A Complete Axiomatisation of Partial Differentiation

Gordon Plotkin

ACT Seminar, May 2020

Cartesian differential categories

The goal of the present paper is to develop an axiomatization which directly characterizes the smooth maps: in other words, to characterize the coKleisli structure of differential categories directly. This leads us to the notion of a Cartesian differential category. This notion embodies the multi-variable differential calculus which, being a fundamental tool of modern mathematics, is well worth studying in its own right.

Blute, Cockett & Seely, *Cartesian differential categories*, 2009

Left additive cartesian categories

- Each hom set is a commutative monoid.
- Composition is left additive:

$$0f = 0 \qquad (f+g)h = fh + gh$$

A morphism f is *additive* iff f- (ie right composition with f) is additive.

- The product structure is compatible:
  - The projections

    $$x \leftarrow x \times y \rightarrow y$$

    are additive
  - Tupling preserves additivity:

    $$\frac{f : x \to y, g : x \to z \text{ additive}}{\langle f, g \rangle : x \to y \times z \text{ additive}}$$

Example: Finite powers of $\mathbb{R}$ and smooth maps

- The gradient

$$\nabla(f) : \mathbb{R}^n \to \mathbb{R}^n$$

of a smooth map $f(x_1, \ldots, x_n)$ of n-arguments is

$$\nabla(f)(\mathbf{v}) = \langle \frac{\partial f}{\partial x_1}(\mathbf{v}), \ldots, \frac{\partial f}{\partial x_n}(\mathbf{v}) \rangle$$

- The differential

$$\mathrm{D}[f] : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$$

of a smooth map

$$f : \mathbb{R}^n \to \mathbb{R}^m = \langle f_1, \ldots, f_m \rangle$$

is

$$\mathrm{D}[f](\mathbf{v}, \mathbf{w}) \;=\; \langle \nabla(f_1)(\mathbf{v}) \cdot \mathbf{w}, \ldots, \nabla[f_n](\mathbf{v}) \cdot \mathbf{w} \rangle \;(= \; \mathrm{J}(f)(\mathbf{v})\mathbf{w}^T)$$

Cartesian differential categories

Differential combinator

$$\frac{f : x \to y}{\mathrm{D}[f] : x \times x \to y}$$

Axioms

1.	$\mathrm{D}[f+g]$	$=$	$\mathrm{D}[f] + \mathrm{D}[g]$	(Additivity)
2.	$\mathrm{D}[f]\langle a, b+c\rangle$	$=$	$\mathrm{D}[f]\langle a, b\rangle + \mathrm{D}[f]\langle a, c\rangle$	(2nd arg. additivity)
3.	$\mathrm{D}[\mathrm{id}]$	$=$	π_1	(Identity)
4.	$\mathrm{D}[\pi_i]$	$=$	$\pi_i \pi_1$	(Projections)
5.	$\mathrm{D}[\langle f, g\rangle]$	$=$	$\langle \mathrm{D}[f], \mathrm{D}[g]\rangle$	(Pairing)
6.	$\mathrm{D}[gf]$	$=$	$\mathrm{D}[g]\langle f\pi_0, \mathrm{D}[f]\rangle$	(Chain Rule)
7.	$\mathrm{D}[\mathrm{D}[f]]\langle\langle a, 0\rangle, \langle b, c\rangle\rangle$	$=$	$\mathrm{D}[f]\langle a, c\rangle$	(2nd arg. linearity)
8.	$\mathrm{D}[\mathrm{D}[f]]\langle\langle 0, b\rangle, \langle a, c\rangle\rangle$	$=$	$\mathrm{D}[\mathrm{D}[f]]\langle\langle 0, a\rangle, \langle b, c\rangle\rangle$	(Symmetry)

Axioms for partial differentiation

Commutative rings + addition and multiplication tables of the reals + the following four partial differentiation axioms:

1. Addition

$$\frac{\partial\, x+y}{\partial x} = 1$$

2. Multiplication

$$\frac{\partial yx}{\partial x} = y$$

3. Binary Chain Rule

$$\frac{\partial\, f(g_0(x), g_1(x))}{\partial x} = \left.\frac{\partial f(x_0, g_1(x))}{\partial x_0}\right|_{x_0=g_0(x)} \frac{\partial g_0(x)}{\partial x} + \left.\frac{\partial f(g_0(x), x_1)}{\partial x_1}\right|_{x_1=g_1(x)} \frac{\partial g_1(x)}{\partial x}$$

