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Jordan Pairs

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INTRODUCTION

The theory of Jordan algebras, compared to that of associative algebras, presents some unusual features. Recall that a unital quadratic Jordan algebra (in the sense of McCrimmon) is a triple $(J,U,1)$ consisting of a module J over some commutative associative ring k of scalars, a quadratic map U from J into the endomorphism ring $\mathrm{End}_k(J)$ of J, and an element $1 \in J$ (the unit element) such that the following identities hold in all scalar extensions.

(1) $\qquad U_1 = \mathrm{Id}$,

(2) $\qquad \{x,y,U_x z\} = U_x\{yxz\}$,

(3) $\qquad U(U_x y) = U_x U_y U_x$.

Here $\{xyz\} = U_{x+z}y - U_x y - U_z y$ is the linearization of $U_x y$. The standard example is an associative algebra with $U_x y = xyx$. Thus for one thing, Jordan algebras are not algebras in the usual sense since they are not based on a bilinear multiplication but rather on the composition $U_x y$ which is quadratic in x and linear in y. More serious is the important role played by the notion of isotope. Let $v \in J$ be invertible with inverse $v^{-1} = U_v^{-1}.v$, and set $U_x^{(v)} = U_x U_v$, for all $x \in J$. Then $(J,U^{(v)},v^{-1})$ is a unital quadratic Jordan algebra, called the v-isotope of J, and denoted by $J^{(v)}$. If v is not invertible then one can still define a non-unital Jordan algebra $J^{(v)}$, the v-homotope of J. We will call two Jordan algebras J and J' isotopic if there exists an isotopism between them, i.e., an isomorphism from J onto some isotope of J'. Several important theorems in the theory of Jordan algebras hold only up to isotopy; i.e., they are of the form " J may not have property (P) but there exists an isotope of J which does". Closely related to this is the fact that the autotopism group of J, usually called the structure group and denoted by $\mathrm{Str}(J)$, plays a more important role than the automorphism group. For example, there is no natural concept of inner automorphism for Jordan algebras (which would be comparable to the inner automorphisms $x \mapsto axa^{-1}$ in an associative algebra) but every invertible element $a \in J$ defines an inner autotopism, namely U_a. All this suggests that there ought to be some algebraic object associated with J which somehow incorporates all homotopes of J and whose automorphism group is just the structure group of J. This object is the Jordan pair (J,J) associated with J.

Let us now describe this concept. A <u>Jordan pair</u> is a pair $V = (V^+, V^-)$ of k-modules together with quadratic maps $Q_+ : V^+ \to \mathrm{Hom}_k(V^-, V^+)$ and $Q_- : V^- \to \mathrm{Hom}_k(V^+, V^-)$ which satisfy the following identities in all scalar extensions.

JP1 $\quad \{x,y,Q_\sigma(x)z\} = Q_\sigma(x)\{yxz\}$,

JP2 $\quad \{Q_\sigma(x)y,y,z\} = \{x,Q_{-\sigma}(y)x,z\}$,

JP3 $\quad Q_\sigma(Q_\sigma(x)y) = Q_\sigma(x)Q_{-\sigma}(y)Q_\sigma(x)$.

Here $\{xyz\} = Q_\sigma(x+z)y - Q_\sigma(x)y - Q_\sigma(z)y$ is, similarly as before, the linearization of $Q_\sigma(x)y$, and the index σ takes the values $+$ and $-$. A standard example is $V^+ = M_{p,q}(R)$, $V^- = M_{q,p}(R)$, rectangular matrices with coefficients in an associative algebra R, with $Q_\sigma(x)y = xyx$. By a homomorphism $h: V \to W$ between Jordan pairs we mean a pair $h = (h_+, h_-)$ of linear maps, $h_\sigma : V^\sigma \to W^\sigma$, such that $h_\sigma(Q_\sigma(x)y) = Q_\sigma(h_\sigma(x))h_{-\sigma}(y)$. From the well-known identity $\{U_x y,y,z\} = \{x,U_y x,z\}$ which holds in any Jordan algebra it is clear that we obtain a Jordan pair from a Jordan algebra J by setting $V^+ = V^- = J$ and $Q_+ = Q_- = U$. This Jordan pair will be denoted by (J,J). Also, if $g \in \mathrm{Str}(J)$ and $g^\# \in \mathrm{Str}(J)$ is defined by $g^\# = g^{-1}U_{g(1)}$ then $(g,(g^\#)^{-1})$ is an automorphism of the Jordan pair (J,J), and it turns out that the map $g \mapsto (g,(g^\#)^{-1})$ is an isomorphism between $\mathrm{Str}(J)$ and the automorphism group of (J,J). Theorems which for Jordan algebras only hold up to isotopy will then hold for the associated Jordan pairs without this restriction.

In an arbitrary Jordan pair $V = (V^+, V^-)$ we still have the concept of homotope as follows. For every $v \in V^-$ the module V^+ becomes a (in general non-unital) Jordan algebra, denoted by V_v^+, with quadratic operators $U_x = Q_+(x)Q_-(v)$ and squaring operation $x^2 = Q_+(x)v$. Thus the space which parametrizes the homotopes (namely V^-) is different from the space in which the homotope lives (namely V^+). By interchanging the roles of V^+ and V^- we can also define a homotope V_u^- for every $u \in V^+$. The condition that a Jordan pair V be of the form (J,J), where J is a unital Jordan algebra, is now that V^- contains an invertible element; i.e., an element v such that $Q_-(v)$ is invertible. In this case, $J = V_v^+$ is a unital Jordan algebra with unit element $Q_-(v)^{-1}v$, and V is isomorphic as a Jordan pair with (J,J). Roughly speaking, therefore, Jordan pairs containing invertible elements are the same as unital Jordan algebras "up to isotopy".

In general, however, a Jordan pair will not contain any invertible elements. To see what is happening in this case, let us first make some remarks on Jordan algebras without unit element. There are two approaches to this: either a non-unital Jordan algebra J is defined in terms of quadratic operators U_x and a squaring operation x^2 (as for example V_v^+ above), or one dispenses with the squaring altogether and retains only the quadratic operators. The first approach leads right back to unital Jordan algebras since J can be imbedded into a unital Jordan algebra $k.1 + J$ by adjoining a unit element 1. The second approach leads to the concept of Jordan triple system, defined as a k-module T together with a quadratic map $U\colon T \to \mathrm{End}_k(T)$ satisfying the following identities in all scalar extensions.

JT1 $\{x,y,U_x z\} = U_x\{yxz\}$,

JT2 $\{U_x y,y,z\} = \{x,U_y x,z\}$,

JT3 $U(U_x y) = U_x U_y U_x$.

(The terminology "triple system" is due to the fact that in case 2 is invertible in k the theory can be based on the trilinear composition $\{xyz\} = U_{x+z}y - U_x y - U_z y$). If we compare these identities with those for a Jordan pair then it is obvious that T gives rise to a Jordan pair (T,T) by setting $V^+ = V^- = T$ and $Q_+ = Q_- = U$. Not every Jordan pair is of this form, however, since it is easy to construct examples of Jordan pairs for which V^+ and V^- are not isomorphic as k-modules. To obtain a Jordan triple system from a Jordan pair, we must have some way of identifying V^+ and V^- . More precisely, we define an involution of a Jordan pair to be a module isomorphism α: $V^+ \to V^-$ such that $Q_-(\alpha(x)) = \alpha Q_+(x)\alpha$ for all $x \in V^+$. Then a Jordan pair with involution gives rise to a Jordan triple system by setting $T = V^+$ and $U_x y = Q_+(x)\alpha(y)$, and this establishes a one-to-one correspondence between Jordan triple systems and Jordan pairs with involution.

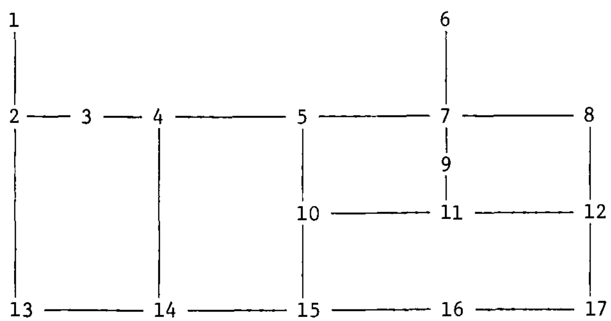
So far we have shown that Jordan pairs provide a unifying framework for both the theory of Jordan algebras and Jordan triple systems. Let us now point out some of the advantages which the the Jordan pair concept offers over both these theories.

In contrast to the case of Jordan algebras and triple systems, there is a natural way of defining inner automorphisms of Jordan pairs. Let $V = (V^+,V^-)$ be a Jordan pair, and consider a pair (x,y) where $x \in V^+$ and $y \in V^-$ (for which we simply write $(x,y) \in V$). We say that (x,y) is quasi-invertible if x is quasi-invertible in the Jordan algebra V^+_y ; i.e., if $1 - x$ is invertible in the Jordan algebra obtained from V^+_y by adjoining a unit element. In this case, (x,y) defines an inner automorphism $\beta(x,y)$ (cf. 3.9). Thus the quasi-invertible pairs are analogous to the invertible elements in an associative algebra. It is irrelevant for this whether V contains invertible elements or not; in fact, for most of the theory of Jordan pairs there is no difference between the two cases. These inner automorphisms play an important role and can be used to give a computation-free treatment of the Peirce decomposition (§5).

Another reason why Jordan pairs are preferable to Jordan algebras or triple systems is that they always contain sufficiently many idempotents. It may happen even in a finite-dimensional simple Jordan algebra that the unit element cannot be written as the sum of orthogonal division idempotents (the algebra need not have "capacity"). The situation is even worse for Jordan triple systems. Here an idempotent is an element x such that $U_x x = x$. In general, there are no such elements except zero; e.g., consider the real numbers with $U_x y = -x^2 y$. If V is a Jordan pair we define an idempotent to be a pair $(x,y) \in V$ such that $Q_+(x)y = x$ and $Q_-(y)x = y$. Then it turns out that a Jordan pair with dcc on principal inner ideals which is not radical always contains non-zero idempotents (§10). Of course, pairs (x,y) with $U_x y = x$ and $U_y x = y$ have been considered before in the theory of Jordan algebras but their natural place seems to be in the context of Jordan pairs. The scarcity of idempotents in the Jordan triple case is also explained. Indeed, under the correspondence between Jordan triple systems and Jordan pairs with involution, idempotents of the Jordan triple system correspond to idempotents (x,y) of the Jordan pair which are invariant under the involution in the sense that $y = \alpha(x)$, and there may be none of these.

Finally, let us mention that Jordan pairs arise naturally in the Koecher-Tits construction of Lie algebras and the associated algebraic groups, a topic not touched upon in these notes. Indeed, it was in this context that they were first introduced by K. Meyberg. For details, we refer to a forthcoming paper (Loos[7]).

We give now a more detailed description of the contents of these notes. There are 17 sections whose logical interdependence is summarized in the following diagram.



Here j depends on i if it stands below and/or to the right of i .

Chapter I (§§1 - 5) contains the general theory of Jordan pairs, beginning with their relationship to Jordan algebras and triple systems as discussed above. Just as in case of Jordan algebras, a long list of identities is required, and this is derived in §2. After the quasi-inverse (§3) various radicals are discussed in §4. The Jacobson radical of a Jordan pair V is directly based on the quasi-inverse, being defined by $\operatorname{Rad} V = (\operatorname{Rad} V^+, \operatorname{Rad} V^-)$ where $\operatorname{Rad} V^\sigma$ is the set of all properly quasi-invertible elements of V^σ (cf. 4.1). In §5, we introduce the Peirce decomposition

$$V = V_2(e) \oplus V_1(e) \oplus V_0(e)$$

of a Jordan pair with respect to an idempotent $e = (e^+, e^-)$ (5.4). Note that we use the indices 2,1,0 instead of the traditional 1,1/2,0 to label the Peirce spaces. Each Peirce space $V_i(e) = (V_i^+, V_i^-)$ is a subpair of V, and we have $\operatorname{Rad} V_i(e) = V_i(e) \cap \operatorname{Rad} V$ (5.8). There is also a Peirce decomposition with respect to an orthogonal system of idempotents (5.14).

Chapter II (§§6 - 9) is devoted to alternative pairs. An <u>alternative pair</u> is a pair $A = (A^+, A^-)$ of k-modules together with trilinear maps $A^+ \times A^- \times A^+ \to A^+$ and $A^- \times A^+ \times A^- \to A^-$, written $(x,y,z) \mapsto \langle xyz \rangle$, which satisfy the identities

AP1 $\langle uv\langle xyz\rangle\rangle + \langle xy\langle uvz\rangle\rangle = \langle\langle uvx\rangle yz\rangle + \langle x\langle vuy\rangle z\rangle$,

AP2 $\langle uv\langle xyx\rangle\rangle = \langle\langle uvx\rangle yx\rangle$,

AP3 $\langle xy\langle xyz\rangle\rangle = \langle\langle xyx\rangle yz\rangle$.

In analogy with the Jordan case, alternative pairs containing invertible elements correspond to isotopism classes of unital alternative algebras, and alternative pairs with involution correspond to alternative triple systems. In contrast to the situation for alternative algebras, there exist simple properly alternative pairs of arbitrary (even infinite) dimension over their centroids. They can be constructed from alternating bilinear forms (6.6). Just as an alternative algebra gives rise to a Jordan algebra by setting $U_x y = xyx$ so we obtain a Jordan pair A^J from an alternative pair A by setting $Q_\pm(x)y = \langle xyx\rangle$. This relation is exploited in §7 to prove results about alternative pairs by passing to the associated Jordan pair. In §9, we study the Peirce decomposition of alternative pairs which is the tool for their classification in §11.

