π calculus: toward global computing

Christian Williams

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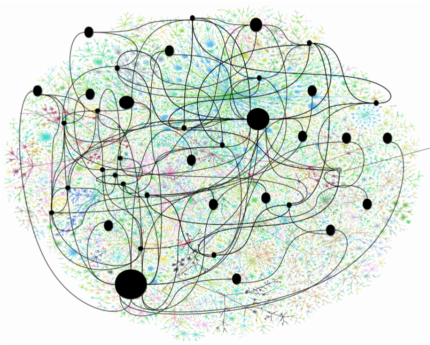
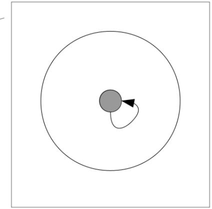


What is the internet?

What is a computer?



π calculus   language of interaction

```
N:=  a, b, c, …                         Names

P:=  0             null                 Processes
     + P|P         parallel
     + a(x).P      input
     + āy          output

     āy|a(x).P => P{y/x}                Communication
```

ϙ̄k | ϙ(x).acs[x] => acs[k]

"… by understanding communicating systems, there is a sense of **completion**; for we can now begin to describe what goes on **outside** a computer in the same terms as what goes on **inside** ... we may say that we inhabit a **global computer**, an informatic world which demands to be understood just as fundamentally as physicists understand the material world."

— Robin Milner, creator

RChain: Secure. Sustainable. Scalable.

(transition)

relaxed intro.

designing communities: magic.

let's interact.

syntax

$N ::= \{a, b, c, \ldots\}$

$P ::=$	0	nul	— do nothing
$\mid$	$\bar{a}b.P$	out	— send b on a, then P
$\mid$	$a(x).P$	inp	— receive (x) on a, then P
$\mid$	$P \mid P$	par	— do in parallel
$\mid$	$\nu x\, P$	new	— create variable with scope P
$\mid$	$!P$	rep	— replicate P
$\mid$	$P + P$	sum	— do first xor second

syntax (in math)

$$N = \{a, b, c, \ldots\}$$

$$P \ni 0$$

$$\overline{} : N^2 \times P \longrightarrow P$$

$$(\,) : N^2 \times P \longrightarrow P$$

$$| : P^2 \longrightarrow P$$

$$\nu : N \times P \longrightarrow P$$

$$! : P \longrightarrow P$$

$$+ : P^2 \longrightarrow P$$

names: free and bound

$\mathrm{fn}(P) := $ free names of P

$$\mathrm{fn}(0) = \emptyset$$

$$\mathrm{fn}(\bar{a}b.P) = \{a,b\} \cup \mathrm{fn}(P)$$

$*$ $$\mathrm{fn}(a(x).P) = \{a\} \cup \mathrm{fn}(P) \setminus \{x\}$$

binding

(conventions)

$$\mathrm{fn}(P \mid Q) = \mathrm{fn}(P) \cup \mathrm{fn}(Q)$$

$*$ $$\mathrm{fn}(\nu x\, P) = \mathrm{fn}(P) \setminus \{x\}$$

$$\mathrm{fn}(!P) = \mathrm{fn}(P)$$

$$\mathrm{fn}(P+Q) = \mathrm{fn}(P) \cup \mathrm{fn}(Q)$$

communication

COMM: $\overline{a}z.P \mid a(x).Q$

$\Rightarrow \quad P \mid Q\{z/x\}$

example: nondeterminism

$$\begin{tikzcd} & \bar{a}y \mid a(x).\bar{x}v \mid \bar{a}z \arrow[dl] \arrow[dr] \arrow[d, phantom, "\text{nondeterminism}"] & \\ \bar{y}v \mid \bar{a}z & & \bar{a}y \mid \bar{z}v \end{tikzcd}$$

$$\bar{a}y.\bar{y}m \mid a(x).x(p).\bar{x}e \mid \bar{a}z.\bar{z}n$$

names are resources

example: restriction

$$\nu s(\overline{x}s.\overline{s}a.\overline{s}b \mid x(u).u(y).u(z).\overline{y}z)$$

