Classification problems in symplectic linear algebra

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Introduction...

Plan:

1. Introduction
2. Symplectic vectors spaces
3. Why symplectic? Connection to dynamical systems
4. More symplectic linear algebra
5. Some classification problems
6. Poset representations
7. A more general picture

Goals:

- Basic introduction to linear symplectic geometry
- Poset representations as a tool for classification problems
- Hint at a category-theoretic picture

A theme:

Connection between symplectic geometry and (twisted) involutions: symplectic structures as fixed points in an appropriate sense

Context

- Baez & team: **black-box functors** often land in categories where
  - objects: symplectic vector spaces
  - morphisms: lagrangian relations
- Weinstein: the **"symplectic category"**
- Scharlau & Co.: developed a **category-theoretic** framework in late 70's with focus on quadratic forms
- School of Kiev (Navarova & Roiter): **representations of posets**, quivers, algebras; Sergeichuk: applications to linear algebra
- Representations of quivers
- Involutions / duality involutions in categories

Symplectic geometry??

A first explanation via (anti)analogy...

A **Euclidean structure** on $V = \mathbb{R}^n$ is a bilinear form

$$B \colon V \times V \longrightarrow \mathbb{R}$$

which is

- non-degenerate: if $B(v, w) = 0 \ \forall w \in V$, then $v = 0$
- symmetric: $B(v, w) = B(w, v) \quad \forall v, w \in V$
- positive definite:
  - $B(v, v) \geq 0 \quad \forall v \in V$
  - If $B(v, v) = 0$, then $v = 0$

A Euclidean structure B on V gives us:

- lengths: $\|v\| := B(v,v)$
- angles: $\cos(\theta) := \frac{B(v,w)}{\|v\|\|w\|}$ for $\theta = v\angle w \in [0,\pi]$

More generally: a **metric structure** on $V = \mathbb{R}^n$ is a bilinear form

$$B\colon V \times V \longrightarrow \mathbb{R}$$

which is non-degenerate and symmetric (but not necessarily positive definite).

From this one can define a “length”, but it might be zero or negative for non-zero vectors.
[E.g.: Lorentzian geometry, as in Einstein’s theories of relativity]

Note: this definition works for a vector space V over any field $\mathbf{k}$.

A **symplectic structure** on V (over $\mathbf{k}$) is a bilinear form

$$\omega \colon V \times V \longrightarrow \mathbf{k}$$

which is non-degenerate and antisymmetric:

$$\omega(v, w) = -\omega(w, v) \quad \forall v, w \in V.$$

Note: if $\mathrm{char}(\mathbf{k}) \neq 2$, then $\omega(v, v) = 0 \quad \forall v \in V.$

We'll stick mostly with $\mathbf{k} = \mathbb{R}$ (and always $\mathrm{char}(\mathbf{k}) \neq 2$).

A **symplectic vector space** is (V, ω), where ω is a symplectic form on V.

Given (V, ω) and (V', ω'), a linear map $f : V \to V'$ is a (linear) **symplectomorphism** if

$$\omega'(fv, fw) = \omega(v, w) \quad \forall v, w \in V.$$

One might also say "isometry" (even though we don't have a "metric").

Fact: Every symplectic vector space is necessarily even dimensional.

Fact: Any two symplectic vector spaces of the same (finite) dimension are symplectomorphic.

Fact: Given any vector space U, the space $U^* \oplus U$ carries a canonical symplectic structure, which I'll usually denote by Ω:

$$\Omega((\xi, v), (\eta, w)) = \xi(w) - \eta(v) \quad \text{for } \xi, \eta \in U^*, \; v, w \in U.$$

Let (V, ω) be symplectic, with $\dim V = 2n$. A basis $(q_1, ..., q_n, p_1, ..., p_n)$ of V is a **symplectic basis** if

$$\omega(q_i, q_j) = 0 \qquad \forall i, j = 1, .., n$$
$$\omega(p_i, p_j) = 0 \qquad \forall i, j = 1, .., n$$
$$\omega(q_i, p_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else.} \end{cases}$$

Every (V, ω) admits a symplectic basis (many, actually).