4. Commutativity

$$\frac{\partial}{\partial y}\frac{\partial}{\partial x} f(x,y) = \frac{\partial}{\partial x}\frac{\partial}{\partial y} f(x,y)$$

Standard equational logic

- Signature

$$\mathrm{op} : n$$

- Expressions (or terms)

$$e ::= x \mid \mathrm{op}(e_0, \ldots, e_{n-1}) \quad (\mathrm{op} : n)$$

- Notation eg: $e + e'$ for $+(e, e')$
- Axioms a set Ax of equations $e_0 = e_1$
- Logic

$$e_0 = e_1 \quad (\text{if } e_0 = e_1 \in \mathrm{Ax})$$

$$e = e \qquad \frac{e_0 = e_1}{e_1 = e_0} \qquad \frac{e_0 = e_1 \quad e_1 = e_2}{e_0 = e_2}$$

$$\frac{e_0 = e'_0, \ldots, e_{n-1} = e'_{n-1}}{f(e_0, \ldots, e_{n-1}) = f(e'_0, \ldots, e'_{n-1})}$$

$$\frac{e_0 = e_1}{e_0[e/x] = e_1[e/x]}$$

- Theorems

$$\vdash_{\mathrm{Ax}} e_0 = e_1$$

Notation for partial differentiation

- In, for example, $\frac{\partial e}{\partial x}$, the variable x should occur in e rather than elsewhere.
  So,
  $$\frac{\partial f(x,y)}{\partial x}$$
  rather than
  $$\frac{\partial f}{\partial x} \text{ for a function } f(x,y)$$
- Most general notation is
  $$\frac{\partial e_0}{\partial x}(e_1)$$
- Others:
  $$\left.\frac{\partial e_0}{\partial x}\right|_{x=e_1} = \frac{\partial e_0}{\partial x}(e_1) \qquad \frac{\partial}{\partial x}e = \frac{\partial e}{\partial x} = \frac{\partial e}{\partial x}(x)$$
- Boring official expression for partial differentiation
  $$\mathrm{PDiff}(x.e_0, e_1) = \left.\frac{\partial e_0}{\partial x}\right|_{x=e_1}$$

Equational theory for partial differentiation: Syntax

- Expressions

$$e \quad ::= \quad r\ (r \in \mathbb{R}) \mid x \mid +(e_0, e_1) \mid \times(e_0, e_1) \mid f(e_0, \ldots, e_{n-1}) \quad (f : n) \mid \mathrm{PDiff}(x.e_0, e_1)$$

- Expression substitution

$$f(e_0, \ldots, e_{n-1})[e/x] \quad = \quad f(e_0[e/x], \ldots, e_{n-1}[e/x])$$

$$\mathrm{PDiff}(y.e_0, e_1)[e/x] \quad = \quad \mathrm{PDiff}(y.e_0[e/x], e_1[e/x])$$

$$\left.\frac{\partial e_0}{\partial y}\right|_{y=e_1}[e/x] \quad = \quad \left.\frac{\partial e_0[e/x]}{\partial y}\right|_{y=e_1[e/x]}$$

Syntax cntnd.

- Abstract substitution

$$f(e_0, \ldots, e_{n-1})[x_0, \ldots, x_{n-1}.e/f] = e[e_0/x_0] \ldots [e_{n-1}/x_{n-1}]$$

$$g(e_0, \ldots, e_{n-1})[x_0, \ldots, x_{n-1}.e/f] = g(e_0[x_0, \ldots, x_{n-1}.e/f], \ldots, e_{n-1}[x_0, \ldots, x_{n-1}.e/f])$$

$$\mathrm{PDiff}(y.e_0, e_1)[x_0, \ldots, x_{n-1}.e/f] = \mathrm{PDiff}(y.e_0[x_0, \ldots, x_{n-1}.e/f], e_1[x_0, \ldots, x_{n-1}.e/f])$$

$$\left.\frac{\partial e_0}{\partial x}\right|_{x=e_1}[x_0, \ldots, x_{n-1}.e/f] = \left.\frac{\partial e_0[x_0,\ldots,x_{n-1}.e/f]}{\partial x}\right|_{x=e_1[x_0,\ldots,x_{n-1}.e/f]}$$

Equational theory for partial differentiation: Axioms and logic

- Axioms As above, plus those for commutative rings and the addition and multiplication tables for reals.

