Symmetries of the Kepler problem

Jesper Göransson

March 8, 2015

Abstract

By choosing a nonstandard parameterisation in the n-dimensional Kepler problem, the energy equation for negative energies will denote an n-dimensional sphere in the spacetime and the $(n+1)$-velocities will describe great circles on this sphere. It follows that the symmetry group for negative energies is $O(n+1)$. On similar grounds it is seen that the symmetry groups for positive energies respectively zero energy are $O^+(1,n)$ and $E(n)$.

1 Introduction

In the Kepler problem a particle with mass m and position vector $\boldsymbol{r}$ is moving under the influence of a central force $\boldsymbol{F} = -k\boldsymbol{r}/r^3$, where k is a constant that determines the strength of the force field and $r = |\boldsymbol{r}|$ is the distance to the origin. The equation of motion becomes

$$m\frac{d^2\boldsymbol{r}}{dt^2} = -k\frac{\boldsymbol{r}}{r^3}.$$

By taking the scalar product with $d\boldsymbol{r}/dt$ and integrating, we get the conservation of energy:

$$m\frac{d^2\boldsymbol{r}}{dt^2}\cdot\frac{d\boldsymbol{r}}{dt} = -k\frac{\boldsymbol{r}}{r^3}\cdot\frac{d\boldsymbol{r}}{dt}$$
$$\frac{m}{2}\left|\frac{d\boldsymbol{r}}{dt}\right|^2 - \frac{k}{r} = E,$$

where the integration constant E is the total energy of the system.

2 Elliptic case

We now assume that the energy is negative, $E < 0$, and we can then define a characteristic speed $V = \sqrt{-2E/m}$, a characteristic radius $R = -k/(2E)$ and a characteristic time $T = R/V = -k\sqrt{-m/(8E^3)}$.

We will consider the Kepler problem in the spacetime. In the spacetime there is no canonical parameterisation of the solution curves and we use this freedom to change the parameterisation of the solution curves to be with respect to some parameter λ instead of the usual time t. We define that a prime will denote differentiating with respect to λ: $t' = dt/d\lambda$, $\boldsymbol{r}' = d\boldsymbol{r}/d\lambda$ and similar for higher derivatives.

Instead of defining what geometric quantity λ is and give the parameterisation explicitly, we state a constraint $t' = r/V$. This specifies the parameterisation indirectly in that it fixes the velocities of the solution curves and thus determines the parameterisation up to a constant. It still leaves open what quantity λ represent, and we will return to this question shortly.

Using the parameterisation constraint, the energy equation can be expressed as

$$
\begin{aligned}
\frac{m}{2}\left|\frac{\boldsymbol{r}'}{t'}\right|^2 - \frac{k}{r} &= E \\
\frac{m|\boldsymbol{r}'|^2}{2(t')^2} - \frac{k}{Vt'} &= E \\
-\frac{2E}{m}(t')^2 - \frac{2k}{mV}t' + |\boldsymbol{r}'|^2 &= 0 \\
V^2(t')^2 - 2VRt' + |\boldsymbol{r}'|^2 &= 0 \\
V^2(t'-T)^2 + |\boldsymbol{r}'|^2 &= R^2.
\end{aligned}
$$

Thus, the energy equation turns into a quadratic equation which defines a canonical metric on the spacetime, making it into a Euclidean space. The length of a spacetime displacement vector $\Delta\boldsymbol{x} = \Delta t\boldsymbol{e}_t + \Delta\boldsymbol{r}$ in this metric is given by

$$
||\Delta\boldsymbol{x}|| = \sqrt{(V\Delta t)^2 + |\Delta\boldsymbol{r}|^2}.
$$

Our choice of parameterisation has a peculiar feature. Since the right hand side of the defining constraint $t' = r/V$ has dimension of time, it follows that differentiating with respect to λ does not change the dimension and so λ must be a dimensionless quantity. This leads to that although $\boldsymbol{r}'$ and $\boldsymbol{r}''$ are to be regarded as a kind of velocity and acceleration vectors, they have the dimension of length and are valid space vectors. In particular, $\boldsymbol{r}$, $\boldsymbol{r}'$ and $\boldsymbol{r}''$ can be added or subtracted from each other.