The main reason why alternative pairs are of interest to us, however, is that they arise naturally in the study of Jordan pairs without invertible elements. To explain this connection, let e be an idempotent of a Jordan pair V with the property that $V_0(e) = 0$. Then $V_1(e)$ becomes an alternative pair by setting $\langle xyz\rangle = \{\{xye^\sigma\}e^{-\sigma}z\}$ (8.2). Conversely, every alternative pair can be obtained in this way by means of the standard imbedding (8.12). Consider now a simple and semisimple Jordan pair with acc and dcc on principal inner ideals. Then we can always find an idempotent e with $V_0(e) = 0$. If $V_1(e)$ is also zero then $V = V_2(e)$ contains invertible elements and is therefore essentially a unital Jordan algebra up to isotopy. In view of the work of N. Jacobson and K. McCrimmon, this case may be considered as well known. If, on the other hand, $V_1(e) \neq 0$ then V is isomorphic with the standard imbedding of $V_1(e)$ (12.5).

In Chapter III (§§10 - 12) we present the structure theory of alternative and Jordan pairs with chain conditions on principal inner ideals. Inner ideals are introduced in §10. The theory follows the one for Jordan algebras but is actually simpler since the minimal inner ideals of type II have no analogue for Jordan pairs (10.5). In §11, we classify simple alternative pairs A containing an idempotent e with $A_{00}(e) = 0$ (11.11), and also under various chain conditions (11.16, 11.18). Finally, §12 contains the classification of semisimple Jordan pairs with dcc and acc on principal inner ideals (12.12), based on the connection with alternative pairs as explained above.

In Chapter IV (§§13 - 17) we consider finite-dimensional Jordan pairs over a field. After introducing universal enveloping algebras in §13 (which properly belongs to §2) the main result of §14 is that the radical of a finite-dimensional Jordan pair is nilpotent (14.11). It is an outstanding problem to extend this result to Jordan pairs with chain conditions on inner ideals. Unfortunately, there seems to be little hope to generalize the present proof since it uses the finite-dimensionality of the universal envelope and Engel's theorem (14.9). In §15, we study Cartan subpairs of Jordan pairs. They are defined as associator nilpotent subpairs which are equal to their own normalizers. Using techniques similar to those in the theory of Cartan subgroups of algebraic groups, we show that any finite-dimensional Jordan pair contains Cartan subpairs (15.20), and that any two Cartan subpairs are conjugate by an inner automorphism, provided the base field is algebraically closed (15.17). The proofs depend on the fact that the orbit of a Cartan subpair under the inner automorphism group is dense in the Zariski topology (15.15). This also allows us to compute the generic minimum polynomial of a Jordan pair by its restriction to a Cartan subpair (16.15). The generic minimum polynomial is defined as the exact denominator of a suitable rational map (essentially the quasi-inverse, cf. 16.2). In contrast to the case of Jordan algebras, the degree of the generic minimum polynomial of a Jordan pair V does in general not coincide with the degree of V. This is the case, however, if V contains invertible elements or is separable. The generic trace of a separable Jordan pair may be degenerate in characteristic two, a phenomenon familiar from the theory of quadratic Jordan algebras. The generic norm, however, defined as the exact denominator of the quasi-inverse, is always non-degenerate in a certain sense (16.13). Finally, in §17, we work out the classification of simple finite-dimensional Jordan pairs over algebraically closed fields, using the results of §12. It turns out that such a Jordan pair is uniquely determined by three numerical invariants, dimension, rank, and genus, and also that the classification is independent of the characteristic of the base field (17.12).

In the Notes at the end of each chapter I have tried to make some historical comments, give credit where it is due, and also point out some open problems. I apologize in advance for any omissions or inaccuracies. In order to keep the text at a reasonable length, I have assumed as known the theory of quadratic Jordan algebras, to the extent of N. Jacobson's Tata Lecture Notes. In particular, the classification of semisimple unital Jordan algebras with dcc on principal inner ideals is not reproduced here from the Jordan pair point of view.

Most of the material was presented in a seminar at the University of British Columbia during the academic year 1973/74, and I wish to thank C.T. Anderson and M. Slater for their patience as my audience, and for many valuable remarks and suggestions. I am also indebted to E. Goodaire for proofreading the manuscript.

Vancouver, Summer 1974

O. Loos

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IV. FINITE-DIMENSIONAL JORDAN PAIRS

NOTATIONS AND CONVENTIONS

0.1. $\underline{\underline{Z}}$ is the ring of integers and $\underline{\underline{N}}$ denotes the set of non-negative integers. The unspecified term "ring" or "algebra" always means an associative (but not necessarily unital or commutative) ring or algebra.The commutator of two elements a and b of a ring is denoted by $[a,b] = ab - ba$. Fields are always commutative. Jordan algebras are always quadratic (but not necessarily unital) Jordan algebras in the sense of McCrimmon.

0.2. Throughout, k denotes a commutative unital ring of scalars. By an extension of k we mean a commutative unital k-algebra. If K is an extension of k then the natural homomorphism $k \to K$, $a \mapsto a.1$, need not be injective (e.g., $k = \underline{\underline{Z}}$ and $K = \underline{\underline{Z}}/p\underline{\underline{Z}}$). An extension field of k is an extension which is a field. The category of commutative unital k-algebras is denoted by k-$\underline{\underline{alg}}$. The symbol T usually stands for an indeterminate. Thus $k[T]$ is the polynomial algebra in one variable over k . The truncated polynomial rings $k[T]/(T^n)$ are denoted by $k(\varepsilon)$, $\varepsilon^n = 0$. For $n = 2$ this is the algebra of dual numbers over k .

0.3. All k-modules are unital. The symbol $\otimes$ stands for $\otimes_k$. If V is a k-module and R is an extension of k then the R-module

$$V_R = V \otimes R$$

is called the module obtained from V by extending the scalars to R , or simply a scalar extension of V. The image of an element x of V under the map $x \mapsto x \otimes 1$ from V into V_R is also denoted by x_R or even simply by x, although this map is in general not injective.

Let $k' \to k$ be a homomorphism of commutative rings with unity. Then the k-module V, considered as a module over k', is called the module obtained from V by restricting the scalars to k', and is denoted by ${}_{k'}V$. In particular, ${}_{\underline{\underline{Z}}}V$ is just the underlying abelian group of V.

If V and W are k-modules then $\mathrm{Hom}_k(V,W)$ or simply $\mathrm{Hom}(V,W)$ is the k-module of k-linear maps from V to W. Also, $\mathrm{End}(V) = \mathrm{Hom}(V,V)$ is the k-algebra of k-linear endomorphisms of V, and $GL(V)$ is the group of invertible elements of $\mathrm{End}(V)$.

0.4. A map $Q\colon V \to W$ between k-modules is called quadratic if $Q(\lambda x) = \lambda^2 Q(x)$ for all $\lambda \in k$, and if

$$Q(x,y) = Q(x+y) - Q(x) - Q(y)$$

is a bilinear map from $V \times V$ into W. For every extension R of k there is a unique quadratic map $Q_R\colon V_R \to W_R$ such that $Q_R(x_R) = Q(x)_R$ (see Jacobson[3] for a proof). Usually, we simply write Q instead of Q_R. A quadratic map q from V into k is called a quadratic form. We say q is non-degenerate if $q(x) = q(x,y) = 0$ for all y implies $x = 0$.

0.5. Let R be a non-associative (i.e., not necessarily associative) k-algebra We denote by R^{op} the opposite algebra, having the same underlying k-module and multiplication $a \cdot b = ba$. The identity maps $R \to R^{op}$ and $R^{op} \to R$ are antiisomorphisms, usually written $a \mapsto \bar{a}$.

The set of $p \times q$ matrices with entries in R is denoted by $M_{p,q}(R)$.

Instead of $M_{p,p}(R)$ we simply write $M_p(R)$. The transpose of a matrix x in $M_{p,q}(R)$ is denoted by ${}^t x$; it belongs to $M_{q,p}(R)$. We denote by $x^* = {}^t\bar{x}$ the transpose with coefficients in R^{op} so that $x^* \in M_{q,p}(R^{op})$. Then we have

$$(x^*)^* = x \quad \text{and} \quad (xy)^* = y^* x^* .$$

(Note that ${}^t(xy) \neq {}^t y \, {}^t x$ unless R is commutative, and then ${}^t x = x^*$).

0.6. A matrix $x = (x_{ij}) \in M_n(R)$ is called alternating if ${}^t x = -x$ and $x_{ii} = 0$. The set of alternating matrices in $M_n(R)$ is denoted by $A_n(R)$.

Assume that R is alternative with unity, and that it has an involution; i.e., an antiautomorphism of period 2 . If R_0 is an ample subspace of R (cf. Jacobson[3]) then we denote by $H_n(R,R_0)$ the set of hermitian matrices in $M_n(R)$ with diagonal entries in R_0 . If $n \leq 3$ or R is associative this is a unital Jordan algebra.

0.7. The notation a.b.c refers to formula (c) in section a.b .

CHAPTER I

JORDAN PAIRS

§1. Definitions and relations with Jordan algebras and triple systems

1.1. Let V^+ and V^- be k-modules, and let

$$Q_\sigma\colon V^\sigma \to \mathrm{Hom}(V^{-\sigma}, V^\sigma)$$

be quadratic maps (here and in the sequel, the index σ always takes the values $+$ and $-$). We define trilinear maps $V^\sigma \times V^{-\sigma} \times V^\sigma \to V^\sigma$, $(x,y,z) \mapsto \{xyz\}$, and bilinear maps $D_\sigma\colon V^\sigma \times V^{-\sigma} \to \mathrm{End}(V^\sigma)$, by the formulas

(1) $$\{xyz\} = D_\sigma(x,y)z = Q_\sigma(x,z)y$$

where $Q_\sigma(x,z) = Q_\sigma(x+z) - Q_\sigma(x) - Q_\sigma(z)$ (cf. 0.4). Obviously, we have $\{xyz\} = \{zyx\}$ and

(2) $$\{xyx\} = 2Q_\sigma(x)y \ .$$

1.2. DEFINITION. A Jordan pair over k is a pair $V = (V^+, V^-)$ of k-modules together with a pair (Q_+, Q_-) of quadratic maps $Q_\sigma\colon V^\sigma \to \mathrm{Hom}(V^{-\sigma}, V^\sigma)$ such that, with the notations introduced above, the following identities hold in all scalar extensions V_R of V :

JP1 $$D_\sigma(x,y)Q_\sigma(x) = Q_\sigma(x)D_{-\sigma}(y,x) \ ,$$

JP2 $$D_\sigma(Q_\sigma(x)y,y) = D_\sigma(x,Q_{-\sigma}(y)x) \ ,$$

JP3 $$Q_\sigma(Q_\sigma(x)y) = Q_\sigma(x)Q_{-\sigma}(y)Q_\sigma(x) \ .$$

The validity of JP1 - JP3 in all scalar extensions is equivalent with the condition that all linearizations of JP1 - JP3 hold in V . For an identity like JP2 which is of degree at most two in each variable this is automatic. For JP1 it suffices to know the validity for $R = k(\varepsilon) = k[T]/(T^2)$, the dual numbers, and for JP3 similarly $R = k[T]/(T^3)$ is sufficient. In any case, if JP1 - JP3 hold for all extensions R of k which are finitely generated and free as k-modules then they hold in arbitrary scalar extensions. - From the definition it is obvious that V_K is a Jordan pair over K , for every extension K of k . Also, if $k' \to k$ is a ring homomorphism then ${}_{k'}V$ (cf. 0.3) is a Jordan pair over k' .

1.3. A <u>homomorphism</u> $h\colon V \to W$ of Jordan pairs is a pair $h = (h_+,h_-)$ of k-linear maps $h_\sigma\colon V^\sigma \to W^\sigma$ such that

(1) $$h_\sigma(Q_\sigma(x)y) = Q_\sigma(h_\sigma(x))h_{-\sigma}(y) \ ,$$

for all $x \in V^\sigma$, $y \in V^{-\sigma}$, $\sigma = \pm$. By linearization, this implies

(2) $$h_\sigma(\{xyz\}) = \{h_\sigma(x),h_{-\sigma}(y),h_\sigma(z)\} \ .$$

Isomorphisms and automorphisms are defined in the obvious way. Jordan pairs over k form a category, denoted by $\underline{\underline{\mathrm{JP}}}_k$. A pair $U = (U^+,U^-)$ of submodules of a Jordan pair V is called a <u>subpair</u> (resp. an <u>ideal</u>) if $Q_\sigma(U^\sigma)U^{-\sigma} \subset U^\sigma$ (resp. $Q_\sigma(U^\sigma)V^{-\sigma} + Q_\sigma(V^\sigma)U^{-\sigma} + \{V^\sigma,V^{-\sigma},U^\sigma\} \subset U^\sigma$). If U is an ideal then $V/U = (V^+/U^+,V^-/U^-)$ is a Jordan pair in the obvious way. We say V is <u>simple</u> if it has only the trivial ideals V and 0 and if Q_+ and Q_- are not zero. An <u>outer ideal</u> is a subpair U satisfying $Q_\sigma(V^\sigma)U^{-\sigma} + \{V^\sigma, V^{-\sigma}, U^\sigma\} \subset U^\sigma$. If $1/2 \in k$ then an outer ideal is an ideal, in view of 1.1.2.