$$\Rightarrow \nu s(\overline{s}a.\overline{s}b \mid s(y).s(z).\overline{y}z)$$

$$\Rightarrow \nu s(\overline{s}b \mid s(z).\overline{a}z)$$

$$\Rightarrow \nu s(\overline{a}b)$$

(passing a private channel)

congruence

SC-SUM-ASSOC	$M_1 + (M_2 + M_3)$	$\equiv$	$(M_1 + M_2) + M_3$
SC-SUM-COMM	$M_1 + M_2$	$\equiv$	$M_2 + M_1$
SC-SUM-INACT	$M + \mathbf{0}$	$\equiv$	M
SC-COMP-ASSOC	$P_1 \mid (P_2 \mid P_3)$	$\equiv$	$(P_1 \mid P_2) \mid P_3$
SC-COMP-COMM	$P_1 \mid P_2$	$\equiv$	$P_2 \mid P_1$
SC-COMP-INACT	$P \mid \mathbf{0}$	$\equiv$	P
SC-RES	$\nu z\, \nu w\, P$	$\equiv$	$\nu w\, \nu z\, P$
SC-RES-INACT	$\nu z\, \mathbf{0}$	$\equiv$	$\mathbf{0}$
SC-RES-COMP	$\nu z\, (P_1 \mid P_2)$	$\equiv$	$P_1 \mid \nu z\, P_2$, if $z \notin \mathsf{fn}(P_1)$
SC-REP	$!P$	$\equiv$	$P \mid !P$

Table 1.1. *The axioms of structural congruence*

REFL	$P = P$
SYMM	$P = Q$ implies $Q = P$
TRANS	$P = Q$ and $Q = R$ implies $P = R$
CONG	$P = Q$ implies $C[P] = C[Q]$

Table 1.2. *The rules of equational reasoning*

reduction

$$\text{R-INTER} \quad \frac{}{(\overline{x}y.\, P_1 + M_1) \mid (x(z).\, P_2 + M_2) \longrightarrow P_1 \mid P_2\{y/z\}}$$

$$\text{R-TAU} \quad \frac{}{\tau.\, P + M \longrightarrow P}$$

$$\text{R-PAR} \quad \frac{P_1 \longrightarrow P_1'}{P_1 \mid P_2 \longrightarrow P_1' \mid P_2} \qquad \text{R-RES} \quad \frac{P \longrightarrow P'}{\nu z\, P \longrightarrow \nu z\, P'}$$

$$\text{R-STRUCT} \quad \frac{P_1 \equiv P_2 \longrightarrow P_2' \equiv P_1'}{P_1 \longrightarrow P_1'}$$

Table 1.3. *The reduction rules*

example: replication

$$\nu s \quad s(v).!\bar{s}v \qquad ?$$

$$!\,\nu p \quad \bar{n}p.p(z).!\bar{p}z \qquad ?$$

(! ~ permanent, ν ~ local)

what about tuples?

$$x(y,z) \overset{\text{def}}{\underset{?}{=}} x(y).x(z)$$

$$\overline{x}\langle a,b\rangle \overset{\text{def}}{=} \overline{x}a.\overline{x}b$$

$$¿\quad x(y,z).\overline{y}z \mid \overline{x}\langle a,b\rangle \mid x(y,z).\overline{z}y \quad ?$$

polyadicity

$$\overline{a}\langle b_1, b_2 \cdots b_n \rangle \stackrel{\text{def}}{=} \nu w\ \overline{a}w.\overline{w}b_1.\cdots\overline{w}b_n$$

$$a(x_1, x_2 \cdots x_n) \stackrel{\text{def}}{=} a(w).w(x_1).\cdots w(x_n)$$

polyadic: send + receive lists of names

recursion

recursion

$A[x] \stackrel{def}{=} P$ $\quad fn(P) \subseteq \{x\}$, P contains A's

$\longrightarrow A[x] = !a(x).P\{\overline{a}y/A[y]\}$

$A[z] \in S$

$\longrightarrow \nu a(\overline{a}z \mid !a(x).P\{''\}) \in S$

(calling yourself)