Given a symplectic basis, the associated coordinate matrix of ω is a block matrix of the form

$$\begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}.$$

Any symplectic form ω on V induces an isomorphism

$$\tilde{\omega} : V \to V^*, \quad v \mapsto \omega(v, -).$$

Note: f symplecto $\Leftrightarrow$ $f^* \tilde{\omega} f = \tilde{\omega}$.

Note: if $(q_1, .., q_n, p_1, .., p_n)$ is a symplectic basis, and $(q_1^*, .., q_n^*, p_1^*, .., p_n^*)$ the dual basis in V^*, the coordinate matrix of $\tilde{\omega}$ is

$$\begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix},$$

the inverse of which is

$$\begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}.$$

Why symplectic??

Origins of symplectic geometry: classical mechanics (planetary motion, projectiles, etc.).

More precisely: origins are in **Hamiltonian mechanics**

- Newton's mechanics: from ca. 1687
- Lagrange's mechanics: from ca. 1788
- Hamilton's mechanics: from ca. 1833

Very quick sketch: from Newtonian to Hamiltonian

Example: Harmonic oscillator

(e.g. a mass attached to a coil spring).

Newton: (“F = ma”)

$$m\ddot{x} = -Cx.$$

We can rewrite as a system of 1st order ODEs. Set:

$$q(t) := x(t) \qquad p(t) := m\dot{x}(t),$$

and get

$$\begin{aligned}\dot{q}(t) &= \frac{1}{m}p(t)\\ \dot{p}(t) &= -Cq(t)\end{aligned}$$

Reformulate the equations as:

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} Cq(t) \\ \frac{1}{m}p(t) \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial q}H(q,p) \\ \frac{\partial}{\partial p}H(q,p) \end{pmatrix}$$

where $H(p,q) := \frac{1}{2}Cq^2 + \frac{1}{2}\frac{1}{m}p^2$.

The function H is called the **Hamiltonian** of the dynamical system, and **Hamilton's equations** are

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial p}H(q,p) \\ -\frac{\partial}{\partial q}H(q,p) \end{pmatrix} =: X_H(q,p).$$

$X_H(q,p)$ is called the **hamiltonian vector field** associated to H.

The set of all possible (generalized) positions q and (generalized) momenta p in a dynamical system is called **phase space**.

In general: phase space modelled as a symplectic manifold (M, ω), or Poisson manifold; we'll stick with (V, ω).

A hamiltonian vector field $X_H : V \to V$ is related to the function H by

$$X_H(v) = \tilde{\omega}^{-1} \circ dH(v) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial q} H(q,p) \\ \frac{\partial}{\partial p} H(q,p) \end{pmatrix},$$

thinking of $dH(v)(-) : V \to \mathbb{R}$ as a 1-form. Equivalently:

$$\tilde{\omega} \circ X_H(v) = dH(v)$$

i.e.

$$\omega(X_H(v), \, - \,) = dH(v)(-).$$

Role of symplectomorphisms:

- Symmetries of phase space: solutions of Ham. equations are mapped to solutions.
- Time-evolution/flow of a Ham. system (V, ω, H):

  Given a time interval $[t_0, t_1]$, we have a symplectomorphism

  $$V \longrightarrow V, \ (q_0, p_0) \mapsto (q(t_1), p(t_1))$$

  where $c(t) = (q(t), p(t))$ is the solution to the Ham. initial value problem

  $$\begin{cases} \dot{c}(t) = X_H(c(t)) \\ c(0) = (q_0, p_0) \end{cases}$$

Upshots of Hamiltonian mechanics:

(compared to Newtonian; comparing with Lagrangian is more complicated!)

- a framework which is more general/abstract/conceptual/geometric
- has a variational formulation ("principle of stationary action")
- beautiful interplay between geometry and physics; e.g. symmetries $\leftrightarrow$ conserved quantities

Example benefit: even if one can't "*solve*" a Hamiltonian system, one can often prove *qualitative* aspects.

More symplectic linear algebra...

(V, ω) symplectic.