- Logic As above, plus:

$$\frac{e_0 = e_0' \qquad e_0 = e_0'}{\mathrm{PDiff}(x.e_0, e_1) = \mathrm{PDiff}(x.e_0', e_1')}$$

$$\frac{e_0 = e_1}{e_0[x_0, \ldots, x_{n-1}.e/f] = e_1[x_0, \ldots, x_{n-1}.e/f]}$$

- Theorems Just write

$$\vdash e_0 = e_1$$

Example theorems

Constant and Unary Chain Rule

$$\frac{\partial y}{\partial x} = 0 \qquad \frac{\partial f(g(x))}{\partial x} = \left.\frac{\partial f(y)}{\partial y}\right|_{y=g(x)} \frac{\partial g(x)}{\partial x}$$

Proof For Constant case: Substitute $x_0, x_1.y$ for f and $z.0$ for g_0 and g_1 in binary chain rule

$$\frac{\partial f(g_0(x), g_1(x))}{\partial x} = \left.\frac{\partial f(x_0, g_1(x))}{\partial x_0}\right|_{x_0=g_0(x)} \frac{\partial g_0(x)}{\partial x} + \left.\frac{\partial f(g_0(x), x_1)}{\partial x_1}\right|_{x_1=g_1(x)} \frac{\partial g_1(x)}{\partial x}$$

to get:

$$\begin{aligned}
\frac{\partial y}{\partial x} &= \left.\frac{\partial y}{\partial x_0}\right|_{x_0=0} \frac{\partial 0}{\partial x} + \left.\frac{\partial y}{\partial x_1}\right|_{x_1=0} \frac{\partial 0}{\partial x} \\
&= \left.\frac{\partial y}{\partial x_0}\right|_{x_0=0} 0 + \left.\frac{\partial y}{\partial x_1}\right|_{x_1=0} 0 \\
&= 0
\end{aligned}$$

Another theorem: general chain rule

Standard version

$$\frac{\partial\, \alpha(\beta_0(x),\ldots,\beta_{n-1}(x))}{\partial x} = \sum_{i=0}^{n-1} \left.\frac{\partial \alpha(\beta_0(x),\ldots,\beta_{i-1}(x),x_i,\beta_{i+1}(x),\ldots,\beta_{n-1}(x))}{\partial x_i}\right|_{x_i=\beta_i(x)} \frac{\partial \beta_i(x)}{\partial x}$$

Substitution version

$$\frac{\partial e[e_0/x_0,\ldots,e_{n-1}/x_{n-1}]}{\partial x} = \sum_{i=0}^{n-1} \frac{\partial e}{\partial x_i}[e_0/x_0,\ldots,e_{n-1}/x_{n-1}]\frac{\partial e_i}{\partial x}$$

Equational logic with binders in general

Example binding operations

$\int_a^b e.\mathrm{d}x \;=\; \mathrm{INT}(x.e, a, b) \quad \lambda x.\, e = \lambda(x.\, e)$

$\forall x.\, \varphi = \forall(x : \iota.\, \varphi)$ most P's are Q's $= \mathrm{MOST}(x : \iota.\, P(x), x : \iota.\, Q(x))$

Example equations

$$(\beta) \quad \mathrm{ap}(\lambda(x.\, f(x)), y) = f(y) \qquad (\eta) \quad \lambda(y.\, \mathrm{ap}(x, y)) = x$$

Operations $\mathrm{op} : m_0, \ldots, m_{k-1}; n$

Syntax Abstracts a and operation expressions e

Abstracts $a ::= x_0, \ldots, x_{m_1}.e$

Op. Expressions $e ::= \mathrm{op}(a_0, \ldots, a_{k-1}, e_0, \ldots, e_{n-1})$
$(\mathrm{op} : m_0, \ldots, m_{k-1}; n,\ a_i : m_i)$

- Fiore, Marcelo. and Chung-Kil Hur, *Second-order equational logic*, 2010.
- Fiore, Marcelo and Ola Mahmoud, *Second-order algebraic theories*, 2010.