example: update

$$S[v] = !(\bar{s}v + s(w).S[w])$$

$$= !s(v).\,!(\bar{s}v + s(w).\bar{s}w)$$

updateable storage

$$(P \mid S[i])$$

$$= (P \mid \nu s(\bar{s}i \mid !s(v).\,!(\bar{s}v + s(w).\bar{s}w)))$$

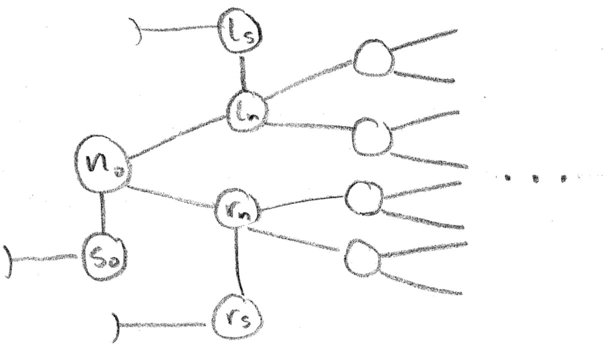
example: binary tree

$$
\begin{aligned}
&\mathrm{tree}[\mathit{node}, \mathit{store}] := \\
&\quad \mathbf{new}\ \mathit{left}_n, \mathit{left}_s, \mathit{right}_n, \mathit{right}_s \\
&\quad (\,!\,\overline{\mathit{node}}\langle \mathit{left}_n, \mathit{left}_s, \mathit{right}_n, \mathit{right}_s \rangle \\
&\quad \mid \mathit{store}(\mathit{value}).\ !\,\overline{\mathit{store}}\langle \mathit{value} \rangle \\
&\quad \mid \mathrm{Tree}[\mathit{left}_n, \mathit{left}_s] \mid \mathrm{Tree}[\mathit{right}_n, \mathit{right}_s]\,)
\end{aligned}
$$

(evolving system of cells)

internal

example: binary tree



$$T \equiv \overline{t}\langle n_0, s_0\rangle \mid !\,t(n,s).\,(\nu\, l_n, l_s, r_n, r_s)$$
$$!\,\overline{n}\langle l_n, l_s, r_n, r_s\rangle \mid s(v).\,!\,\overline{s}v \mid \overline{t}\langle l_n, l_s\rangle \mid \overline{t}\langle r_n, r_s\rangle)$$

example: HIGUL

a pair of processes cooperate
to build two unbounded chains of storage cells

$$P \overset{\text{def}}{=} (\nu n,s)(H \mid I \mid G \mid U \mid L)$$

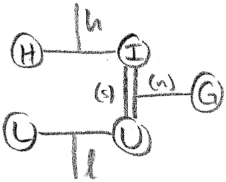
$$H \overset{\text{def}}{=} h(z).\,!\bar{h}z$$

$$L \overset{\text{def}}{=} \ell(z).\,!\bar{\ell}z$$

$$G \overset{\text{def}}{=} !\nu p\ \bar{n}p.p(z).\,!\bar{p}z$$

$$I \overset{\text{def}}{=} \nu i\,(\bar{i}h \mid !i(z).n(a).\bar{z}a.\bar{s}a.s(c).\bar{i}c)$$

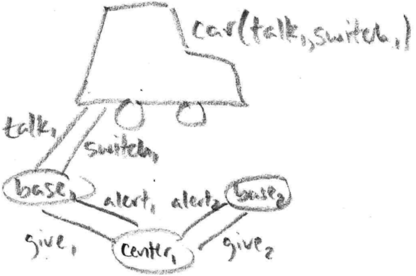
$$U \overset{\text{def}}{=} \nu u\,(\bar{u}\ell \mid !u(z).n(b).\bar{z}b.\ n(c).\bar{z}c.s(a).\bar{s}c.\bar{u}a)$$