Given a subspace $U \subseteq V$, its (symplectic) **orthogonal** is the subspace

$$U^\omega = \{v \in V \mid \omega(v, u) = 0 \ \forall u \in U\}.$$

Special subspaces:

- symplectic $U^\omega \cap U = 0$
- isotropic $U \subseteq U^\omega$
- coisotropic $U^\omega \subseteq U$
- lagrangian $U = U^\omega$.

The operation $(-)^\omega$ defines an **order-reversing involution** on the poset $\Sigma(V)$ of subspaces of V.

Given (V, ω) and (V', ω') symplectic $\rightsquigarrow (V \oplus V', \omega \oplus \omega')$.

Given a (linear) symplectomorphism $f \colon V \to V'$, its **graph** $\Gamma(f) \subseteq V \oplus V'$ is a lagrangian subspace of

$$(V \oplus V', (-\omega) \oplus \omega').$$

Notation: for V with symplectic ω,

$$\overline{V} := \text{same vector space but with "} -\omega \text{"}.$$

A (linear) **lagrangian relation** $V \to V'$ is a lagrangian subspace

$$L \subseteq \overline{V} \oplus V'.$$

Note: these form the morphisms of a category; composition is the same as for set-relations.

(V, ω) symplectic.

Def: A vector field $X : V \to V$ is **hamiltonian** if $\tilde{\omega} \circ X(v) = dH(v)$ for some function H. Call it **linear** when X is a linear map.

Fact: $\quad X$ lin. ham. $\;\Leftrightarrow\;$ $\tilde{\omega}X = -X^*\tilde{\omega}$.

Symplectomorphisms $V \to V$ form the **symplectic group** $Sp(V, \omega)$; it's a Lie group.

Fact: The set $\mathfrak{sp}(V, \omega)$ of linear hamiltonian vector fields on V corresponds to the Lie algebra of $Sp(V, \omega)$.

Def: denote by $Lag(V, \omega)$ the set of lagrangian relations $L : V \to V$.

Some classification problems...

$Sp(V,\omega)$ acts on itself, $Lag(V,\omega)$, and $\mathfrak{sp}(V,\omega)$ by conjugation:

$$Sp(V,\omega) \times Sp(V,\omega) \to Sp(V,\omega), \quad (f,g) \mapsto fgf^{-1}.$$
$$Sp(V,\omega) \times Lag(V,\omega) \to Lag(V,\omega), \quad (f,L) \mapsto fLf^{-1}.$$
$$Sp(V,\omega) \times \mathfrak{sp}(V,\omega) \to \mathfrak{sp}(V,\omega), \quad (f,X) \mapsto fXf^{-1}.$$

Compare with: $GL(V) \times End(V) \to End(V), \quad (f,\eta) \mapsto f\eta f^{-1}.$

Typical questions:

- what are the orbits?
- can we find representatives given in a normal form?

Common theme in algebra:

- objects of study (often) *decompose* into basic building blocks, and this decomposition is sometimes *essentially unique*
- strategy: classify the indecomposable building blocks.

Example: $GL(V) \times End(V) \to End(V), \quad (f, \eta) \mapsto f\eta f^{-1}$.

Consider a category we'll call $\mathbf{End_k}$:

- **Objects**: (U, η), with $\eta \in End(U)$
- **Morphisms**: a map $f : (U, \eta) \to (U', \eta')$ is a linear map $f : U \to U'$ such that

$$\begin{array}{ccc} U & \xrightarrow{f} & U' \\ {\scriptstyle \eta}\downarrow & & \downarrow{\scriptstyle \eta'} \\ U & \xrightarrow{f} & U' \end{array} \qquad \text{commutes.}$$

In particular: (U, η) and (U', η') are isomorphic if there exists $f \in GL(V)$ such that $f\eta f^{-1} = \eta'$.

Direct sums: $(U, \eta) \oplus (U', \eta') := (U \oplus U, \eta \oplus \eta')$.
Indecomposable = not isomorphic to some direct sum with (at least) two non-zero summands.

Fact: (Krull-Schmidt holds) Every (U, η) is isomorphic to a direct sum of indecomposable pieces, and such a decomposition is essentially unique.