1.4. Let $\mathrm{Aut}(V)$ denote the group of automorphisms of a Jordan pair V . Clearly, this is a subgroup of $GL(V^+) \times GL(V^-)$. Note that we have a homomorphism from the group k^* of invertible elements of k into the center of $\mathrm{Aut}(V)$ given by $t \mapsto (t.\mathrm{Id}_{V^+}, t^{-1}.\mathrm{Id}_{V^-})$.

Let $k(\varepsilon)$ be the algebra of dual numbers. A pair $\Delta = (\Delta_+, \Delta_-) \in \mathrm{End}(V^+) \times \mathrm{End}(V^-)$ is called a <u>derivation</u> of V if $\mathrm{Id} + \varepsilon\Delta$ is an automorphism of $V_{k(\varepsilon)}$. A simple computation shows that this is the case if and only if

(1) $$\Delta_\sigma(Q_\sigma(x)y) = \{\Delta_\sigma(x), y, x\} + Q_\sigma(x)\Delta_{-\sigma}(y) ,$$

for all $x \in V^\sigma$, $y \in V^{-\sigma}$. The derivations of V form a Lie subalgebra of $\mathrm{End}(V^+) \times \mathrm{End}(V^-)$, denoted by $\mathrm{Der}(V)$.

1.5. The <u>opposite</u> V^{op} of a Jordan pair V is the Jordan pair (V^-, V^+) with quadratic maps (Q_-, Q_+) . By an <u>antihomomorphism</u> from V to W we mean a homomorphism $\eta: V \to V^{\mathrm{op}}$. Thus $\eta_\sigma: V^\sigma \to W^{-\sigma}$ satisfies

(1) $$\eta_\sigma Q_\sigma(x) = Q_{-\sigma}(\eta_\sigma(x))\eta_{-\sigma} .$$

An antiautomorphism η of V is called an <u>involution</u> if $\eta_{-\sigma} \circ \eta_\sigma$ is the identity on V^σ . The <u>direct product</u> of two Jordan pairs V and W is $V \times W = (V^+ \times W^+, V^- \times W^-)$ with componentwise operations. In $V \times V^{\mathrm{op}} = (V^+ \times V^-, V^- \times V^+)$ we have the <u>exchange involution</u> given by $(x,y) \mapsto (y,x)$.

1.6. The Jordan pair associated with a Jordan algebra. Let J be a quadratic Jordan algebra over k ; i.e., J is a k-module with quadratic maps $U\colon J \to \mathrm{End}(J)$ and ${}^2\colon J \to J$ (squaring) satisfying the following identities in all scalar extensions:

$$V_{x,x}y = x^2 \circ y \,, \tag{1}$$

$$U_x(x \circ y) = x \circ U_x y \,, \tag{2}$$

$$U_x x^2 = (x^2)^2 \,, \tag{3}$$

$$U_x U_y x^2 = (U_x y)^2 \,, \tag{4}$$

$$U_{x^2} = U_x^{\,2} \,, \tag{5}$$

$$U_{U_x y} = U_x U_y U_x \,. \tag{6}$$

Here $x \circ y = (x + y)^2 - x^2 - y^2$ and $V_{x,y}z = U_{x,z}y = U_{x+z}y - U_x y - U_z y$ (cf. McCrimmon [6]). Then (J,J) is a Jordan pair over k with quadratic maps $Q_\sigma(x) = U_x$. Indeed, JP3 is (6), and it is well known that the identities

$$V_{x,y}U_x = U_x V_{y,x} \,, \tag{7}$$

$$V_{U_x y,y} = V_{x,U_y x} \tag{8}$$

hold in any quadratic Jordan algebra which proves JP1 and JP2. We call (J,J) <u>the Jordan pair associated with the Jordan algebra</u> J .

A homomorphism $f\colon J \to J'$ of Jordan algebras induces a homomorphism $(f,f)\colon (J,J) \to (J',J')$ of Jordan pairs. Also note that $(J,J) = (V^+,V^-)$ has a natural involution η given by the identity maps $\mathrm{Id}\colon V^\sigma \to V^{-\sigma}$. If K is an ideal of J then obviously (K,K) is an ideal of (J,J) . Conversely, if J is unital and (K^+,K^-) is an ideal of (J,J) then $K^+ = K^-$ is an ideal of J . Indeed, $K^\sigma = U_1 K^\sigma \subset K^{-\sigma}$, and it follows easily that K^+ is an ideal of J . Therefore, if J is unital then J is simple if and only if (J,J) is simple.

1.7. Let J be a Jordan algebra, and let $v \in J$. Recall that the v-homotope of J is the Jordan algebra $J^{(v)}$ having the same underlying k-module as J, but squaring and quadratic operators given by

(1) $$x^{(2,v)} = U_x v ,$$

(2) $$U_x^{(v)} = U_x U_v .$$

If J is unital and v is invertible then $J^{(v)}$ is called an isotope of J. In this case, v^{-1} is the unit element of $J^{(v)}$. An <u>isotopy</u> $g: J \to J'$ between unital Jordan algebras is an isomorphism from J onto an isotope of J'. Since $g(1)$ must be the unit element of this isotope, this means that g is an isomorphism from J onto $J'^{(v)}$ where $v = g(1)^{-1}$. The set of autotopies of J is a group, the <u>structure group</u> $\mathrm{Str}(J)$ of J. Clearly $g \in GL(J)$ belongs to the structure group if and only if

(3) $$U_{g(x)} = gU_x g^{\#} ,$$

for all $x \in J$, where $g^{\#} = g^{-1}U_{g(1)}$.

1.8. PROPOSITION. <u>Let J and J' be unital Jordan algebras with associated Jordan pairs V and V'. Then the map $g \mapsto (g, U_{g(1)}^{-1}\cdot g)$ is a bijection between the set of isotopies from J to J' and the set of isomorphisms from V to V'. In particular, V is isomorphic with V' if and only if J is isotopic with J', and the structure group of J is isomorphic with the automorphism group of V.</u>

<u>Proof</u>. Let $h_+ = g: J \to J'$ be an isotopism, let $v = g(1)^{-1}$, and let $h_- = U_v \cdot g: J \to J'$. Then $h = (h_+, h_-) : V \to V'$ is an isomorphism: indeed, $h_+(Q_+(x)y) = h_+(U_x y) = U^{(v)}_{g(x)}\cdot g(y) = U_{g(x)}U_v\cdot g(y) = Q_+(h_+(x))h_-(y)$, and $h_-(Q_-(y)x) = h_-(U_y x)$

$= U_v g(U_y x) = U_v U_{g(y)} U_v g(x) = U(U_v g(y)).g(x) = Q_-(h_-(y))h_+(x)$. Conversely, if (h_+,h_-) is an isomorphism between V and V' then $h_+(U_x y) = h_+(U_x U_1 y)$ $= U_{h_+(x)} \cdot U_{h_-(1)} \cdot h_+(y)$, and hence h_+ is an isotopism from J to J' . It is easily verified that the two constructions are inverses of each other.

1.9. <u>The Jordan algebras</u> V_v^+ . Let V be a Jordan pair, and let $v \in V^-$. On the k-module V^+ we define a squaring and quadratic operators by

(1) $$x^2 = x^{(2,v)} = Q_+(x)v \ ,$$

(2) $$U_x = U_x^{(v)} = Q_+(x)Q_-(v) \ .$$

With these definitions, V^+ becomes a Jordan algebra which will be denoted by V_v^+ Indeed, by definition we have $x \circ y = \{xvy\}$, and hence

$V_{x,x}y = \{x,Q_-(v)x,y\} = \{Q_+(x)v,v,y\} = x^2 \circ y$, by JP2. Also, $U_x(x \circ y)$ $= Q_+(x)Q_-(v)\{xvy\} = Q_+(x)\{v,x,Q_-(v)y\} = \{x,v,Q_+(x)Q_-(v)y\} = x \circ U_x y$ by JP1, and $U_x . x^2 = Q_+(x)Q_-(v)Q_+(x)v = Q_+(Q_+(x)v)v = (x^2)^2$ by JP3. Similarly, (4) - (6) of 1.6 follow from JP3.

If $h: V \to W$ is a homomorphism of Jordan pairs then it is readily checked that

(3) $$h_+ : V_v^+ \to W_{h_-(v)}^+$$

is a homomorphism of Jordan algebras. Also, we have the formula

(4) $$(V_v^+)^{(u)} = V_{Q_-(v)u}^+$$

for all $u \in V^+$. Finally, by passing to the opposite Jordan pair we can analogously define Jordan algebras V_w^- for every $w \in V^+$.

1.10. <u>Invertible elements in Jordan pairs</u>. Let V be a Jordan pair. An element $u \in V^{\sigma}$ is called <u>invertible</u> if $Q_{\sigma}(u): V^{-\sigma} \to V^{\sigma}$ is invertible. In this case, we define the inverse $u^{-1} \in V^{-\sigma}$ by

(1) $$u^{-1} = Q_{\sigma}(u)^{-1}(u) .$$

From JP3 it follows easily that

(2) $$Q_{\sigma}(u)^{-1} = Q_{-\sigma}(u^{-1}) ,$$

and also

(3) $$(u^{-1})^{-1} = u .$$

In general, a Jordan pair will contain no invertible elements.

We say V is a <u>Jordan division pair</u> if $V \neq 0$ and if every non-zero element of V is invertible. More generally, V is called <u>local</u> if the non-invertible elements of V form a proper ideal, say N. Then V/N is a division pair.

If J is a unital Jordan algebra then invertibility in J and in the associated Jordan pair (J,J) are obviously equivalent. In particular, J is local (resp. a division algebra) if and only if (J,J) is local (resp. a division pair).

1.11. PROPOSITION. <u>Let</u> V <u>be a Jordan pair, let</u> $v \in V^-$ <u>be invertible, and let</u> $u = v^{-1} \in V^+$. <u>Then the Jordan algebras</u> $J = V^+_v$ <u>and</u> $J' = V^-_u$ <u>are unital with unit elements</u> u <u>and</u> v , <u>respectively, and</u> $Q_-(u): J \to J'$ <u>is an isomorphism with inverse</u> $Q_+(u): J' \to J$. <u>Moreover, the map</u>

$$(\mathrm{Id}_J, Q_-(v)) : (J,J) \to (V^+,V^-)$$

<u>is an isomorphism of Jordan pairs</u>.

Proof. In $\mathcal{J}$ we have $U_u = Q_+(u)Q_-(v) = \mathrm{Id}$ and $U_x(u) = Q_+(x)Q_-(v) = Q_+(x)v = x^2$ by 1.10, proving that u is the unit element of $\mathcal{J}$. Similarly, one shows that v is the unit element of $\mathcal{J}'$. Now $Q_-(v)(U_x y) = Q_-(v)Q_+(x)Q_-(v)y = Q_-(Q_-(v)x)y = Q_-(Q_-(v)x)Q_+(u)Q_-(v)y = U_{Q_-(v)x}\cdot(Q_-(v)y)$ shows that $Q_-(v) : \mathcal{J} \to \mathcal{J}'$ is an isomorphism whose inverse is $Q_+(u)$ by 1.10. Finally, $\mathrm{Id}(U_x y) = Q_+(x)Q_-(v)y$ and $Q_-(v)U_x y = Q_-(v)Q_+(x)Q_-(v)y = Q_-(Q_-(v)x)y$, and hence $(\mathrm{Id},Q_-(v))$ is an isomorphism.

1.12. COROLLARY. The map $\mathcal{J} \to (\mathcal{J},\mathcal{J})$ induces a bijection between isotopism classes of unital Jordan algebras and isomorphism classes of Jordan pairs containing invertible elements. The inverse map is induced by $\mathcal{V} \to \mathcal{V}^+_v$ where v is any invertible element of $\mathcal{V}^-$.

This follows immediately from 1.8 and 1.11.

1.13. Jordan triple systems and Jordan pairs with involution. Let $\mathcal{T}$ be a Jordan triple system over k , i.e., $\mathcal{T}$ is a k-module with a quadratic map $P : \mathcal{T} \to \mathrm{End}(\mathcal{T})$ such that the following identities hold in all scalar extensions:

(1) $$L(x,y)P(x) = P(x)L(y,x) ,$$

(2) $$L(P(x)y,y) = L(x,P(y)x) ,$$

(3) $$P(P(x)y) = P(x)P(y)P(x) ,$$

where $L(x,y)z = P(x,z)y = P(x + z)y - P(x)y - P(z)y$ (cf. Meyberg[6]). A k-linear map f between Jordan triple systems $\mathcal{T}$ and $\mathcal{T}'$ is called a homomorphism if $f(P(x)y) = P(f(x))f(y)$. Clearly, every Jordan algebra can be considered as a Jordan triple system simply by "forgetting" the squaring operation and setting

$P(x) = U_x$. From the definitions it also obvious that every Jordan triple system T gives rise to a Jordan pair $(V^+,V^-) = (T,T)$ with quadratic maps $Q_\sigma(x) = P(x)$, and that the identity map $\mathrm{Id}\colon V^+ \to V^-$ defines a canonical involution κ of (T,T). Conversely, let V be a Jordan pair with involution η , let $T = V^+$ as a k-module, and define $P\colon T \to \mathrm{End}(T)$ by

(4) $$P(x) = Q_+(x)\eta_+ .$$

Then one checks readily that T is a Jordan triple system, that the map (Id,η_+): $(T,T) \to (V^+,V^-)$ is an isomorphism of Jordan pairs, and that the canonical involution κ of (T,T) corresponds to the given involution η . If $f\colon T \to T'$ is a homomorphism of Jordan triple systems then $(f,f)\colon (T,T) \to (T',T')$ is a homomorphism of Jordan pairs, commuting with the involutions κ and κ' . Conversely, if $h\colon (V,\eta) \to (V',\eta')$ is a homomorphism of Jordan pairs with involution (i.e., a homomorphism commuting with the involutions) then $h_+\colon V^+ \to V'^+$ is a homomorphism of Jordan triple systems, where the Jordan triple structure is as in (4). Thus we see that the category of Jordan triple systems is equivalent with the category of Jordan pairs with involution.