2.1 Equations of motion

To see how the system evolves with respect to the new parameterisation we rewrite the equation of motion using the parameterisation constraint:

$$
\begin{aligned}
m\frac{t'\boldsymbol{r}'' - t''\boldsymbol{r}'}{(t')^3} &= -k\frac{\boldsymbol{r}}{r^3} \\
mV^2\frac{r\boldsymbol{r}'' - r'\boldsymbol{r}'}{r^3} &= -k\frac{\boldsymbol{r}}{r^3} \\
\boldsymbol{r}'' &= -\frac{R\boldsymbol{r}}{r} + \frac{r'\boldsymbol{r}'}{r}.
\end{aligned}
$$

The radial component of $\boldsymbol{r}''$ is $r'' - (|\boldsymbol{r}'|^2 - (r')^2)/r$ and the radial equation of motion becomes

$$\begin{aligned} r'' - \frac{|\boldsymbol{r}'|^2 - (r')^2}{r} &= -R + \frac{(r')^2}{r} \\ r'' &= \frac{|\boldsymbol{r}'|^2}{r} - R \\ &= \frac{R^2 - (R-r)^2}{r} - R \\ &= -(r - R), \end{aligned}$$

which is equivalent to

$$t''' = -(t' - T).$$

By differentiating the equation of motion and using the result for the radial equation of motion we further get that

$$\boldsymbol{r}''' = -\boldsymbol{r}'.$$

2.2 Symmetry group

Let $\boldsymbol{v} = (t' - T)\boldsymbol{e}_t + \boldsymbol{r}'$ be a vector denoting the $(n+1)$-velocity relative to the characteristic $(n+1)$-velocity $T\boldsymbol{e}_t$. The energy equation and the equations of motion can then be expressed as

$$\begin{cases} ||\boldsymbol{v}|| = R \\ \boldsymbol{v}'' = -\boldsymbol{v} \end{cases}$$

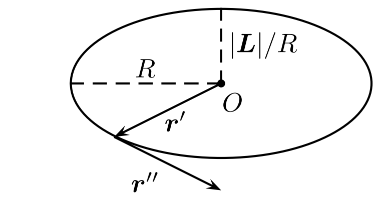
These equations are invariant under the symmetry group $O(n+1)$ and we see that the $\boldsymbol{v}$-vector will describe a great circle on the sphere $||\boldsymbol{v}|| = R$ with constant speed $||\boldsymbol{v}'|| = R$. We also see that the great circles are parameterised by their angle and that the parameter λ is the angle of the circle.

2.3 Conserved spacetime bivector

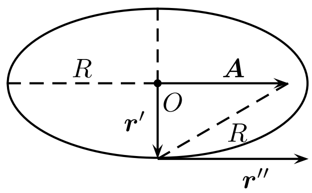
Since $\boldsymbol{v}$-curves are great circles with constant speed we have that $\boldsymbol{\Gamma} = \boldsymbol{v} \wedge \boldsymbol{v}'$ is a conserved spacetime bivector. It can be computed as

$$\begin{aligned} \boldsymbol{\Gamma} &= \boldsymbol{v} \wedge \boldsymbol{v}' \\ &= T\boldsymbol{e}_t \wedge \frac{1}{T}((t' - T)\boldsymbol{r}'' - t''\boldsymbol{r}') + \boldsymbol{r}' \wedge \boldsymbol{r}'' \\ &= T\boldsymbol{e}_t \wedge -\frac{1}{t'}((t' - T)\boldsymbol{r} + t''\boldsymbol{r}') + \frac{T}{t'}\boldsymbol{r} \wedge \boldsymbol{r}' \\ &= T\boldsymbol{e}_t \wedge -\frac{1}{r}((r - R)\boldsymbol{r} + r'\boldsymbol{r}') + \frac{R}{r}\boldsymbol{r} \wedge \boldsymbol{r}'. \end{aligned}$$

The bivector consists of a conserved spatial bivector $\boldsymbol{L} = \frac{R}{r}\boldsymbol{r} \wedge \boldsymbol{r}'$ and a conserved space vector $\boldsymbol{A} = \frac{1}{r}((r - R)\boldsymbol{r} + r'\boldsymbol{r}')$. The great circles are projected onto the spatial subspace as centered ellipses. Such an ellipse lies in the plane given by the bivector $\boldsymbol{L}$ and has major semiaxis R and area $\pi|\boldsymbol{L}|$.



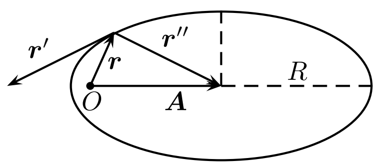
To see the meaning of $\boldsymbol{A}$ we observe that when $\boldsymbol{r}'$ points at a covertex of the ellipse we have that $\boldsymbol{A} = (r-R)\boldsymbol{r}''/R$ and hence is directed along the major axis. Its length is then $|\boldsymbol{A}| = r - R = V|t'-T| = \sqrt{R^2 - (|\boldsymbol{L}|/R)^2}$ since $V|t'-T|$ is the height from the covertex to the corresponding point on the great circle. This length relationship is one of the defining properties of a focus of an ellipse and we conclude that $\boldsymbol{A}$ points to one of the foci of the ellipse.