For general $\mathbf{k}$, the indecomposable objects are (up to iso):

$$(\mathbf{k}[X]/(p^m), \mu_X) \qquad p \in \mathbf{k}[X] \text{ monic irreducible}, m \in \mathbb{N},$$

where the endomorphism μ_X is "multiplication by X".

For $\mathbf{k} = \mathbb{C}$: monic irreducibles p are $p(X) = X - \lambda$ for any $\lambda \in \mathbb{C}$.

For $\mathbf{k} = \mathbb{R}$:

$$\begin{cases} p(X) = X - \lambda, & \lambda \in \mathbb{R}, \text{ or} \\ p(X) = X^2 - 2\Re(\lambda)X + |\lambda|^2 & \lambda \in \mathbb{C} \backslash \mathbb{R}. \end{cases}$$

Normal forms: e.g. Jordan canonical form.

For $Sp(V,\omega)$, $Lag(V,\omega)$ and $\mathfrak{sp}(V,\omega)$:

- Direct sums: are *orthogonal* direct sums
  E.g. $(V,\omega,g) \oplus (V',\omega',g') := (V \oplus V', \omega \oplus \omega', g \oplus g')$.
- Indecomposability: analogously
- Define classes of objects as (V,ω,g), (V,ω,L), (V,ω,X), respectively
- For morphisms: want isomorphisms to be *symplectomorphisms*
- Krull-Schmidt: objects decompose into indecomposables; essential uniqueness depends on further hypotheses. For $\mathbb{C}$ (and $\mathbb{R}$?) we have essentially uniqueness.

Poset representations...

Let $(P, \leq)$ be a finite poset (with elements labeled 1 through n)

A **representation** of P is a vector space V and subspaces $\{U_i\}_{i=1}^n$ of V such that

$$\text{if } i \leq j \text{ in } P, \quad \text{then } U_i \subseteq U_j.$$

So: a representation is a monotone map

$$\psi : P \to \Sigma(V).$$

Two representations $(V; U_1, ..., U_n)$ and $(V'; U_1', ..., U_n')$ of P are **isomorphic** if there exists a linear isomorphism $f : V \to V'$ such that $f(U_i) = U_i'$ (for all $i = 1, ..., n$).

Representations of a fixed poset P form a category, $\mathbf{Rep_k}(P)$.

Direct sums of poset reps: defined in the obvious way

Krull-Schmidt holds: any $\psi \in \mathbf{Rep_k}(P)$ is isomorphic to a direct sum of indecomposable poset reps, and such a decomposition is essentially unique.

Many classification problems of linear algebra can be encoded using poset representations .

Example: Given an endomorphism (U, η), consider the poset $P = \{1, 2, 3, 4\}$ with empty ordering and associate to (U, η) the following poset representation in $V = U \oplus U$:

$$(U \oplus U; U \oplus 0, 0 \oplus U, \Gamma(Id), \Gamma(\eta)).$$

Fact: objects (U, η) and (U', η') are isomorphic iff their associated poset reps are isomorphic; and indecomposables correspond to indecomposables

Symplectic poset representations:

Start with a poset P equipped with an order-reversing ("twisted") involution $(-)^{\perp} : P \to P^{op}$.

Def: a symplectic poset rep of $(P, \perp)$ on a symplectic space (V, ω) is a monotone map

$$\varphi : P \to \Sigma(V),$$

such that

$$\varphi(i^{\perp}) = \varphi(i)^{\omega} \quad \forall i \in P.$$

Example: If $P = \{1 \leq 2\}$, with $1^{\perp} = 2$, then a symplectic poset rep φ of $(P, \perp)$ corresponds to an isotropic subspace of (V, ω):

$$\varphi(1) \subseteq \varphi(2) = \varphi(1^{\perp}) = \varphi(1)^{\omega}.$$

Objects such as (V, ω, g), where $g \in Sp(V, \omega)$, can be encoded in symplectic poset reps:

To (V, ω, g), associate the system of subspaces

$$(\overline{V} \oplus V; V \oplus 0, 0 \oplus V, \Gamma(Id), \Gamma(g)).$$

Note:

- $V \oplus 0$ and $0 \oplus V$ are symplectic subspaces of $\overline{V} \oplus V$,
- $\Gamma(Id)$ and $\Gamma(g)$ are lagrangian subspace of $\overline{V} \oplus V$.