1.14. Polarized Jordan triple systems. It is possible to imbed conversely the category of Jordan pairs into the category of Jordan triple systems as follows. Let V be a Jordan pair over k , and let η be the exchange involution in $V \times V^{op}$ (cf. 1.5). By 1.13, $T = V^+ \times V^-$ is a Jordan triple system with

(1) $$P(x)y = (Q_+(x_+)y_-, Q_-(x_-)y_+)$$

where $x = (x_+,x_-)$ and $y = (y_+,y_-)$ in T . Thus if we write the elements of T as column vectors then we may identify $P(x)$ with the 2×2-matrix

$$\begin{pmatrix} 0 & Q_+(x_+) \\ Q_-(x_-) & 0 \end{pmatrix} \quad \text{and} \quad L(x,y) \quad \text{with} \quad \begin{pmatrix} D_+(x_+,y_-) & 0 \\ 0 & D_-(x_-,y_-) \end{pmatrix}.$$

Moreover, $T = T^+ + T^-$ where $T^+ = V^+ \times 0$ and $T^- = 0 \times V^-$, and by (1) we have

(2) $$P(T^\sigma)T^\sigma = 0 \ , \quad P(T^\sigma)T^{-\sigma} \subset T^\sigma \quad .$$

This leads to the following definition: a polarized Jordan triple system is a Jordan triple system T with a direct sum decomposition into submodules $T = T^+ + T^-$ such that (2) holds. A homomorphism of polarized Jordan triple systems T and U is a homomorphism f respecting the "polarization", i.e., such that $f(T^\sigma) \subset U^\sigma$. Then (T^+, T^-) is a Jordan pair with $Q_\sigma(x) = P(x)| T^{-\sigma}$, and we have an equivalence between the category of Jordan pairs and the category of polarized Jordan triple systems.

1.15. <u>The centroid</u> $Z(V)$ of a Jordan pair V over k is defined as the set of all $a = (a_+, a_-) \in \mathrm{End}(V^+) \times \mathrm{End}(V^-)$ such that

(1) $$a_\sigma Q_\sigma(x) = Q_\sigma(x) a_{-\sigma} \ ,$$

(2) $$a_\sigma D_\sigma(x,y) = D_\sigma(x,y) a_\sigma \ ,$$

(3) $$Q_\sigma(a_\sigma(x)) = a_\sigma^2 Q_\sigma(x) \ ,$$

for all $x,y \in V^\pm$. Clearly, if $\lambda \in k$ then $(\lambda\mathrm{Id}, \lambda\mathrm{Id})$ belongs to $Z(V)$. We say V is <u>central</u> if every element of $Z(V)$ is of this form. If $V = (J,J)$ is the Jordan pair associated with a Jordan algebra J and if s is in the centroid of J (cf. McCrimmon[3]) then (s,s) is in the centroid of V, and the centroids of J and (J,J) may be identified in case J is unital.

We consider $\mathrm{End}(V^+) \times \mathrm{End}(V^-)$ as an algebra over k with componentwise operations. In general, $Z(V)$ will be neither commutative nor even a subalgebra. If it is, however, then we can consider V as a Jordan pair over $Z(V)$.

1.16. PROPOSITION. Let $a,b \in Z(V)$. Then $aba \in Z(V)$, and $[a_\sigma^2,b_\sigma]Q_\sigma(x) = 0$, for all $x \in V^\sigma$, and the image I of $[a,b]$ (i.e., $I = ([a_+,b_+]V^+,[a_-,b_-]V^-)$) is a trivial ideal of V. Moreover, $a + b \in Z(V)$ if and only if $[a_\sigma,b_\sigma]Q_\sigma(x) = 0$, and $ab \in Z(V)$ if and only if $a_\sigma[a_\sigma,b_\sigma]b_\sigma Q_\sigma(x) = 0$, for all $x \in V^\sigma$.

Proof. To show that $aba \in Z(V)$ it suffices to verify 1.15.3 since (1) and (2) of 1.15 are linear in a and hence are automatically satisfied. Now we have, dropping subscripts σ for easier notation:

$$(aba)^2Q(x) = aba^2baQ(x) = aba^2Q(x)ba = abQ(ax)ba = ab^2Q(ax)a = aQ(bax)a$$
$$= a^2Q(bax) = Q(abax).$$

Similarly, $a^2bQ(x) = a^2Q(x)b = Q(ax)b = qQ(ax) = ba^2Q(x)$, and hence

$$Q([a,b]x) = Q(abx) - Q(abx,bax) + Q(bax) = a^2Q(bx) - aQ(bx,ax) + b^2Q(ax)$$
$$= a^2b^2Q(x) - a^2Q(x,x)b^2 + b^2a^2Q(x) = (a^2b^2 - 2a^2b^2 + b^2a^2)(Q(x)$$
$$= [b^2,a^2]Q(x) = 0,$$

which proves $Q_\sigma(I^\sigma) = 0$. From (1) and (2) of 1.15 it is clear that I is an outer ideal, and hence it follows that I is a trivial ideal of V. Finally, $a + b$ (resp. ab) belong to the centroid if and only if they satisfy 1.15.3. Now

$$Q((a + b)x) = Q(ax) + Q(ax,bx) + Q(bx) = a^2Q(x) + abQ(x,x) + b^2Q(x)$$
$$= (a^2 + 2ab + b^2)Q(x) \quad \text{whereas} \quad (a + b)^2Q(x) = (a^2 + ab + ba + b^2)Q(x).$$

Similarly, $Q(abx) - (ab)^2Q(x) = (a^2b^2 - (ab)^2)Q(x) = a[a,b]bQ(x)$. This completes the proof.

In spite of the somewhat pathological character of the centroid (which is due to the fact that we are dealing with a non-linear structure) we have

1.17. PROPOSITION. (a) If V contains no trivial ideals then $Z(V)$ is a commutative k-algebra.

(b) If V is simple then $Z(V)$ is an extension field of k.

Proof. (a) This follows from 1.16.

(b) A simple Jordan pair contains no trivial ideals and hence $Z(V)$ is a commutative k-algebra by (a). From the defining properties of the centroid it is immediate that the image $aV = (a_+V^+, a_-V^-)$ of every $a \in Z(V)$ is an ideal of V. Therefore either $aV = 0$ or $aV = V$. Also, $\mathrm{Ker}(a) = (\mathrm{Ker}(a_+), \mathrm{Ker}(a_-))$ is an outer ideal of V, and if $aV = V$ then it is an ideal since $a_\sigma(Q_\sigma(x)V^{-\sigma}) = a_\sigma Q_\sigma(x)a_{-\sigma}V^{-\sigma} = a_\sigma^2 Q_\sigma(x)V^{-\sigma} = Q_\sigma(a_\sigma x)V^{-\sigma} = 0$ for $x \in \mathrm{Ker}(a_\sigma)$. By simplicity of V it follows that $Z(V)$ is a field.

§2. Identities and representations

2.0. Notational convention. To simplify notation, the index $\pm\sigma$ in expressions like $D_\sigma(x,y)$, $Q_\sigma(x)$, $Q_\sigma(x,z)$, etc. will be suppressed, and we simply write $D(x,y)$, $Q(x)$, $Q(x,z)$, or even $D_{x,y}$, Q_x, $Q_{x,z}$ instead. This causes no confusion as long as care is taken that the expressions make sense. Thus $D(x,y)Q(x)$ is admissible for $x \in V^\sigma$ and $y \in V^{-\sigma}$ but expressions like $D(x,x)$, $Q(x)x$, $Q(x)D(x,y)$ are not permitted. In any case, the reader will find it easy to supply the missing indices if necessary.

2.1. Basic identities. The defining identities of a Jordan pair are (using the convention introduced above)

JP1 $$D(x,y)Q(x) = Q(x)D(y,x) \quad \text{or} \quad \{x,y,Q(x)z\} = Q(x)\{yxz\} \ ,$$

JP2 $$D(Q(x)y,y) = D(x,Q(y)x) \quad \text{or} \quad \{Q(x)y,y,z\} = \{x,Q(y)x,z\} \ ,$$

JP3 $$Q(Q(x)y) = Q(x)Q(y)Q(x) \ .$$

Since the right hand side of JP1 is symmetric in y and z so must be the left hand side. This implies

(1) $$\{x,y,Q(x)z\} = \{x,z,Q(x)y\} = Q(x)\{yxz\} \ ,$$

or, in operator form,

JP4 $$D(x,y)Q(x) = Q(x,Q(x)y) = Q(x)D(y,x) \ .$$

Let $k(\varepsilon)$ be the ring of dual numbers. If we replace x by $x + \varepsilon u$ in (1), expand, and compare the terms with ε we obtain

(2) $$\begin{aligned}\{xy\{xzu\}\} + \{u,y,Q(x)z\} &= \{xz\{xyu\}\} + \{u,z,Q(x)y\} \\ &= \{x\{yxz\}u\} + Q(x)\{yuz\} \ .\end{aligned}$$

(This procedure is justified since by definition the identities remain valid in any scalar extension, and furthermore $k(\varepsilon) = k.1 \oplus k.\varepsilon$ is free over k so that $V_{k(\varepsilon)} = V \oplus \varepsilon V$). After a change of notation, this can be written in operator form as follows:

JP5 $$\begin{aligned}Q(x,z)D(y,x) + Q(x)D(y,z) &= Q(x,\{xyz\}) + Q(z,Q(x)y) \\ &= D(x,y)Q(x,z) + D(z,y)Q(x) \ ,\end{aligned}$$

JP6 $$\begin{aligned}D(x,\{yxz\}) + Q(x)Q(y,z) &= D(x,z)D(x,y) + D(Q(x)y,z) \\ &= D(x,y)D(x,z) + D(Q(x)z,y) \ .\end{aligned}$$

Similarly we linearize JP2 and obtain

(3) $$\{\{xyu\}yz\} = \{u,Q(y)x,z\} + \{x,Q(y)u,z\} ,$$

(4) $$\{x\{yxu\}z\} = \{Q(x)y,u,z\} + \{Q(x)u,y,z\} ,$$

which in operator form becomes

JP7 $$D(\{xyz\},y) = D(z,Q(y)x) + D(x,Q(y)z) ,$$

JP8 $$D(x,\{yxz\}) = D(Q(x)y,z) + D(Q(x)z,y) ,$$

JP9 $$D(x,y)D(z,y) = Q(x,z)Q(y) + D(x,Q(y)z) ,$$

JP10 $$Q(x,z)D(y,x) = Q(Q(x)y,z) + D(z,y)Q(x) .$$

Addition resp. subtraction of JP5 and JP10 gives

JP11 $$D(x,y)Q(x,z) = Q(Q(x)y,z) + Q(x)D(y,z) ,$$

JP12 $$D(x,y)Q(z) + Q(z)D(y,x) = Q(z,\{xyz\}) ,$$

and addition of JP6 and JP8 results in

JP13 $$D(x,y)D(x,z) = D(Q(x)y,z) + Q(x)Q(y,z) .$$

We linearize JP12 and apply it to an element $v \in V^{\pm}$:

$$\{x\{yzv\}u\} + \{zy\{xvu\}\} = \{xv\{uyz\}\} + \{uv\{xyz\}\} .$$

Changing x to z and conversely we have

JP14 $$\{xy\{uvz\}\} - \{uv\{xyz\}\} = \{\{xyu\}vz\} - \{u\{yxv\}z\} ,$$

or in operator form,

JP15 $$[D(x,y),D(u,v)] = D(\{xyu\},v) - D(u,\{yxv\}) .$$

Note that all the identities derived so far are a consequence of JP1 and JP2.

2.2. PROPOSITION. (a) *If V has no 2-torsion then JP3 follows from JP1 and JP2.* (b) *If V has no 6-torsion then JP1 - JP3 all follow from JP14.*

Proof. (a) Since the left hand side of JP14 changes sign if we interchange (x,y) and (u,v) so does the right hand side. This implies

JP16 $$\{\{xyu\}vz\} - \{u\{yxv\}z\} = \{x\{vuy\}z\} - \{\{uvx\}yz\} .$$

Now we have by JP14 and 1.1.2:

$$8Q(Q(x)y)z = \{\{xyx\}z\{xyx\}\} = 2\{\{\{xyx\}zx\}yx\} - \{x\{z\{xyx\}y\}x\} .$$

By JP16 it follows that $\{z\{xyx\}y\} = 2\{yx\{zxy\}\} - \{y\{xzx\}y\}$. Hence

$$\{\{xyx\}z\{xyx\}\} = 2\{\{\{xyx\}zx\}yx\} - 2\{x\{yx\{zxy\}\}x\} + \{x\{y\{xzx\}y\}x\} =$$

$= 8Q(x)Q(y)Q(x)z$, since

$$\{\{\{xyx\}zx\}yx\} = 2D(x,y)D(x,z)Q(x)y = 2Q(x)D(y,x)D(z,x)y = \{x\{yx\{zxy\}\}x\},$$

by JP1.