dynamic topology

<u>mobility</u>

- $car[talk, switch]$
  $\stackrel{def}{=} talk.car[talk, switch]$
  $+\ switch(t', s').car[t', s']$
- $base[t, s, g, a]$
  $\stackrel{def}{=} t.base[t, s, g, a]$
  $+\ g(t', s').\bar{s}\langle t', s'\rangle.idlebase(t, s, g, a)$
- $idlebase[t, s, g, a] \stackrel{def}{=} a.base(t, s, g, a)$
- $center_1 \stackrel{def}{=} \overline{give_1}\langle talk_2, switch_2\rangle.alert_2.center_2$
  $center_2 \stackrel{def}{=} \overline{give_2}\langle talk_1, switch_1\rangle.alert_1.center_1$



dynamic topology

$$\mathit{system}_1 \equiv (\nu \vec{c})(\mathit{car}(\mathit{talk}_1, \mathit{switch}_1) \mid \mathit{base}_1 \mid \mathit{idle}_2 \mid \mathit{center}_1)$$

$$\Rightarrow (\nu \vec{c})(\mathit{car}(\mathit{talk}_1, \mathit{switch}_1) \mid \overline{\mathit{switch}_1}\langle \mathit{talk}_2, \mathit{switch}_2 \rangle.\mathit{idle}_1 \mid \mathit{idle}_2 \mid \mathit{alert}_2.\mathit{center}_2)$$

$$\Rightarrow (\nu \vec{c})(\mathit{car}(\mathit{talk}_2, \mathit{switch}_2) \mid \mathit{idle}_1 \mid \mathit{idle}_2 \mid \mathit{alert}_2.\mathit{center}_2)$$

$$\Rightarrow (\nu \vec{c})(\mathit{car}(\mathit{talk}_2, \mathit{switch}_2) \mid \mathit{idle}_1 \mid \mathit{base}_2 \mid \mathit{center}_2) \equiv \mathit{system}_2$$

categories

$$\bar{a}b.P \mid a(x).Q \Rightarrow P \mid Q\{b/x\}$$

$$\begin{tikzcd}
 & (N^2P)^2 \arrow[r] \arrow[dr, phantom, "\Downarrow \mathrm{comm}"] & P^2 \arrow[dr] & \\
N^3P^2 \arrow[ur] \arrow[drr] & & & P \\
 & & P^2 \arrow[ur] &
\end{tikzcd}$$

?

enriched algebraic theories

— with binding

reflection

reflection

code $\Longleftrightarrow$ data

$$p(n) = \overset{\text{nul}}{1} + \overset{\text{out}}{n^2 p(n)} + \overset{\text{inp}}{n^2 p(n)} + \overset{\text{par}}{p(n)^2}$$

$$\underset{\text{new}}{n\,p(n)} + \underset{\text{rep}}{p(n)} + \underset{\text{sum}}{p(n)^2}$$

$p(r) = r$ (least fixed point — "Y combinator")

ρ calculus

<u>rho</u>

$P ::= 0 \mid a?(\vec{x}).P \mid a!(P) \mid P|P \mid {}^{*}a$

evaluate

$N ::= @P$ reference

COMM: $a!(P) \mid a?(x).Q \Rightarrow Q\{@P/x\}$

application: blockchain

application: biology [1]

Biology	Process calculi
Entity	Process
Interaction capability	Channel
Interaction	Communication
Modification/evolution	State change

Table 1: Process calculi abstraction for systems biology

application: biology [2]

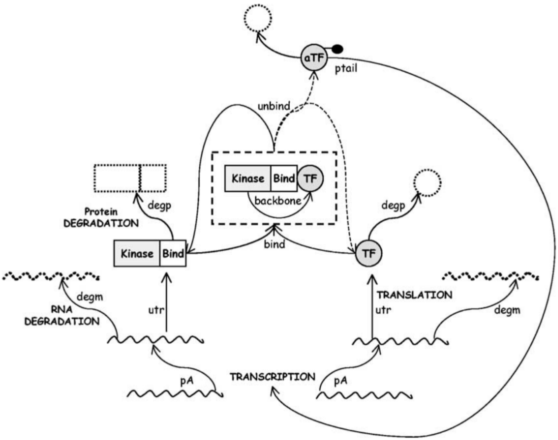


Fig. 1. A simple biomolecular process: Transcriptional regulation by positive feedback.