Semantics of partial differentiation theory

Denotations

$$
\begin{array}{lcl}
\mathcal{S}[\![r]\!]\varphi\rho & = & r \\
\mathcal{S}[\![x]\!]\varphi\rho & = & \rho(x) \\
\mathcal{S}[\![e + e']\!]\varphi\rho & = & \mathcal{S}[\![e]\!]\varphi\rho + \mathcal{S}[\![e']\!]\varphi\rho \\
\mathcal{S}[\![ee']\!]\varphi\rho & = & \mathcal{S}[\![e]\!]\varphi\rho \times \mathcal{S}[\![e']\!]\varphi\rho \\
\mathcal{S}[\![f(e_0, \ldots, e_{n-1})]\!]\varphi\rho & = & \varphi(f)(\mathcal{S}[\![e_0]\!]\varphi\rho, \ldots, \mathcal{S}[\![e_{n-1}]\!]\varphi\rho) \\
\mathcal{S}[\![\mathrm{PDiff}(x.\, e, e')]\!]\varphi\rho & = & \frac{\mathrm{d}\mathcal{S}[\![e]\!]\varphi\rho[r/x]}{\mathrm{d}r}(\mathcal{S}[\![e']\!]\varphi\rho)
\end{array}
$$

Environments

$$\rho(x) \in \mathbb{R} \qquad \varphi(f) : \mathbb{R}^n \xrightarrow{\text{smooth}} \mathbb{R} \;\; (f : n)$$

Validity

$$\models e_0 = e_1 \iff \mathcal{S}[\![e_0]\!] = \mathcal{S}[\![e_1]\!]$$

General interpretation of a binding algebraic theory

- An abstract clone $C(p)$ ($p \geq 0$)

- For each operation $\mathrm{op}:(m_1, \ldots, m_k; n)$ a *uniform* family of maps

$$\mathrm{op}_C : C(p + m_1) \times \ldots \times C(p + m_k) \times C(p)^n \to C(p)$$

  for $p \geq 0$.

- For each expression in context

$$\varphi_1 : m_1, \ldots, \varphi_k : m_k \mid x_1, \ldots, x_n \vdash e$$

  one then defines its denotation, viz a map

$$C(m_1) \times \ldots \times C(m_k) \xrightarrow{\mathcal{D}[\![e]\!]} C(n)$$

Consistency and completeness

Consistency

$$\vdash e_0 = e_1 \implies \models e_0 = e_1$$

Completeness

$$\models e_0 = e_1 \implies \vdash e_0 = e_1$$

Polynomial Function Environments

$$\varphi(f) : \mathbb{R}^n \xrightarrow{\text{poly}} \mathbb{R} \quad (f : n)$$

Polynomial Completeness

$$\models_{\text{poly}} e_0 = e_1 \implies \vdash e_0 = e_1$$

Proof strategy

- Define canonical forms $c = \mathrm{CF}(e)$ of expressions such that

$$\vdash e = \mathrm{CF}(e)$$

- Define a relation $c \approx c'$ between canonical forms such that

$$c \approx c' \implies \vdash c = c'$$

- Show

$$c \not\approx c' \implies \exists\, \text{poly}\ \varphi, \rho.\, \mathcal{S}[\![c]\!]\varphi\rho \neq \mathcal{S}[\![c']\!]\varphi\rho$$

- And so obtain contrapositive of completeness

$$\begin{array}{llll} \not\vdash e = e' & \implies \not\vdash \mathrm{CF}(e) = \mathrm{CF}(e') & \implies & \not\vdash \mathrm{CF}(e) \approx \mathrm{CF}(e') \\ & \implies \not\models_{\text{poly}} \mathrm{CF}(e) = \mathrm{CF}(e') & \stackrel{con}{\implies} & \not\models_{\text{poly}} e = e' \end{array}$$

- With consistency have further that:

$$\vdash e = e' \iff \mathrm{CF}(e) \approx \mathrm{CF}(e')$$

(and so, potentially, decidability)