With the help of the vector $\boldsymbol{A}$ the equation of motion for $\boldsymbol{r}''$ can be expressed as:

$$\boldsymbol{r}'' = -(\boldsymbol{r} - \boldsymbol{A})$$

Also the $\boldsymbol{r}''$-vector describe the same ellipse as the $\boldsymbol{r}'$-vector and hence we conclude that the orbits for negative energies of the Kepler problem are ellipses with the origin as one of the foci.



The conserved bivector $\boldsymbol{L}$ can be identified with the angular momentum bivector

$$\boldsymbol{L}_{\mathrm{AM}} = \boldsymbol{r} \wedge m\frac{d\boldsymbol{r}}{dt} = \boldsymbol{r} \wedge m\frac{\boldsymbol{r}'}{t'} = \frac{m}{T}\boldsymbol{L}$$

and the the conserved vector $\boldsymbol{A}$ that gives the displacement of the force center corresponds to the so called Laplace-Runge-Lenz vector which in three dimensions is defined as

$$\boldsymbol{A}_{\mathrm{LRL}} = -m\frac{d\boldsymbol{r}}{dt} \times \boldsymbol{L}_{\mathrm{AM}} - mk\frac{\boldsymbol{r}}{r}.$$

In n dimensions the cross product generalises to the geometric product and we get that

$$\boldsymbol{A}_{\mathrm{LRL}} = -m\frac{d\boldsymbol{r}}{dt}\boldsymbol{L}_{\mathrm{AM}} - mk\frac{\boldsymbol{r}}{r} = \frac{m^2}{V^2}\boldsymbol{A}.$$

3 Hyperbolic case

When the energy is positive, $E > 0$, we instead define the characteristic speed, radius and time as $V = \sqrt{2E/m}$, $R = k/(2E)$ respectively $T = R/V = k\sqrt{m/(8E^3)}$. Using the constraint $t' = r/V$ then gives that the energy equation in this case becomes

$$V^2(t'+T)^2 - |\boldsymbol{r}'|^2 = R^2,$$

which is the equation of a hyperboloid of two sheets centered at $-T\boldsymbol{e}_t$. Of the the two sheets we only consider the one in the positive halfspace since we have that $t' > 0$. This hyperboloid has its vertex at the origin. The energy equation defines a canonical semimetric on the spacetime and turns it into a Minkowski space with scalar product

$$\Delta\boldsymbol{x}^2 = (V\Delta t)^2 - |\Delta\boldsymbol{r}|^2$$

The equations of motions are calculated in a similar way as in the elliptic case with the only difference that there is a change of sign. We thus get that also in the hyperbolic case we have that

$$\begin{cases} \boldsymbol{v} = (t'+T)\boldsymbol{e}_t + \boldsymbol{r}' \\ \boldsymbol{v}^2 = R^2 \\ (\boldsymbol{v}')^2 = R^2 \\ \boldsymbol{v}'' = \boldsymbol{v} \end{cases}$$

The $\boldsymbol{v}$-vector will describe a great hyperbola, i.e. a hyperbola resulting from intersecting the hyperboloid with planes through $-T\boldsymbol{e}_t$, and parameterised such that the spacetime bivector $\boldsymbol{\Gamma} = \boldsymbol{v} \wedge \boldsymbol{v}'$ also in this case is conserved. The parameter λ has the interpretation as the hyperbolic angle of the great hyperbola.

The symmetry group is $O^+(1,n)$ which is $O(1,n)$ without the reflection that interchanges the two sheets.

4 Parabolic case

For the case $E = 0$ we use the parameterisation constraint $t' = r\sqrt{m/k}$. The energy equation turns into

$$2t' - \sqrt{\frac{m}{k}}|\boldsymbol{r}'|^2 = 0$$

and denotes a paraboloid with its vertex at the origin. The equation of motion becomes

$$\boldsymbol{r}'' = -\frac{\boldsymbol{r}}{r} + \frac{r'\boldsymbol{r}'}{r}$$

and the radial acceleration turns out to be constant

$$r'' = 1 \quad \text{and thus} \quad t''' = \sqrt{\frac{m}{k}}.$$

From this it can be calculated that

$$\boldsymbol{r}''' = 0$$

and it follows that if $\boldsymbol{v} = t'\boldsymbol{e}_t + \boldsymbol{r}'$ is a point on the paraboloid then $\boldsymbol{v}'' = \sqrt{m/k}\,\boldsymbol{e}_t$ and $\boldsymbol{v}$ will describe a parabola on this paraboloid cut out by a plane parallel to $\boldsymbol{e}_t$. The parabola projects to the spatial subspace as a line and it follows that the symmetry group is the Euclidean group $E(n)$.