This is a symplectic poset rep of $P = \{1, 2, 3, 4\}$, with empty order, and

$$1^\perp = 2 \qquad 2^\perp = 1 \qquad 3^\perp = 3 \qquad 4^\perp = 4.$$

We can also treat $Lag(V, \omega)$ and $\mathfrak{sp}(V, \omega)$ with symplectic poset reps.

Symplectic reps of a fixed $(P, \perp)$ form a category, $\mathbf{SRep_k}(P, \perp)$.

Direct sums: again, *orthogonal*

Krull-Schmidt?: any $\varphi \in \mathbf{SRep_k}(P, \perp)$ is isomorphic to a direct sum of indecomposable poset reps; essential uniqueness depends on further hypotheses.

A basic task: classify indecomposables!

Strategy: relate $\mathbf{SRep_k}(P, \perp)$ and $\mathbf{Rep_k}(P)$.

Caveat: depending on P, it can be that $\mathbf{Rep_k}(P)$ is not well-understood.

Given: $(P, \perp)$, (V, ω).

Def: A **linear** (ordinary) representation of $(P, \perp)$ on V is a monotone map

$$\psi : P \to \Sigma(V).$$

Any symplectic poset rep φ has an **underlying** linear rep $\hat{\varphi}$.

Given a linear rep ψ of $(P, \perp)$ on V, define **dual representation** on V^* by

$$\psi^*(i) = \psi(i^\perp)^\circ = \{\xi \in V^* \mid \xi|_{\psi(i^\perp)} \equiv 0\}.$$

Symplectification: building symplectic reps from linear reps.

Given a linear rep of $(P, \perp)$, its **symplectification** is

$$\psi^- : P \longrightarrow \Sigma(V^* \oplus V, \Omega)$$

$$\psi^-(x) := \psi^*(x) \oplus \psi(x).$$

Fact: ψ^- is a symplectic representation. We call an indecomposable symplectic rep **split** if it is (isomorphic to) a symplectification.

Some indecomposable symplectic reps are **non-split**: they come from an ordinary indecomposable rep

$$\psi : P \to \Sigma(V)$$

such that V happens to admit a symplectic form which is **compatible** with ψ (making ψ symplectic). We call such an ω a **compatible form**.

Magic Lemma (Sergeichuk / Scharlau et. al): Let φ be an indecomposable symplectic representation. Then φ is either split or non-split (but not both):

1. $\varphi \simeq \psi^-$, the symplectification of some indecomposable linear rep ψ.
2. $\hat{\varphi}$ is linearly indecomposable.

Consequence: we can classify indecomposables of $\mathbf{SRep}_k P$ using indecomposables of $\mathbf{Rep}_k P$, by

1. identifying which linear indecomposables admit compatible symplectic structures, and classifying these.

   Tricky part: a given linear indecomposable ψ might admit multiple non-equivalent compatible forms!

2. For those that don't admit compatible symplectic forms: symplectify!

Current work (Hermann, L., Weinstein): Classification of triples of isotropic subspaces.

A more general picture...

Def: A category with **twisted involution** (a tCat) is $(\mathcal{C}, \delta, \eta)$, where

$$\delta \dashv \delta^{op} : \mathcal{C} \to \mathcal{C}^{op}$$

is an adjoint equivalence, with unit η.

Example: $\mathcal{C} = \mathsf{FinVect}_{\mathbf{k}}$, with $\delta(V) = V^*$, $\delta(f) = f^*$ and $\eta_V = \iota : V \to V^{**}$ the canonical isomorphism. A variant: take $\eta_V = -1 \cdot \iota$.

Example: $(\mathcal{C}, \delta, id)$ where $\mathcal{C}$ is a poset with twisted involution δ.