Interpolation I: Unary functions

What f, g, x make:

$$g(f(f(g(x)))) \neq g(f(g(x))$$

Solution: Number distinct subexpressions:

$$g(f(f(g(x = 1) = 2) = 3) = 4) = 5 \neq g(f(g(x = 1) = 2) = 3) = 6$$

Requirements

$$f(2) = 3 \quad f(3) = 4$$

$$g(1) = 2 \quad g(4) = 5 \quad g(3) = 6$$

Solution Legendre interpolation

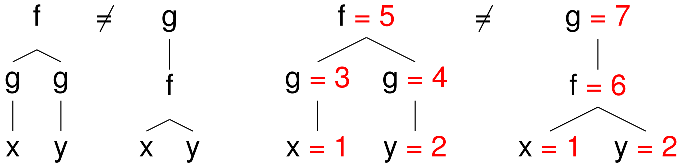
$$g(x) = \frac{(x-4)(x-3)}{(1-4)(1-3)}2 + \frac{(x-1)(x-3)}{(4-1)(4-3)}5 + \frac{(x-1)(x-4)}{(3-1)(3-4)}6$$

Interpolation II: Functions of several arguments

What f, g, x make:

$$f(gx, gy) \neq g(f(x, y))$$

Solution: Number distinct subexpressions



Requirements

$$f(3,4) = 5 \quad f(1,2) = 6$$

Multivariate Legendre Interpolation

$$f(\mathbf{x}) = \frac{\mathrm{d}^2(\mathbf{x},(1,2))}{\mathrm{d}^2((3,4),(1,2))}5 + \frac{\mathrm{d}^2(\mathbf{x},(3,4))}{\mathrm{d}^2((1,2),(1,2))}6$$

Interpolation III: Adding partial derivatives

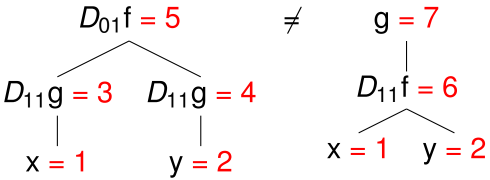
Notation For (eg) binary smooth f and $m = i_1 \ldots i_k \in \{0,1\}^*$ define:

$$\mathrm{D}_m f(x_0, x_1) = \; = \; \frac{\partial f(x_0, x_1)}{\partial x_{i_1} \ldots \partial x_{i_k}}$$

What f, g, x make:

$$D_{01} f(D_{11} g(x), D_{11} g(y)) \neq g(D_{11} f(x, y))$$

Solution: Number distinct subexpressions



Requirements

$$D_{01} f(3, 4) = 5 \quad D_{11} f(1, 2) = 6$$

Multivariate Hermite Interpolation

A *multivariate Hermite interpolation problem of dimension* $d \geq 0$ is given by:

1. A finite set of *nodes* $x_i \in \mathbb{R}^d$ ($i = 1.k$), and
2. For each node x_i, finitely many *conditions*

$$D_{m_{ij}}(f)(x_i) = r_{ij}$$

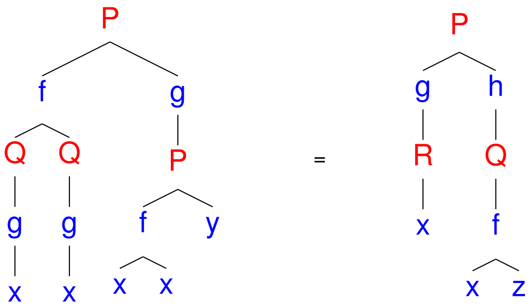
The set of conditions must be *consistent*, in that each r_{ij} is determined by the node x_i and the permutation equivalence class of m_{ij}.

Theorem (Severi 1930)

Every such problem has a solution by a polynomial of degree $k(\max(|m_{ij}|) + 1) - 1$.

R. A. Lorentz, *Multivariate Birkhoff Interpolation*, Lecture Notes in Mathematics, **1516**, Springer, 1992.

Interpolation: Polynomially layered expressions



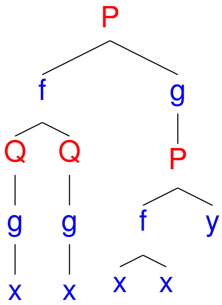
Outputs of Polynomials = Inputs of Functions + Outputs of expressions

Inputs of Polynomials = Outputs of Functions + Inputs of expressions

Interpolation Idea

- Choose inputs of polynomials to differentiate their outputs.
- Solve resulting interpolation problems for functions.