application: biology [2]

Table 2
Specification of a biomolecular system

$$\mathrm{Sys} = \mathrm{Gene\_A}|\mathrm{Gene\_TF}|\mathrm{Transcr}|\mathrm{Transl}|\mathrm{RNA\_Deg}|\mathrm{Protein\_Deg}$$

$$\mathrm{Gene\_A} = (\mathrm{basal}(), 4).(\mathrm{Gene\_A}|\mathrm{RNA\_A}) + (\mathrm{pA}(), 40).(\mathrm{Gene\_A}|\mathrm{RNA\_A})$$

$$\mathrm{RNA\_A} = (\mathrm{utr}(), 1).(\mathrm{RNA\_A}|\mathrm{Protein\_A}) + (\mathrm{degm}(), 1)$$

$$\mathrm{Protein\_A} = (\nu \mathrm{bb1}, \mathrm{bb2}, \mathrm{bb3})(\mathrm{Binding\_Site}|\mathrm{Kinase})$$

$$\mathrm{Binding\_Site} = (\overline{\mathrm{bind}}\langle \mathrm{bb1}, \mathrm{bb2}, \mathrm{bb3}\rangle, 0.1).\mathrm{Bound\_Site} + (\mathrm{degp}(), 0.1).(\overline{\mathrm{bb3}}, \infty)$$

$$\mathrm{Bound\_Site} = (\overline{\mathrm{bb1}}, 10).\mathrm{Binding\_Site} + (\mathrm{degp}(), 0.1).(\overline{\mathrm{bb3}}, \infty).(\overline{\mathrm{bb3}}, \infty)$$

$$\mathrm{Kinase} = (\overline{\mathrm{bb2}}\langle \mathrm{ptail}\rangle, 10).\mathrm{Kinase} + (\mathrm{bb3}(), \infty)$$

$$\mathrm{Gene\_TF} = (\mathrm{basal}(), 4).(\mathrm{Gene\_TF}|\mathrm{RNA\_TF}) + (\mathrm{pA}(), 40).(\mathrm{Gene\_TF}|\mathrm{RNA\_TF})$$

$$\mathrm{RNA\_TF} = (\mathrm{utr}(), 1).(\mathrm{RNA\_TF}|\mathrm{Protein\_TF}) + (\mathrm{degm}(), 1)$$

$$\mathrm{Protein\_TF} = (\mathrm{bind}(\mathrm{c\_bb1}, \mathrm{c\_bb2}, \mathrm{c\_bb3}), 0.1).\mathrm{Bound\_TF} + (\mathrm{degp}(), 0.1)$$

$$\begin{aligned}\mathrm{Bound\_TF} = {} & (\mathrm{c\_bb1}(), 10).\mathrm{Protein\_TF} + (\mathrm{c\_bb3}(), \infty) + \\ & (\mathrm{c\_bb2}(\mathrm{tail}), 10).((\mathrm{c\_bb1}(), 10).\mathrm{Active\_TF}(\mathrm{tail}) + (\mathrm{c\_bb3}(), \infty))\end{aligned}$$

$$\mathrm{Active\_TF}(\mathrm{tail}) = (\overline{\mathrm{tail}}, 100).\mathrm{Active\_TF}(\mathrm{tail}) + (\mathrm{degp}(), 0.1)$$

$$\mathrm{Transcr} = (\overline{\mathrm{basal}}, 4).\mathrm{Transcr} + (\mathrm{ptail}(), 100).(\overline{\mathrm{pA}}, 40).\mathrm{Transcr}$$

$$\mathrm{Transl} = (\overline{\mathrm{utr}}, 1).\mathrm{Transl}$$

$$\mathrm{RNA\_Deg} = (\overline{\mathrm{degm}}, 1).\mathrm{RNA\_Deg}$$

$$\mathrm{Protein\_Deg} = (\overline{\mathrm{degp}}, 0.1).\mathrm{Protein\_Deg}$$

mathematics

linear logic [3]
game semantics [4]
double categories [5]
bigraphs [6]
operational semantics
categorical semantics

references

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