(b) Using JP14 we have

$$2D(x,y)Q(x)z = \{xy\{xzx\}\} = \{\{xyx\}zx\} - \{x\{yxz\}x\} + \{xz\{xyx\}\} = 2\{xz\{xyx\}\}$$
$$- \{x\{yxz\}x\} = 2\{\{xzx\}yx\} - 2\{x\{zxy\}x\} + 2\{xy\{xzx\}\} - \{x\{yxz\}x\} = 4\{xy\{xzx\}\}$$
$$- 3\{x\{zxy\}x\} = 8D(x,y)Q(x)z - 6Q(x)D(y,x)z .$$

This proves JP1, and JP2 follows from JP15 by setting $u = x$ and $v = y$. Finally, JP3 holds by (a).

2.3. *Representations*. Let A be an associative k-algebra with unit element 1 , and let $e = e_+$ be an idempotent of A . We set $e_- = 1 - e_+$ and $A^{\sigma\tau} = e_\sigma A e_\tau$ $(\sigma,\tau = \pm)$ so that $A = A^{++} \oplus A^{+-} \oplus A^{-+} \oplus A^{--}$ is the Peirce decomposition of A with respect to e_+ and e_- . Let V be a Jordan pair over k . A *representation* (d,q) *of* V *in* A consists of bilinear maps $d_\sigma: V^\sigma \times V^{-\sigma} \to A^{\sigma\sigma}$ and quadratic maps $q_\sigma: V^\sigma \to A^{\sigma,-\sigma}$ such that the following identities hold in all scalar extensions.

$$(1)\qquad d_\sigma(x,y)q_\sigma(x) = q_\sigma(x)d_{-\sigma}(y,x) = q_\sigma(x,Q(x)y)$$

$$(2)\qquad q_\sigma(x)d_{-\sigma}(y,z) + d_\sigma(z,y)q_\sigma(x) = q_\sigma(x,\{xyz\}),$$

$$(3)\qquad d_\sigma(x,y)d_\sigma(x,z) = d_\sigma(Q(x)y,z) + q_\sigma(x)q_{-\sigma}(y,z),$$

$$(4)\qquad d_\sigma(z,x)d_\sigma(y,x) = d_\sigma(z,Q(x)y) + q_\sigma(y,z)q_{-\sigma}(x),$$

$$(5)\qquad q_\sigma(Q(x)y) = q_\sigma(x)q_{-\sigma}(y)q_\sigma(x).$$

(here $q_\sigma(x,z) = q_\sigma(x+z) - q_\sigma(x) - q_\sigma(z)$). We will often apply the conventions of 2.0 to representations as well and simply write $q(x)$, $d(x,y)$, etc. if there is no danger of confusion.

Let $M = (M^+,M^-)$ be a pair of k-modules, and let $E = \mathrm{End}(M^+ \times M^-)$. We write the elements of $M^+ \times M^-$ as column vectors and the elements of E as 2×2-matrices, and set $e_+ = \begin{pmatrix} \mathrm{Id}_{M^+} & 0 \\ 0 & 0 \end{pmatrix}$ and $e_- = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{Id}_{M^-} \end{pmatrix}$.

If (d,q) is a representation of V in E then we also say that M is a V-<u>module</u>, or that (d,q) is a representation of V on M. With any representation (d,q) of V in A we can associate a V-module M in a natural way by setting $M^+ = A^{++} \oplus A^{+-}$, $M^- = A^{-+} \oplus A^{--}$, and combining the given representation (d,q) with the left regular representation of A.

2.4. <u>The regular representation of</u> V <u>on itself</u> is defined by

$$d_+(x,y) = \begin{pmatrix} D_+(x,y) & 0 \\ 0 & 0 \end{pmatrix}, \qquad d_-(y,x) = \begin{pmatrix} 0 & 0 \\ 0 & D_-(y,x) \end{pmatrix},$$

$$q_+(x) = \begin{pmatrix} 0 & Q_+(x) \\ 0 & 0 \end{pmatrix}, \qquad q_-(y) = \begin{pmatrix} 0 & 0 \\ Q_-(y) & 0 \end{pmatrix}.$$

By the identities derived in 2.1, this is indeed a representation of V , and hence V is a V-module. The subalgebra of $\mathrm{End}(V^+ \times V^-)$ generated by the $d_\sigma(x,y)$'s and $q_\sigma(x)$'s and $e_\pm$ is called the <u>multiplication algebra</u> of V and denoted by $M(V)$. Clearly, $M(V) = M^{++} \oplus M^{+-} \oplus M^{-+} \oplus M^{--}$ where the $M^{\sigma\tau}$ are the Peirce spaces with respect to e_+ and e_- .

2.5. <u>The duality principle</u>. Let (d,q) be a representation of V in A . If A has an involution $*$ such that $e_\sigma^* = e_{-\sigma}$, $d_\sigma(x,y)^* = d_{-\sigma}(y,x)$, and $q_\sigma(x)^* = q_\sigma(x)$ then we say that (d,q) is a $*$-representation. From a given representation (d,q) of V in A we can always construct a $*$-representation $(\bar{d},\bar{q})$ in $A \times A^{op}$ (with the exchange involution) by setting

$$\bar{e}_\sigma = (e_\sigma, e_{-\sigma}),\quad \bar{d}_\sigma(x,y) = (d_\sigma(x,y), d_{-\sigma}(y,x)),\quad \bar{q}_\sigma(x) = (q_\sigma(x), q_\sigma(x)).$$

The proof consists in a straightforward verification and is omitted. As a consequence, we get the following duality principle: If F is any identity in $d_\sigma(x,y)$ and $q_\sigma(z)$ which is valid for every representation of a Jordan pair V then its dual F^* , obtained by replacing $d_\sigma(x,y)$ by $d_{-\sigma}(y,x)$ and reversing the order of the factors, is also valid for every representation of V . Indeed, F holds in particular for the $*$-representation of V in $A \times A^{op}$. By applying the involution of $A \times A^{op}$ and projecting onto the first factor we see that F^* holds in A .

2.6. LEMMA. <u>For a representation</u> (d,q) <u>of a Jordan pair</u> V <u>the following identities hold</u> (with σ's omitted; cf. 2.0).

(1) $$d(Q(x)y,y) = d(x,Q(y)x),$$

(2) $$q(x,z)d(y,x) = d(z,y) + q(Q(x)y,z),$$

(3) $$d(x,y)q(x,z) = q(x)d(y,z) + q(Q(x)y,z),$$

(4) $$d(Q(x)y,z)q(x) = q(x)d(y,Q(x)z),$$

(5) $$q(x)q(y)d(x,z) + d(x,Q(y)Q(x)z) = d(Q(x)y,z)d(x,y),$$

(6) $$q(Q(x)y,\{xyz\}) = q(x)q(y)q(x,z) + q(x,z)q(y)q(x),$$

(7) $$q(\{xyz\}) + q(Q(x)y,Q(z)y) = q(x)q(y)q(z) + q(z)q(y)q(x) + q(x,z)q(y)q(x,z).$$

Proof. We set $y = z$ in 2.3.3 and 2.3.4 and subtract. This proves (1). Linearize 2.3.1:

$$q(x)d(y,z) + q(x,z)d(y,x) = q(x,\{xyz\}) + q(z,Q(x)y)$$

and subtract 2.3.2 which proves (2). Now (3) follows from (2) by the duality principle. By (2) and (1) of 2.3 and the linearization of (2) we have

$$q(x)d(y,Q(x)z) + d(Q(x)z,y)q(x) = q(x,\{x,y,Q(x)z\}) = q(x,Q(x)\{yxz\})$$
$$= d(x,\{yxz\})q(x) = d(Q(x)y,z)q(x) + d(Q(x)z,y)q(x)$$

which proves (4). For (5) we use (3) and 2.3.4 and have

$$q(x)q(y)d(x,z) + d(x,Q(y)Q(x)z) = q(x)(d(y,x)q(y,z) - q(Q(y)x,z))$$
$$+ d(x,y)d(Q(x)z,y) - q(x)d(z,x)q(y) = d(x,y)d(x,z)d(x,y) - q(x)q(Q(y)x,z)$$
$$- q(x)d(z,x)q(y) = d(Q(x)y,z)d(x,y) + q(x)(q(y,z)d(x,y) - q(Q(y)x,z) - d(z,x)q(y))$$
$$= d(Q(x)y,z)d(x,z) .$$

To prove the remaining formulas, let $R = k(\varepsilon)$ with $\varepsilon^3 = 0$, and replace x by $x + \varepsilon z$ in 2.3.5, computing in the scalar extension V_R of V. By equating the coefficients of ε and ε^2 in the resulting identity we obtain (6) and (7).

2.7. PROPOSITION. Let $M = (M^+,M^-)$ be a module for the Jordan pair V. Then $V \oplus M = (V^+ \oplus M^+, V^- \oplus M^-)$ becomes a Jordan pair by setting

$$Q_\sigma(x \oplus m)(y \oplus n) = Q_\sigma(x)y \oplus (q_\sigma(x)n + d_\sigma(x,y)m)$$

for $(x,y) \in V^\sigma \times V^{-\sigma}$, $(m,n) \in M^\sigma \times M^{-\sigma}$, called the split null extension of V by M.

This follows from 2.6.1 - 2.6.7 by a straightforward verification, using the fact that any product containing more than one element from M is zero. The details are left to the reader.

2.8. PROPOSITION. (Permanence principle) *If* F *is any identity in* $D_\sigma(x,y)$ *and* $Q_\sigma(z)$ *which is valid for the regular representation of all Jordan pairs over* k *then the identity obtained from* F *by replacing* D_σ, Q_σ *with* d_σ, q_σ *is valid for all representations of Jordan pairs over* k .

Indeed, let (d,q) be a representation of V in A, and let $M = (A^{++} \oplus A^{+-}, A^{-+} \oplus A^{--})$ be the associated V-module (cf. 2.3). By assumption, F is valid for the regular representation of the split null extension $V \oplus M$. If we restrict to M and apply F to the unit element of A the assertion follows.

Combining this with 2.5 we obtain the duality principle in the following form:

2.9. PROPOSITION. *If* F *is an identity in* $D_\sigma(x,y)$ *and* $Q_\sigma(z)$ *valid for all Jordan pairs over* k *then its dual* F* , *obtained by replacing* $D_\sigma(x,y)$ *by* $D_{-\sigma}(y,x)$ *and reversing the order of the factors, is also valid for all Jordan pairs over* k .

2.10. *More identities*. By specializing 2.6.4 - 2.6.7 to the regular representation we get

JP17 $$D(Q_x y,z)Q_x = Q_x D(y,Q_x z),$$

JP18 $$D(Q_x y,z)D(x,y) = Q_x Q_y D(x,z) + D(x,Q_y Q_x z),$$

JP19 $$Q(Q_x y,\{xyz\}) = Q_x Q_y Q(x,z) + Q(x,z)Q_y Q_x,$$

JP20 $$Q(\{xyz\}) + Q(Q_x y,Q_z y) = Q_x Q_y Q_z + Q_z Q_y Q_x + Q(x,z)Q_y Q(x,z).$$

We will also need

JP21 $$Q(\{xyz\}) + Q(Q_xQ_yz,z) = Q_xQ_yQ_z + Q_zQ_yQ_x + D(x,y)Q_zD(y,x),$$

JP22 $$Q(Q_xQ_yz,\{xyz\}) = Q_xQ_yQ_zD(y,x) + D(x,y)Q_zQ_yQ_x.$$

Proof of JP21: If we compare with JP20 we see that we have to show

$$D(x,y)Q_zD(y,x) = Q(x,z)Q_yQ(x,z) + Q(Q_xQ_yz,z) - Q(Q_xy,Q_zy).$$

Using JP10, JP13 and again JP10 we have

$$\begin{aligned}D(x,y)Q_zD(y,x) &= Q(z,x)D(y,z)D(y,x) - Q(Q_zy,x)D(y,x)\\ &= Q(z,x)Q_yQ(z,x) + Q(z,x)D(Q_yz,x) - Q(Q_zy,x)D(y,x)\\ &= Q(z,x)Q_yQ(z,x) + Q(Q_xQ_yz,z) + D(z,Q_yz)Q_x - Q(Q_xy,Q_zy) - D(Q_zy,y)Q_x ,\end{aligned}$$

and the third and fifth term cancel by JP2.

Proof of JP22: We linearize JP19 with respect to y and get

$$Q(Q_xu,\{xyz\}) + Q(Q_xy,\{xuz\}) = Q_xQ(y,u)Q(x,z) + Q(x,z)Q(y,u)Q_x .$$

Next replace u by Q_yz and use JP1 and JP2:

$$Q(Q_xQ_yz,\{xyz\}) = Q_xQ_yD(z,y)Q(x,z) + Q(x,z)D(y,z)Q_yQ_x - Q(Q_xy,\{x,Q_yz,z\})$$

(by using JP11 on the first term, JP10 on the second term and JP2 on the third term)

$$= Q_xQ_y(Q_zD(y,x) + Q(Q_zy,x)) + (D(x,y)Q_z + Q(Q_zy,x)) - Q(Q_xy,\{x,y,Q_zy\}),$$

and this proves JP22 if we observe JP19 with z replaced by Q_zy .

2.11. The transformations $B(x,y)$. For $(x,y) \in V^\sigma \times V^{-\sigma}$ we define $B_\sigma(x,y) \in \mathrm{End}(V^\sigma)$ by

$$B_\sigma(x,y) = \mathrm{Id}_{V^\sigma} - D_\sigma(x,y) + Q_\sigma(x)Q_{-\sigma}(y),$$

or, in simplified notation (cf 2.0),

$$B(x,y) = \mathrm{Id}_{V^\sigma} - D(x,y) + Q_xQ_y.$$

The $B(x,y)$ play a fundamental rôle in the theory of Jordan pairs. Obviously, we have $B(\lambda x,y) = B(x,\lambda y)$ for all $\lambda \in k$. Next we prove some identities for them.