Polynomially layered expressions (cntnd)



Atoms	$f(Q(gx), Q(gx))$	gx	$g(P(f(x,x),y))$	$f(x,x)$	x	y
Interpolation Variables	v_1	v_2	v_3	v_4	v_5	v_6

Node Polynomials	$P(v_1, v_3)$	$Q(v_2)$	$P(v_4, v_6)$	v_5	v_6
Separation values	$P(a_1, a_3) : b_1$	$Q(a_2) : b_3$	$P(a_4, a_6) : b_4$	$a_5 = b_5$	$a_6 = b_6$

Interpolation for f	$f(b_3, b_3) = a_1$	$f(b_5, b_5) = a_4$

Canonical forms and atomic expressions

Notation:

$$f_m(e_0, \ldots, e_{n-1}) \quad = \quad \frac{\partial f(x_0, \ldots, x_{n-1})}{\partial x_{i_1} \ldots \partial x_{i_k}}[e_0/x_0, \ldots, e_{n-1}/x_{n-1}] \quad (m = i_1, \ldots, i_k)$$

- $$\text{A variable } x \text{ is an atomic expression}$$
- $$\frac{c_0, \ldots, c_{n-1} \text{ are canonical forms}}{f_m(c_0, \ldots, c_{n-1}) \text{ is an atomic expression}}$$
- For a polynomial $P(x_0, \ldots, x_{n-1})$ (all x_i in P)

  $$\frac{a_0, \ldots, a_{n-1} \text{ atomic expressions}}{P(a_0, \ldots, a_{n-1}) \text{ is a canonical expression}}$$

  (and $a_0, \ldots, a_{n-1}$ are its *immediate* atomic subexpressions).

Lemma (Canonicalisation)

For every e there is a canonical form $\mathrm{CF}(c)$ such that $\vdash e = \mathrm{CF}(e)$.

Equivalence of atomic expressions

$$\frac{f = f' \quad m \sim m' \quad c_i \approx c'_i \;\; (i = 1, n)}{f_m(c_1, \ldots, c_n) \approx f'_{m'}(c'_1, \ldots, c'_{n'})}$$

Equivalence of canonical forms

1. Fix a finite set of atomic expressions A
2. Set

$$[a] = \{a' \in A \mid \vdash a' = a\} \quad (a \in A)$$

3. Choose distinct *interpolation* variables $v_{[a]}$.
4. For any canonical form $c = P(a_0, \ldots, a_{n-1})$ with immediate atomic expressions included in A, its node polynomial is

$$P_c = P(v_{[a_0]}, \ldots, v_{[a_{n-1}]})$$

5. For two such c and c' set:

$$c \approx c' \iff P_c \sim P_{c'}$$

Separation, completeness, and decidability

Completeness follows from:

Theorem (Separation theorem)

For any finite set C of canonical forms there is a polynomial function environment φ and an environment ρ such that, for all $c, c' \in C$:

$$c \not\approx c' \iff \mathcal{S}[\![c]\!]\varphi\rho \neq \mathcal{S}[\![c']\!]\varphi\rho$$

As canonicalisation is effective, and $\approx$ is decidable (by induction on size of canonical expressions), we have:

Theorem

For any two expressions e_0, e_1, it is decidable whether or not

$$\vdash e_0 = e_1$$

Equational completeness

- An equational theory is *inconsistent* if its proves $x = y$.
- An equational theory is *equationally complete* (or *Hilbert-Post complete*) if it has no consistent extension.
- The theory of Semilattices (Commutative Monoids & $x + x = x$) is equationally complete.
- Proof: Consider adding an equation, no variables repeating on either side:

$$\sum_{i=1}^{m} x_i = \sum_{j=1}^{n} y_j$$

- If a variable x on one side not on the other then get

$$x = 0$$

Equational completeness

First note that, for closed e, by completeness one has $\vdash e = r$ where $r =_{\mathrm{def}} \mathcal{S}[\![e]\!]$ (for $\models e = r$). Then:

