamics

Associate to each node, its neighborhood: $\texttt{nhood}\colon N \to 2^N$.

Recursively define a function: $\alpha\colon N \times \mathbb{N} \to \{0, 1\}$

Set $\alpha(-, 0) = 0$.

Perturb the initial state:

$$\text{set } \alpha(N', 1) = 1 \text{ for some } N' \subseteq N \text{ with } |N'|/|N| << 1.$$

Update rule:

- if $\alpha(x, k) = 1$, do nothing
- if $\alpha(x, k) = 0$ and
$$\theta(x) < \frac{|\alpha(-, k)^{-1}(1) \bigcap \texttt{nhood}(n)|}{|\texttt{nhood}(n)|}$$
set $\alpha(n, k + 1) = 1$, else do nothing.

This binary decision model is similar to some well-known models in the literature, but with important differences.

(ex1 – disease)

Disease spread models are similar.

Important difference is Watt's model introduces *local dependency*. For instance, the effect of an activated neighbor on updating depends on the remaining neighbors.

examples w. deficiencies

This binary decision model is similar to some well-known models in the literature, but with important differences.

(ex2 – bootstrap)

Bootstrap percolation on a network with a random initial configuration of active nodes. Each node has a threshold for activating, e.g. a node activates at time $k+1$ if $j \in \mathbb{N}$ nodes are active at times k.

Watt's *threshold is fractional*, not raw. Thus, value of activation decreases.

This binary decision model is similar to some well-known models in the literature, but with important differences.

(ex3 – Ising)

Ising model from statistical mechanics consists of a lattice on which each node has an associated (particle) spin. Using a lattice forces each node to have the same number of neighbors. Watt's model allows for *variable neighborhood sizes*.

Local dependency, fractional thresholds, and heterogeneity are central features to the Watt's model.

Two important quantities

Watts' model focuses on:

- probability a global cascade is triggered by a small seed of active nodes
- expected size of a global cascade once triggered

global cascade?

First, a word on percolation theory.

(finite percolation)

Consider a finite random network initialized with a seed A of active nodes and an update rule for activating nodes in the future.

Fix $B \subset N$. Percolation occurs when all nodes in B are activated in finite time.

(e.g.) Water, poured on a square of porous material, trying to reach the bottom of the material. A is top of the square and B is the bottom

global cascade?

(infinite percolation)

The mathematics of the model significantly simplifies when the random network is infinite.

Percolation occurs when infinitely many nodes are activated in finite time.

In Watts' model, we will use an infinite network and say a *global cascade* occurs when infinitely many nodes are active.

Solving infinite Watts

Start with a Watt's model ($G(N, p_{(-)}, \theta)$) with infinite N.

We use infinite N for easier analysis. Computer simulations with large N are qualitatively similar.

How growth?

Define the active seed: $\alpha(N, 0)$

An node x is *vulnerable* if it neighbors an active node and has threshold smaller than $1/\texttt{deg}(x)$.

To grow activation, there must exist a vulnerable node.

Thus, growth depends on vulnerability.

(conjecture) A sufficient condition for global cascades is that a sub-network of vulnerable nodes must percolate through the network.

Assuming this conjecture, studying global cascades isn't a network dynamics problem, but a percolation problem. (nice!)

starting Watts analysis

Recall ingredients:

- degree probability distribution on $\mathbb{N}$: $p_{(-)}$
- threshold probability distribution on $[0, 1]$: f
- threshold function $\theta \colon N \to [0, 1]$ defined using f

Facts:

- For a random node x: $P(\mathtt{deg}(x) = k) = p_k$
- Probability x is vulnerable:

$$\rho_{\mathtt{deg}(x)} := P(\theta(x) \leq 1/\mathtt{deg}(x)) = \int_0^{1/\mathtt{deg}(x)} f$$

- Probability x has degree k and is vulnerable: $\rho_k p_k$

starting Watts analysis

From this last fact, the generating function for the degree of a vulnerable node is

$$G_0(z) := \sum_j \rho_j p_j z^j$$

- $G_0(1^-) :=$ fraction of nodes that are vulnerable
- $G_0'(1^-) :=$ average degree of vulnerable nodes

more generating functions

We are interested in a cascade propagating from an active node x to a random neighboring node y.

The larger $\mathtt{deg}(y)$, the more likely y neighbors x.

Define the normalized moment generating function

$$G_1(z) := \frac{\sum_j j\rho_j p_j z^{j-1}}{\sum_j j p_j}$$

corresponding to a neighbor of an initial active node

Note $G_1(z) = G_0'(z)/$'avg. node degree'.

vulnerable clusters

(Handwavily) define *clusters* to be a group of highly connected nodes.

We are interested in the behavior of clusters of vulnerable nodes.

To calculate properties of such clusters, we define generating functions

$$H_0(z) := \sum_j q_j z^j \quad \text{and} \quad H_1(z) := \sum_j r_j z^j$$

- q_j is the probability a randomly chosen node belongs to a vulnerable cluster of size j
- r_j is the probability that a neighbor of an initially active node belongs to a vulnerable cluster of size j

some equations

The following two equations are shown to hold

- $H_1(z) = (1 - G_1(1)) + zG_1(H_1(z))$
- $H_0(z) = (1 - G_0(1)) + zG_0(H_1(z))$

For each,

- first term: probability a node is not vulnerable
- second term: accounts for size distribution of vulnerable clusters attached to another vulnerable node

some arithmetic

Using the equations

- $G_0(z) \coloneqq \sum_j \rho_j p_j z^j$
- $G_1(z) \coloneqq G_0'(z) / \sum_j j p_j$
- $H_1(z) = (1 - G_1(1)) + z G_1(H_1(z))$
- $H_0(z) = (1 - G_0(1)) + z G_0(H_1(z))$

we get an equation for the *average vulnerable cluster size*

$$H_0'(1) = G_0(1) + \frac{G_0'(1))^2}{\left(\sum_j j p_j\right) - G_0''(1)}$$

This is extremely important because *it diverges at percolation*.

i.e. when $G_0''(1) = \sum_j j(j-1)\rho_j p_j = \sum_j j p_j$.

interpretation

Interpret

$$H_0'(1) = G_0(1) + \frac{G_0'(1))^2}{\sum_j jp_j - G_0''(1)}$$

as follows

- $\sum_j jp_j < G_0''(1)$ means the average size of a vulnerable cluster is infinite, so a global cascade has positive chance of being triggered
- $\sum_j jp_j > G_0''(1)$ means the average size of a vulnerable cluster is small, so vulnerable clusters are likely far apart and exert little influence on each other. Thus, global cascade is impossible.

discussion

The Watts model and analysis is based on having infinite nodes.

This is not something computers deal with well

But, simulations using a large number of nodes ($> 10,000$) correlate closely with the Watts model.

Hence, the Watts model provides testable predictions about global cascades on real-world networks.

the end