JP23 $$B(x,y)Q(x) = Q(x)B(y,x) = Q(x - Q(x)y),$$

JP24 $$B(Q(x)y,y) = B(x,Q(y)x) = B(x,y)B(x,-y),$$

JP25 $$B(x,y)^2 = B(2x - Q(x)y,y) = B(x,2y - Q(y)x),$$

JP26 $$Q(B(x,y)z) = B(x,y)Q(z)B(y,x),$$

JP27 $$\begin{aligned}Q(B(x,y)z,x - Q(x)y) &= B(x,y)(Q(x,z) - D(z,y)Q(x)) \\ &= (Q(x,z) - Q(x)D(y,z))B(y,x).\end{aligned}$$

Identity JP23 is an immediate consequence of JP3 and JP4, and JP24 and JP25 follow easily from JP2, JP3, and JP13 (observe that it suffices to prove one equality in each of these cases; the other equality follows from the duality principle since the dual of $B(x,y)$ is $B(y,x)$). The proof of JP26 consists in expanding both sides, comparing terms of equal degree and using JP3, JP12, JP21 and JP22 to see that they are equal. For JP27 we have

$$\begin{aligned}&Q(B(x,y)z,x - Q_xy) = Q(x,z) - Q(\{xyz\},x) + Q(Q_xQ_yz,x) - Q(z,Q_xy) \\ &+ Q(\{xyz\},Q_xy) - Q(Q_xQ_yz,Q_xy) = \text{(by JP4, JP5, JP19 and JP3)} \\ &= Q(x,z) - D(x,y)Q(x,z) - D(z,y)Q_x + D(x,Q_yz)Q_x + Q_xQ_yQ(x,z) \\ &+ Q(x,z)Q_yQ_x - Q_xQ(Q_yz,y)Q_x = B(x,y)Q(x,z) + (D(x,Q_yz) - D(z,y) \\ &+ Q(x,z)Q_y - Q_xQ(Q_yz,y))Q_x = \text{(by JP9 and JP4) } B(x,y)Q(x,z) \\ &+ (D(x,y)D(z,y) - D(z,y) - Q_xQ_yD(z,y))Q_x = B(x,y)(Q(x,z) - D(z,y)Q_x) ,\end{aligned}$$

and the second equality follows from the duality principle.

2.12. Assume that V has invertible elements (cf. 1.10). Then for x (resp. y) invertible we have

(1) $$B(x,y) = Q(x)Q(x^{-1} - y) = Q(x - y^{-1})Q(y).$$

Indeed,

$$Q(x)Q(x^{-1} - y) = Q(x)(Q(x^{-1}) - Q(x^{-1},y) + Q(y))$$

$= \mathrm{Id} - Q(x)Q(x^{-1},y) + Q(x)Q(y).$

By JP13 we have

$$Q(x)Q(x^{-1},y) = D(x,x^{-1})D(x,y) - D(Q(x)x^{-1},y) = 2D(x,y) - D(x,y) = D(x,y)$$

since $D(x,x^{-1}) = D(x,x^{-1})Q(x)Q(x^{-1}) = Q(x,Q(x)x^{-1})Q(x^{-1}) = Q(x,x)Q(x^{-1})$ $= 2Q(x)Q(x^{-1}) = 2\mathrm{Id}$ by JP1. Similarly one proves the second formula.

If $V = (J,J)$ is the Jordan pair associated with a unital Jordan algebra J then formula (1) reads

(2) $$B(x,y) = U_x U(x^{-1} - y) = U(x - y^{-1})U_y.$$

§3. The quasi-inverse

3.1. DEFINITION. Let V be a Jordan pair. Instead of $x \in V^+$ and $y \in V^-$ we simply write $(x,y) \in V$. A pair $(x,y) \in V$ is called quasi-invertible if x is quasi-invertible in the Jordan algebra V^+_y (cf. 1.9); in other words, if $1 - x$ is invertible in the unital Jordan algebra $k.1 \oplus V^+_y$ obtained from V^+_y by adjoining a unit element. In this case, $(1 - x)^{-1} = 1 + z$ where $z \in V^+$, and we set $z = x^y$ and call it the quasi-inverse of (x,y).

Recall that an element a of a unital Jordan algebra J is invertible if and only if the following equivalent conditions hold (cf. Jacobson[3]).

(i) There exists $b \in J$ such that $U_a b = a$ and $U_a b^2 = 1$;

(ii) U_a is invertible;

(iii) U_a is surjective;

(iv) 1 belongs to the image of U_a.

For the quasi-inverse in a Jordan pair, the analogous statement is

3.2. PROPOSITION. *For* $(x,y) \in V$ *the following conditions are equivalent.*

(i) (x,y) *is quasi-invertible;*

(ii) *there exists* $z \in V^+$ *such that*

(1) $$B(x,y)z = x - Q(x)y \quad \text{and} \quad B(x,y)Q(z)y = Q(x)y;$$

(iii) $B(x,y)$ *is invertible;*

(iv) $B(x,y)$ *is surjective;*

(v) $2x - Q(x)y$ *belongs to the image of* $B(x,y)$.

If these conditions are satisfied then

(2) $$z = x^y = B(x,y)^{-1}(x - Q(x)y)$$

is the quasi-inverse of (x,y) .

Proof. In $k.1 \oplus V_y^+$ we have $U_{1-x}.1 = 1 - 2x + Q(x)y$ and $U_{1-x}.w = B(x,y)w$ for all $w \in V^+$. From this and 3.1, (i) - (iv) it follows easily that (i) - (v) are equivalent, and (2) is a consequence of (1).

3.3. PROPOSITION. (Symmetry principle) *Let* $(x,y) \in V$. *Then* (x,y) *is quasi-invertible if and only if* (y,x) *is quasi-invertible in* V^{op} , *and in this case,*

$$x^y = x + Q(x)y^x.$$

Proof. Let (y,x) be quasi-invertible in V^{op} , and set $z = x + Q(x)y^x$. We show that z satisfies 3.2.1. By 3.2.2, JP23 and JP26 we have

$$B(x,y)z = B(x,y)(x + Q(x)y^x) = B(x,y)x + Q_x B(y,x)y^x$$

$$= x - \{xyx\} + Q_x Q_y x + Q_x(y - Q_y x) = x - Q_x y,$$

and

$$B(x,y)Q_z y = Q(B(x,y)z)B(y,x)^{-1}y = Q(x - Q_x y)B(y,x)^{-1}y = Q_x B(y,x)B(y,x)^{-1}y = Q_x y.$$

3.4. LEMMA. (a) _Let_ $(z,y) \in V$. _Then_ $Q(y)\colon V^+_{Q(y)z} \to V^-_z$ _is a homomorphism of Jordan algebras._

(b) _Let_ $(u,v) \in V$ _and_ $y \in V^-$. _Then_ $B(u,v)\colon V^+_{B(v,u)y} \to V^+_y$ _is a homomorphism of Jordan algebras._

This follows by a straightforward computation, using JP3 and JP26.

3.5. PROPOSITION. (Shifting principle) (a) _Let_ $x,z \in V^+$ _and_ $y \in V^-$. _Then_ $(x,Q(y)z) \in V$ _is quasi-invertible if and only if_ $(Q(y)x,z) \in V^{op}$ _is quasi-invertible, and in this case_

(1) $$Q(y)(x^{Q(y)z}) = (Q(y)x)^z.$$

(b) _Let_ (x,y) _and_ (u,v) _be in_ V . _Then_ $(x,B(v,u)y)$ _is quasi-invertible if and only if_ $(B(u,v)x,y)$ _is quasi-invertible, and in this case_

(2) $$B(u,v)(x^{B(v,u)y}) = (B(u,v)x)^y.$$

Proof. (a) Let $(x,Q(y)z)$ be quasi-invertible. Since a homomorphism of Jordan algebras maps quasi-invertible elements into quasi-invertible elements it follows from 3.4 (a) that $(Q(y)x,z)$ is quasi-invertible and that (1) holds. If conversely $(Q(y)x,z)$ is quasi-invertible then by the symmetry principle and what we just proved it follows that $(x,Q(y)z)$ is quasi-invertible.

(b) This follows similarly from 3.4 (b).

3.6. Identities involving the quasi-inverse. Let (x,y) be quasi-invertible in V. From (ii) of 3.2 it follows that (x,y) is still quasi-invertible in any Jordan pair containing V as a subpair. In particular, if (d,q) is a representation of V in A and M is the associated V-module (cf. 2.3) then the quasi-invertibility of (x,y) in the split null extension $V \oplus M$ (cf. 2.7) implies that $b_+(x,y) = e_+ - d_+(x,y) + q_+(x)q_-(y)$ is invertible in A^{++} and $b_-(y,x) = e_- - d_-(y,x) + q_-(y)q_+(x)$ is invertible in A^{--}. For this reason, the permanence and duality principle (2.8, 2.9) may be applied to identities like the following whose proof requires the cancelling of a factor $B(x,y)$.

JP28 $$B(x,y)Q(x^y) = Q(x^y)B(y,x) = Q(x),$$

JP29 $$B(x,y)Q(x^y,z) + Q(x)D(y,z) = Q(x^y,z)B(y,x) + D(z,y)Q(x) = Q(x,z),$$

JP30 $$B(x,y)D(x^y,z) = D(x,z) - Q(x)Q(y,z),$$

JP31 $$D(z,x^y)B(y,x) = D(z,x) - Q(y,z)Q(x),$$

JP32 $$D(x^y,y - Q(y)x) = D(x - Q(x)y,y^x) = D(x,y),$$

JP33 $$B(x,y)B(x^y,z) = B(x,y + z),$$

JP34 $$B(z,x^y)B(y,x) = B(y + z,x),$$

JP35 $$B(x,y)^{-1} = B(x^y,-y) = B(-x,y^x).$$

Proof. By applying Q to 3.2.1 and observing JP23 and JP26 we get

$$B(x,y)Q(x^y)B(y,x) = B(x,y)Q(x) = Q(x)B(y,x),$$

and by the invertibility of $B(x,y)$ and $B(y,x)$ we have JP28. By JP26 and JP27 we have

$$B(x,y)Q(x^y,z)B(y,x) = Q(B(x,y)x^y,B(x,y)z) = Q(x - Q(x)y,\ B(x,y)z)$$
$$= B(x,y)(Q(x,z) - D(z,y)Q(x)) = (Q(x,z) - Q(x)D(y,z))B(y,x)$$

which implies JP29. Applying JP29 to an element u and reading the result as a transformation in z gives JP30, and JP31 follows by the duality principle. For JP32, set $z = y - Q(y)x$ in JP30 and use JP9 and JP4 which yields

$$B(x,y)D(x^y,y - Q(y)x) = -Q(x)Q(y,y) + Q(x)Q(y,Q(y)x) + D(x,y) - D(x,Q(y)x)$$
$$= D(x,y) - D(x,y)^2 + Q(x)Q(y)D(x,y) = B(x,y)D(x,y).$$

If we cancel $B(x,y)$ we get the first identity, and the other one follows by the duality principle. For JP33 we have by JP28 and JP30,

$$B(x,y)B(x^y,z) = (B(x,y)(Id - D(x^y,z) + Q(x^y)Q(z))$$
$$= B(x,y) + Q(x)Q(y,z) - D(x,z) + Q(x)Q(z)$$
$$= Id - D(x,y + z) + Q(x)Q(y + z) = B(x,y + z).$$

Now JP34 follows by duality and JP35 by setting $z = -y$ in JP33 and JP34.

3.7. THEOREM. Let $(x,y) \in V$ be quasi-invertible.

(a) For all $z \in V^-$ we have $(x,y + z)$ quasi-invertible if and only if (x^y,z) is quasi-invertible, and in this case,

(1) $$x^{(y + z)} = (x^y)^z.$$

(b) For all $z \in V^+$ we have $(x + z,y)$ quasi-invertible if and only if (z,y^x) is quasi-invertible, and in this case,

(2) $$(x + z)^y = x^y + B(x,y)^{-1}. z^{(y^x)}.$$

Proof. (a) The first statement follows immediately from JP33 and (iv) of 3.2. For (1) we have by 3.2.2, JP33 and JP28

$$x^{(y + z)} = B(x,y + z)^{-1}(x - Q(x)(y + z))$$
$$= B(x^y,z)^{-1}B(x,y)^{-1}(x - Q(x)(y + z))$$
$$= B(x^y,z)^{-1}(x^y - B(x,y)^{-1}Q(x)z)$$
$$= B(x^y,z)^{-1}(x^y - Q(x^y)z) = (x^y)^z .$$

(b) The first statement follows from (a) and the symmetry principle. To prove (2) we have by 3.2.2

$$(x + z)^y = B(x + z,y)^{-1}(x + z - Q(x + z)y)$$

and

$$x^y + B(x,y)^{-1}(z^{(y^x)}) = B(x,y)^{-1}(x - Q(x)y + B(z,y^x)^{-1}(z - Q(z)y^x))$$
$$= B(x,y)^{-1}B(z,y^x)^{-1}(B(z,y^x)(x - Q(x)y) + z - Q(z)y^x).$$

By JP34 it suffices therefore to show that

$$x + z - Q(x)y - \{xyz\} - Q(z)y = B(z,y^x)(x - Q(x)y) + z - Q(z)y^x.$$

The right hand side is

$$x + z - Q(x)y - D(z,y^x)(x - Q(x)y) - Q(z)(y^x - Q(y^x)(x - Q(x)y)).$$

By JP32 we have

$$D(z,y^x)(x - Q(x)y) = D(x - Q(x)y,y^x)z = D(x,y)z = \{xyz\},$$

and by JP28 and 3.3 we have

$$y^x - Q(y^x)(x - Q(x)y) = y^x - Q(y)B(x,y)^{-1}(x - Q(x)y)$$
$$= y^x - Q(y)x^y = y.$$

This finishes the proof.