- Suppose, for unary f and otherwise closed $e_0(f), e_1(f)$, we add the unprovable equation

$$e_0 = e_1$$

- Then, *by polynomial completeness* there is a unary polynomial $P(x)$ such that

$$\mathcal{S}[\![e_0]\!]0[P/f]] \neq \mathcal{S}[\![e_1]\!]0[P/f]]$$

- So we have

$$\mathcal{S}[\![e_0[(x).\,P(x)/f]]\!] = \mathcal{S}[\![e_0]\!]0[P/f]] \neq \mathcal{S}[\![e_1]\!]0[P/f]] = \mathcal{S}[\![e_0[(x).\,P(x)/f]]\!]$$

- But, by above remark & assumed equation, can prove

$$r =_{\mathrm{def}} \mathcal{S}[\![e_0[(x).\,P(x)/f]]\!] = \mathcal{S}[\![e_0[(x).\,P(x)/f]]\!] =_{\mathrm{def}} s$$

- So can prove, in succession:

$$r - s = 0 \qquad 1 = 0 \qquad x = 0 \qquad x = y$$

Equational incompleteness of basic theory (= no constants)

- It is consistent to add

$$1 + 1 = 0$$

- Can take as model the boolean ring (= 2-element field) $(\mathbb{B}, \oplus, \wedge)$, with differentiation of functions $f : \mathbb{B}^n \to \mathbb{B}$ defined by:

$$\frac{\partial f(x_0,\ldots,x_{n-1})}{\partial x_i} = \begin{array}{l} f(x_0, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_{n-1}) \\ + \; f(x_0, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_{n-1}) \end{array}$$

  equivalently

$$\frac{\partial f(x_0,\ldots,x_{n-1})}{\partial x_i} = \frac{f(\ldots,x_i + \mathrm{d}x_i,\ldots) - f(\ldots,x_i,\ldots)}{\mathrm{d}x_i}$$

  setting $\mathrm{d}x_i = 1$.

- Another (different) model: abstract clone of polynomials over $\mathbb{B}$, with usual formal differentiation.

Syntax for cartesian differential categories

Differential at e_1 of the function $f(x) =_{\text{def}} e_0$, applied to e_2:

$$\mathrm{D}(x.e_0, e_1, e_2)$$

Notation after Cockett, Blute, Seely:

$$\frac{\partial e_0}{\partial x}(e_1) \cdot e_2 \quad = \quad \mathrm{D}(x.e_0, e_1, e_2)$$

More notation

$$\frac{\partial}{\partial x} e \cdot e_0 \quad = \quad \frac{\partial e}{\partial x} \cdot e_0 \quad = \quad \frac{\partial e}{\partial x}(x) \cdot e_0$$

More notation

$$\frac{\partial e}{\partial x_{n-1} \ldots \partial x_0} \cdot (e_0, \ldots, e_{n-1}) \quad = \quad \frac{\partial}{\partial x_{n-1}} (\ldots \frac{\partial}{\partial x_0} e \cdot e_0 \ldots) \cdot e_{n-1}$$

Syntax

Axioms for cartesian differential categories

1. $+, 0$ form a commutative monoid

2. $$\frac{\partial 0}{\partial x} \cdot v = 0 \qquad \frac{\partial x + y}{\partial x} \cdot v = v$$

3. $$\frac{\partial f(x)}{\partial x} \cdot 0 = 0 \qquad \frac{\partial f(x)}{\partial x} \cdot (v + w) = \frac{\partial f(x)}{\partial x} \cdot v + \frac{\partial f(x)}{\partial x} \cdot w$$

4. $$\frac{\partial\, f(g_0(x), g_1(x))}{\partial x} \cdot v \quad = \quad \frac{\partial f(x_0, g_1(x)))}{\partial x_0}(g_0(x)) \cdot (\frac{\partial\, g_0(x)}{\partial x} \cdot v) + \frac{\partial f(g_0(x), x_1)}{\partial x_1}(g_1(x)) \cdot (\frac{\partial\, g_1(x)}{\partial x} \cdot v)$$

5. $$\frac{\partial f(x, y)}{\partial x \partial y} \cdot (u, v) = \frac{\partial f(x, y)}{\partial y \partial x} \cdot (v, u)$$

6. $$\frac{\partial}{\partial y}\left(\frac{\partial f(x)}{\partial x} \cdot y\right) \cdot z = \frac{\partial f(x)}{\partial x} \cdot z$$