Def: A **fixed point** in a tCat $(\mathcal{C}, \delta, \eta)$ is (x, h) where

$$h : x \to \delta(x)^{op} \quad \text{is an isomorphism in } \mathcal{C} \text{ such that}$$

$$\begin{array}{ccc} x & \xrightarrow{h} & (\delta x)^{op} \\ & {\scriptstyle \eta_x} \searrow & \uparrow {\scriptstyle (\delta h)^{op}} \\ & & \delta^{op}\delta x \end{array} \qquad \text{commutes.}$$

Def: A **morphism** of fixed points $(x, h) \to (x', h')$ is

$$f : x \to x' \quad \text{in } \mathcal{C} \text{ such that}$$

$$\begin{array}{ccc} x & \xrightarrow{h} & \delta x \\ {\scriptstyle f} \downarrow & & \uparrow {\scriptstyle \delta f} \\ x' & \xrightarrow{h'} & \delta x' \end{array} \qquad \text{commutes.}$$

Example: Take $\mathcal{C} = \mathrm{FinVect}_{\mathbf{k}}$, with $\delta(V) = V^*$, $\delta(f) = f^*$, $\eta_V = -1 \cdot \iota$.

- Fixed points are $(V, \tilde{\omega})$ with $\tilde{\omega} : V \to V^*$ such that

$$\tilde{\omega} = -\tilde{\omega}^* \circ \iota \quad \rightsquigarrow \text{ encodes symplectic spaces } (V, \omega).$$

- Morphisms of fixed points encode symplectomorphisms (isometries):

$$\begin{array}{ccc} V & \xrightarrow{\tilde{\omega}} & V^* \\ {\scriptstyle f}\downarrow & & \uparrow{\scriptstyle f^*} \\ V' & \xrightarrow{\tilde{\omega}'} & V'^* \end{array}$$

Example: $\mathcal{C} = \mathbf{Rep}_{\mathbf{k}}(P, \perp) = [(P, \perp), \mathrm{FinVect}_{\mathbf{k}}]$, with $\delta\psi = \psi^*$ and $\eta_\psi = -\iota : \psi \to \psi^{**}$.

- Fixed points encode symplectic poset representations
- Morphisms of fixed points = morphisms of symplectic poset reps

Example: Take $\mathcal{C} = \mathbf{Aut_k}$ (objects are (V, g) with $g \in \mathrm{Aut}(V)$); set

$$\delta(V, g) := (V^*, (g^*)^{-1}) \quad \text{and } \eta_{(V,g)} := -\iota : V \to V^{**}.$$

- Fixed points are $(V, g, \tilde{\omega})$ with $\tilde{\omega} : V \to V^*$ such that

$$\begin{array}{ccc} V & \xrightarrow{\tilde{\omega}} & V^* \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle (g^*)^{-1}} \\ V & \xrightarrow{\tilde{\omega}} & V^* \end{array} \qquad \text{commutes.}$$

$\rightsquigarrow$ this encodes symplectomorphisms $g \in Sp(V, \omega)$.

- Morphisms of fixed points are symplectomorphisms $f : (V, g, \omega) \to (V', g', \omega')$ such that $fgf^{-1} = g'$.

Example: Take $\mathcal{C} = \mathbf{End_k}$ (objects are (V, X) with $X \in \mathrm{End}(V)$); set

$$\delta(V, X) := (V^*, -X^*) \quad \text{and} \quad \eta_{(V,X)} := -\iota : V \to V^{**}.$$

- Fixed points are $(V, X, \tilde{\omega})$ with $\tilde{\omega} : V \to V^*$ such that

$$\begin{array}{ccc} V & \xrightarrow{\tilde{\omega}} & V^* \\ {\scriptstyle X}\downarrow & & \downarrow{\scriptstyle -X^*} \\ V & \xrightarrow{\tilde{\omega}} & V^* \end{array} \qquad \text{commutes.}$$

$\rightsquigarrow$ this encodes lin. ham. vector fields $X \in \mathfrak{sp}(V, \omega)$.

- Morphisms of fixed points are symplectomorphisms $f : (V, X, \omega) \to (V', X', \omega')$ such that $fXf^{-1} = X'$.

Summary of patterns and themes:

- symplectic (and metric) geometry is linked with (twisted) involutions
- where there are involutions, there are "split" and "non-split" things
- "non-split" things can be built by "doubling" ($\rightsquigarrow$ symplectification)
- beautiful category theory is also lurking

Thanks for listening!