3.8. <u>Powers and nilpotence</u>. Let $(x,y) \in V$. Then we denote by $x^{(n,y)}$ the n-th power of x in the Jordan algebra V_y^+, and by $y^{(n,x)}$ the n-th power of y in V_x^-. We have the formula

$$x^{(n+1,y)} = Q(x).y^{(n,x)}. \tag{1}$$

Indeed, for $n = 1$ this is the definition of $x^{(2,y)}$ (cf. 1.9), and for $n = 2$ we have $x^{(3,y)} = U_x x = Q(x)Q(y)x = Q(x).y^{(2,x)}$. If we assume (1) to be true for n then we get $x^{(n+3,y)} = U_x.x^{(n+1,y)} = Q(x)Q(y)Q(x).y^{(n.x)} = Q(x).y^{(n+2,x)}$. Here we use the fact that in a Jordan algebra the formula $U_x.x^n = x^{n+2}$ holds.

We say that a pair $(x,y) \in V$ is <u>nilpotent</u> if x is nilpotent in V_y^+. From (1) it is clear that (x,y) is nilpotent if and only if $(y,x) \in V^{op}$ is nilpotent. Also, if (x,y) is nilpotent then it is quasi-invertible, and the quasi-inverse is given by

$$x^y = \sum_{n=1}^{\infty} x^{(n,y)} . \tag{2}$$

For the proof, we use the fact that for a nilpotent element x of a Jordan algebra the operators U_x and V_x (cf. 1.6) are nilpotent and commute. In our case, $V_x = D(x,y)$ and $U_x = Q(x)Q(y)$ which implies that $B(x,y) = \mathrm{Id} - D(x,y) + Q(x)Q(y)$ is unipotent and hence in particular invertible. In order to prove (2) we apply $B(x,y)$ to the right hand side and have (setting $x^n = x^{(n,y)}$ and using $x \circ x^n = 2x^{n+1}$)

$$\begin{aligned}\sum_{n=1}^{\infty} (x^n - 2x^{n+1} + x^{n+2}) &= x + x^2 + \dots - 2(x^2 + x^3 + \dots) + x^3 + \dots \\ &= x - x^2 = x - Q(x)y\end{aligned}$$

which proves (2) in view of 3.2.2.

3.9. <u>Inner automorphisms</u>. Let $(x,y) \in V$ be quasi-invertible. Then $B(x,y)$ and $B(y,x)$ are invertible and it follows from JP26 that

$$\beta(x,y) = (B(x,y), B(y,x)^{-1}) \tag{1}$$

is an automorphism of V, called <u>the inner automorphism defined by</u> (x,y). From JP33 - JP35 we get the formulas

$$\beta(x,y)\beta(x^y,z) = \beta(x,y+z), \tag{2}$$

$$\beta(z,y^x)\beta(x,y) = \beta(x+z,y), \tag{3}$$

$$\beta(x,y)^{-1} = \beta(x^y,-y) = \beta(-x,y^x). \tag{4}$$

If $h = (h_+,h_-)$ is an automorphism of V then we have

(5) $$h\beta(x,y)h^{-1} = \beta(h_+(x),h_-(y)).$$

Thus we have proved

3.10. PROPOSITION. The group $\mathrm{Inn}(V)$ generated by all $\beta(x,y)$, $(x,y) \in V$ quasi-invertible, is a normal subgroup of $\mathrm{Aut}(V)$, called the inner automorphism group of V.

3.11. Inner derivations. Let $k(\varepsilon)$ be the algebra of dual numbers over k. Then $(x,\varepsilon y)$ is nilpotent and hence quasi-invertible in $V_{k(\varepsilon)}$ for all $(x,y) \in V$, and we have $B(x,\varepsilon y) = \mathrm{Id} - \varepsilon D(x,y)$ and $B(\varepsilon y,x)^{-1} = \mathrm{Id} + \varepsilon D(y,x)$. Thus $\beta(x,\varepsilon y) = \mathrm{Id} - \varepsilon\delta(x,y)$ where

(1) $$\delta(x,y) = (D(x,y),\ -D(y,x))$$

is a derivation of V (cf. 1.4), called the inner derivation defined by (x,y). (The fact that $\delta(x,y)$ is a derivation is also an immediate consequence of JP12). From 3.8.5 it follows

(2) $$h\delta(x,y)h^{-1} = \delta(h_+(x),h_-(y)),$$

for $h \in \mathrm{Aut}(V)$. Let $\Delta = (\Delta_+,\Delta_-) \in \mathrm{Der}(V)$. Then $\mathrm{Id} + \varepsilon\Delta \in \mathrm{Aut}(V_{k(\varepsilon)})$ and from (2) we get by a simple computation

(3) $$[\Delta,\delta(x,y)] = \delta(\Delta_+(x),y) + \delta(x,\Delta_-(y)).$$

(Note that for $\Delta = \delta(u,v)$ this is just JP15). We have proved

3.12. PROPOSITION. The k-module $\mathrm{Inder}(V)$ spanned by all $\delta(x,y)$, $(x,y) \in V$, is an ideal of $\mathrm{Der}(V)$, stable under all automorphisms of V.

We call $\mathrm{Inder}(V)$ the <u>inner derivation algebra</u> of V. It should be noted that $\mathrm{Inder}(V)$ is contained in, but in general not equal to, the set of all $\Delta \in \mathrm{Der}(V)$ such that $\mathrm{Id} + \varepsilon\Delta \in \mathrm{Inn}(V_{k(\varepsilon)})$, in contrast to the situation for the derivation algebra $\mathrm{Der}(V)$ (cf. 1.4).

3.13. <u>The case of a Jordan algebra</u>. Let J be a unital Jordan algebra and let $V = (J,J)$ be the Jordan pair associated with J. Assume that (x,y) is quasi-invertible in V and that x is invertible in J. Then we have $B(x,y) = U_x U_{x^{-1}-y}$ by 2.12 and hence $x^{-1} - y$ is invertible. We claim that

$$x^y = (x^{-1} - y)^{-1} . \tag{1}$$

Indeed, $(x^{-1} - y)^{-1} = U(x^{-1} - y)^{-1}(x^{-1} - y) = B(x,y)^{-1}U_x(x^{-1} - y)$
$= B(x,y)^{-1}(x - U_x y) = x^y$.

§ 4. <u>Radicals</u>

4.1. <u>The Jacobson radical</u>. Let V be a Jordan pair over k. An element $x \in V^+$ is called <u>properly quasi-invertible</u> if (x,y) is quasi-invertible for all $y \in V^-$. Similarly $y \in V^-$ is called properly quasi-invertible if (x,y) is quasi-invertible for all $x \in V^+$. The <u>Jacobson radical</u> (or simply the radical) of V is $\mathrm{Rad}\, V = (\mathrm{Rad}\, V^+, \mathrm{Rad}\, V^-)$ where $\mathrm{Rad}\, V^\sigma$ is the set of properly quasi-invertible elements of V^σ. From the definitions, it is obvious that $\mathrm{Rad}\, V$ is invariant under all automorphisms of V, and that $\mathrm{Rad}\, V^{op} = (\mathrm{Rad}\, V^-, \mathrm{Rad}\, V^+)$. Also, if $k' \to k$ is a ring homomorphism and ${}_{k'}V$ is the Jordan pair over k' obtained by restricting the scalars to k' (cf. 1.2) then $\mathrm{Rad}\, V = \mathrm{Rad}({}_{k'}V)$.

We say V is <u>semisimple</u> if $\mathrm{Rad}\, V = 0$, and V is <u>quasi-invertible</u> or <u>radical</u> if $V = \mathrm{Rad}\, V$.

4.2. THEOREM. <u>The radical of V is both the largest quasi-invertible ideal of V and the smallest among all ideals I of V such that V/I is semisimple; in particular, $\mathrm{Rad}\, V$ is an ideal of V. If I is any ideal of V then $\mathrm{Rad}\, I = I \cap \mathrm{Rad}\, V$.</u>

<u>Proof</u>. We first show that $\mathrm{Rad}\, V$ is an ideal of V. Let $R = \mathrm{Rad}\, V^+$, and let $x,z \in R^+$, $\lambda \in k$. By 3.7(b), $(x+z,y)$ is quasi-invertible for all y, and therefore $x+z \in R^+$. Since $B(\lambda x,y) = B(x,\lambda y)$ we also have $\lambda x \in R^+$. Thus R^+ is a submodule of V^+. Now let $x \in R^+$, $y \in V^-$ and $z \in V^+$. Then $(x,Q(y)z)$ is quasi-invertible, and by the shifting principle (3.5) we have $(Q(y)x,z)$ quasi-invertible. Since z was arbitrary this shows that $Q(y)x \in R^-$, and therefore $Q(V^-).R^+ \subset R^-$. Similarly, let $(u,v) \in V^+ \times V^-$, $y \in V^-$, and $x \in R^+$. Then $(x,B(v,u)y)$ is quasi-invertible, and by 3.5, $(B(u,v)x,y)$ is quasi-invertible which proves $B(u,v)x \in R^+$. Now $B(u,v)x = x - D(u,v)x + Q(u)Q(v)x$ and hence we get $D(u,v)x = \{uvx\} \in R^+$, i.e., $\{V^+,V^-,R^+\} \subset R^+$. Let $x \in R^+$ and $y \in V^-$. By 3.7(a), $x^y \in R^+$, and by 3.2.2, $Q(x)y = x - B(x,y)x^y \in R^+$. Thus $Q(R^+)V^- \subset R^+$. Since all this holds with $+$ and $-$ interchanged as well $\mathrm{Rad}\, V$ is an ideal.

Let $I = (I^+,I^-)$ be a quasi-invertible ideal, let $x \in I^+$ and $y \in V^-$. Then $B(x,y)B(x,-y) = B(x,Q(y)x)$ (JP24) is invertible since $Q(y)x \in I^-$. By (iv) of 3.2, (x,y) is quasi-invertible. Hence $I^+ \subset R^+$, and it follows that $\mathrm{Rad}\, V$ is the largest quasi-invertible ideal of V. Finally, it is clear that a surjective homomorphism $h\colon V \to W$ of Jordan pairs maps $\mathrm{Rad}\, V$ into $\mathrm{Rad}\, W$. It remains to be shown that $V/\mathrm{Rad}\ V$ is semisimple which is a consequence of the following

4.3. LEMMA. Let I be a quasi-invertible ideal of V and let $x \mapsto \bar{x}$ denote the canonical map $V \to V/I$. If $(\bar{x},\bar{y})$ is quasi-invertible in V/I then (x,y) is quasi-invertible in V.

Proof. I^+ is a quasi-invertible ideal in the Jordan algebra V^+_y, and $x \mapsto \bar{x}$ is a homomorphism $V^+_y \to (V/I)^+_{\bar{y}}$ of Jordan algebras. Now the lemma follows from the corresponding fact for Jordan algebras (Jacobson[3], III, Lemma 5).

To complete the proof of 4.2, let I be any ideal of V. Then $I \cap \mathrm{Rad}\ V$ is a quasi-invertible ideal of I and therefore contained in $\mathrm{Rad}\ I$. Conversely, let $x \in \mathrm{Rad}\ I^+$ and $y \in V^-$. Then $Q(y)x \in I^-$ and hence $(x,Q(y)x)$ is quasi-invertible. Since quasi-invertibility in I and in V are equivalent it follows from JP24 and (iv) of 3.2 that (x,y) is quasi-invertible. Thus $x \in \mathrm{Rad}\ V^+$, and a similar proof shows that $\mathrm{Rad}\ I^- \subset \mathrm{Rad}\ V^-$.

4.4. PROPOSITION. (a) If V is a local Jordan pair (cf. 1.10) then $\mathrm{Rad}\ V$ is the set of non-invertible elements of V.

(b) Conversely, if V is a Jordan pair containing invertible elements and such that $V/\mathrm{Rad}\ V$ is a division pair then V is local.

Proof. (a) Let N be the set of non-invertible elements of V. By definition, $N = (N^+,N^-)$ is a proper ideal. Since V/N is a division pair we have $\mathrm{Rad}\ V \subset N$. Now let $x \in N^+$ and let $v \in V^-$ be invertible. Then $x - v^{-1}$ is invertible since otherwise $x - (x - v^{-1}) = v^{-1} \in N^+$. By 2.12, $B(x,v) = Q(x - v^{-1})Q(v)$ is invertible and therefore (x,v) is quasi-invertible. Also, $x^v \in N^+$ since N is an ideal. If $y \in N^-$ then $y - v$ is invertible, and hence $(x^v,y - v)$ is quasi-invertible. By 3.7, $((x^v)^{-v},y) = (x,y)$ is quasi-invertible. Therefore N is a quasi-invertible ideal, and by 4.2 it is contained in Rad V.

(b) We show that Rad V is the set N of non-invertible elements of V. Obviously Rad $V^+ \subset N^+$. Conversely, let x not be in Rad V^+, and let $v \in V^-$ be invertible. Since x is invertible modulo the radical there exists $y \in V^-$ such that $v^{-1} - Q(x)y = z \in \text{Rad } V^+$. By 2.12, $B(z,v) = Q(z - v^{-1})Q(v)$, and this is invertible since $z \in \text{Rad } V^+$. Hence $Q(z - v^{-1}) = Q(Q(x)y) = Q(x)Q(y)Q(x)$ is invertible which proves x invertible. A similar proof applies to N^-.

4.5. The small radical. Let V be a Jordan pair. An element $z \in V^{\pm}$ is called trivial (or an absolute zero divisor) if $Q(z) = 0$. We say V is non-degenerate (or strongly semiprime) if it contains no non-zero trivial elements.

An ideal I of V is called a semiprime ideal if V/I is non-degenerate, in other words, if $Q(z)V^{-\sigma} \subset I^\sigma$ implies $z \in I^\sigma$, for all $z \in V^\sigma$. The intersection $S = (S^+,S^-)$ of all semiprime ideals of V is itself a semiprime ideal. For if $Q(z)V^{-\sigma} \subset S^\sigma$ then $Q(z)V^{-\sigma}$ is contained in every semiprime ideal and hence so is z which shows $z \in S^\sigma$. Thus S is the smallest semiprime ideal of V, called the small radical (or lower radical, strongly semiprime radical) and denoted by rad V. It is obviously invariant under Aut(V). Note that the Jacobson radical is a semiprime ideal and therefore contains rad V which justifies the terminology. Indeed, if x is trivial then (x,y) is nilpotent for all y and by 3.8 it is quasi-invertible which shows that x belongs to the radical.

A Jordan pair is called s-radical if it is its own small radical; in other words, if every non-zero homomorphic image contains non-zero trivial elements. Before we can prove that the small radical is the largest s-radical ideal we need the following

4.6. LEMMA. Let $\mathcal{B}$ and $\mathcal{C}$ be ideals of $\mathcal{V}$, and let $T^\sigma(\mathcal{B},\mathcal{C})$ be the set of all finite sums of elements $z \in \mathcal{V}^\sigma$ such that $Q(z)\mathcal{B}^{-\sigma} \subset \mathcal{C}^\sigma$. Then $T(\mathcal{B},\mathcal{C}) = (T^+(\mathcal{B},\mathcal{C}), T^-(\mathcal{B},\mathcal{C}))$ is an ideal of $\mathcal{V}$ containing $\mathcal{C}$. In particular, $T(\mathcal{V}) = T(\mathcal{V},0)$ is an ideal of $\mathcal{V}$, contained in rad $\mathcal{V}$.

Proof. Let $\mathcal{T}^\sigma = T^\sigma(\mathcal{B},\mathcal{C})$. Clearly $\mathcal{T}^\sigma$ is a submodule of $\mathcal{V}^\sigma$. From JP3 and JP26 it follows that $\mathcal{T}^\sigma$ is invariant under all $Q(y)$ and $B(x,y)$, and hence $Q(\mathcal{V}^\sigma)\mathcal{T}^{-\sigma} \subset \mathcal{T}^\sigma$ and $\{\mathcal{V}^\sigma,\mathcal{V}^{-\sigma},\mathcal{T}^\sigma\} \subset \mathcal{T}^\sigma$. Now let $z = z_1 + \dots + z_n \in \mathcal{T}^\sigma$ where $Q(z_i)\mathcal{B}^{-\sigma} \subset \mathcal{C}^\sigma$. Then $Q(z)y = \Sigma\, Q(z_i)y + \sum_{i<j} \{z_i y z_j\}$. By JP3, $Q(Q(z_i)y)\mathcal{B}^{-\sigma} \subset \mathcal{C}^\sigma$, and $\{z_i y z_j\} \in \mathcal{T}^\sigma$ by what we proved before. Hence $Q(z)y \in \mathcal{T}^\sigma$ and $(\mathcal{T}^+, \mathcal{T}^-)$ is an ideal. The last assertion follows from the following

4.7. PROPOSITION. Let $T_0(\mathcal{V}) = 0$, and define

(1) $$T_\alpha(\mathcal{V}) = \bigcup_{\beta<\alpha} T_\beta(\mathcal{V})$$

if α is a limit ordinal, and

(2) $$T_\alpha(\mathcal{V}) = T(\mathcal{V}, T_{\alpha-1}(\mathcal{V}))$$

otherwise. Then

(3) $$\text{rad } \mathcal{V} = \lim_\alpha T_\alpha(\mathcal{V}) .$$

Proof. By 4.6, the $T_\alpha(\mathcal{V})$ form an increasing sequence of ideals of $\mathcal{V}$. Assume that $T_\beta(\mathcal{V}) \subset \text{rad } \mathcal{V}$ for all $\beta < \alpha$. If α is a limit ordinal then by (1), $T_\alpha(\mathcal{V}) \subset \text{rad } \mathcal{V}$. If not then by (2),

$$T_\alpha(\mathcal{V}) = T(\mathcal{V}, T_{\alpha-1}(\mathcal{V})) \subset T(\mathcal{V}, \text{rad } \mathcal{V}) \subset \text{rad } \mathcal{V}$$

since rad $\mathcal{V}$ is a semiprime ideal. Let ω be the ordinal at which the sequence $T_\alpha(\mathcal{V})$ stabilizes. Then $T_\omega(\mathcal{V}) \subset \text{rad } \mathcal{V}$, and $T_{\omega+1}(\mathcal{V}) = T(\mathcal{V}, T_\omega(\mathcal{V})) = T_\omega(\mathcal{V})$ which means that $T_\omega(V)$ is a semiprime ideal. Since rad V is the smallest such ideal the assertion follows.

4.8. PROPOSITION. <u>If</u> I <u>is an ideal of</u> V <u>then</u> $I \cap T(I,0) \subset T_2(I)$.

<u>Proof</u>. Let $x \in I^+ \cap T^+(I,0)$. Then $x = \Sigma z_i$ where $Q(z_i)I^- = 0$. For $y \in I^-$ we then have $Q(x)y = \sum_{i<j} \{z_i y z_j\}$, and from JP21 it follows that $Q(\{z_i y z_j\})I^- = 0$, i.e., $\{z_i y z_j\} \in T_1^+(I)$. Hence $x \in T_2^+(I)$.

4.9. LEMMA. <u>If</u> I <u>is an ideal of</u> V <u>and</u> S <u>is a semiprime ideal of</u> I <u>then</u> S <u>is an ideal of</u> V .

<u>Proof</u>. Let $(x,y) \in S$. Then $z = Q(x)y \in I^+$ and $Q(z)I^- = Q_x Q_y Q_x I^- \subset Q_x Q_y I^+ \subset Q_x I^- \subset S^+$ implies $z \in S^+$ since S is a semiprime ideal of I . Thus we have $Q(S^+)V^- \subset S^+$.

Next we show that $Q(V^+)S^- \subset S^+$. Let $x \in V^+$, $y \in S^-$, and define $z = Q(x)y$, $z' = Q(z)u$, $z'' = Q(z')v$ where u and v are arbitrary elements of I^- . It suffices to show that $Q(z'')I^- \subset S^+$. Indeed, this implies $z'' \in S^+$ by semiprimeness of S in I , and since v and u are arbitrary in I^- it follows for the same reason that $z' \in S^+$ and $z \in S^+$. Now we have, using JP3 repeatedly,

$Q(z'')I^- = Q(z')Q(v)Q(z')I^- = Q_z Q_u Q_x Q_y (Q_x Q_v Q_x) Q_y Q_x Q_u Q_z I^- \subset Q_z (Q_u Q_x Q_y) Q(Q_x v) I^-$,

and this is contained in $Q_z (Q_u Q_x Q_y) I^+$. Since I is an ideal of V we have $z \in I^+$. Also, if $s \in S^+$ then by JP20, $Q_u Q_x Q_y s = Q(\{uxy\})s + \{Q_u x, s, Q_y x\} - Q_y Q_x Q_u s - \{u, Q_x\{usy\}, y\} \in S^-$, using the fact that I is an ideal of V and S is an ideal of I . It follows that $Q(z'')I^- \subset S^+$.

Finally we prove that $\{V^+, V^-, S^+\} \subset S^+$. Let $x \in V^+$, $x' \in V^-$ and

$y \in S^+$. In order to show that $\{xx'y\} \in S^+$ it suffices to show that $z = B(x,x')y \in S^+$. Let $z' = Q(z)u$, $z'' = Q(z')v$ where $u,v \in I^-$. Argueing as above we see that it suffices to prove $Q(z'')I^- \subset S^+$. By JP3 and JP26 we have $Q(z'')I^- = Q(z')Q(v)Q(z')I^- = Q_zQ_uB(x,x')Q_y(B(x',x)Q_vB(x,x'))Q_yB(x',x)Q_uQ_zI^-$ $\subset Q_zQ_uB(x,x')Q_yQ(B(x,x')v)S^+ \subset Q_zQ_uB(x,x')Q_yS^-$. By what we already proved and the definition of $B(x,x')$ it suffices to show that $D(x,x')Q_yS^- \subset S^+$. But for $s \in S^-$ we have by JP12, $D(x,x')Q_ys = -Q_y\{x'xs\} + \{ys\{xx'y\}\} \in S^+$. Since all this holds as well with $+$ and $-$ interchanged the Lemma follows.

4.10. COROLLARY. Let I be an ideal of V and let I' be an ideal of I such that I is the ideal of V generated by I'. Then I/I' is s-radical.

Proof. Let S be the inverse image of $\operatorname{rad}(I/I')$ under the canonical map $I \to I/I'$. Then S is a semiprime ideal of I containing I'. By 4.9, S is an ideal of V, and hence $S = I$.

4.11. THEOREM. The small radical $\operatorname{rad} V$ is the largest s-radical ideal of V.

Proof. Let $I = \operatorname{rad} V$ and let $S = \operatorname{rad} I$. By 4.9, S is an ideal of V, and $Q(x)V^{-\sigma} \subset S^\sigma$ implies $x \in S^\sigma$ since S is semiprime in I and I is semiprime in V. Hence S is semiprime in V and this implies $S = I$. It follows that $\operatorname{rad} V$ is s-radical. Now let N be any s-radical ideal of V. After dividing by $\operatorname{rad} V$ we may assume that V is non-degenerate and have to show that $N = 0$. If $N \neq 0$ there exists a non-zero trivial element of N, say $x \in N^+$ with $Q(x)N^- = 0$. Since N is an ideal we have for all $y \in V^-$ that $Q(Q(x)y)V^-$ $= Q_xQ_yQ_xV^- \subset Q_xN^- = 0$ which implies $Q_xy = 0$ and hence $x = 0$, a contradiction.

4.12. PROPOSITION. (a) Let $k' \to k$ be a ring homomorphism, and let ${}_{k'}V$ be the Jordan pair over k' obtained by restriction of scalars. Then we have $\mathrm{rad}\ {}_{k'}V = \mathrm{rad}\ V$.

(b) Let K be an extension of k, and define $\phi\colon V \to V_K$ by $\phi(x) = x \otimes 1$. Then $\phi(\mathrm{rad}\ V) \subset \mathrm{rad}\ V_K$.

Proof. (a) Since every semiprime ideal of V is a semiprime ideal of ${}_{k'}V$ we have $\mathrm{rad}\ {}_{k'}V \subset \mathrm{rad}\ V$. Conversely, it suffices to show that $S = \mathrm{rad}\ {}_{k'}V$ is a k-submodule of V. If $x \in S^\sigma$ and $\lambda \in k$ then $Q(\lambda x)V^{-\sigma} = \lambda^2 Q(x)V^{-\sigma} \subset Q(x)V^{-\sigma} \subset S^\sigma$ and this implies $\lambda x \in S^\sigma$.

(b) By 4.9 we are reduced to proving that V_K is s-radical provided V is. Let $\pi\colon V_K \to W \neq 0$ be a surjection. Then $\pi(\phi(V)) = U$ contains a trivial element $z \neq 0$ of U, and since U spans W as a K-module z is a trivial element of W.

We remark that (b) is false for the Jacobson radical.

4.13. THEOREM. If I is an ideal of V then $\mathrm{rad}\ I = I \cap \mathrm{rad}\ V$.

Proof. By 4.9, $\mathrm{rad}\ I$ is a s-radical ideal of V, and therefore is contained in $\mathrm{rad}\ V$ by 4.11. For the converse it suffices to show that every ideal of a s-radical Jordan pair is itself s-radical. Thus let V be s-radical, and let I be an ideal of V. By 4.9, $\mathrm{rad}\ I$ is an ideal of V. After dividing by $\mathrm{rad}\ I$ we may assume that $\mathrm{rad}\ I = 0$ and have to show that $I = 0$. Assume to the contrary that $I \neq 0$. By Zorn's Lemma there exists an ideal M of V which is maximal with respect to the property that $I \cap M = 0$. After factoring M out we may assume that $I \cap K = 0$ implies $K = 0$, for all ideals K of V. Let $z \in V^+$ be a trivial element of V and let $x \in I^-$. Then $Q(x)z$ is a trivial element contained in I^- which implies $Q(x)z = 0$. From this we conclude that $Q(x)T^+ = 0$ where $T^+ = T^+(V)$ is as in 4.6. Now let $z \in I^+ \cap T^+$ and $x \in I^-$.

Then we have $Q(Q(z)x)I^- = Q(z)Q(x)Q(z)I^- \subset Q(z)Q(x)I^+ = 0$, using the fact that $T(V)$ is an ideal (4.6). Since I does not contain trivial elements we get $Q(z)x = 0$ and hence $z = 0$. Therefore $T^+ \cap I^+ = 0$ and in the same way one proves that $T^- \cap I^- = 0$. This is a contradiction since V is s-radical and therefore $T \neq 0$.

4.14. <u>The nil radical.</u> A Jordan pair V is called <u>nil</u> if all $(x,y) \in V$ are nilpotent (cf. 3.8). If I is an ideal of V then V is nil if and only if I and V/I are nil. Hence the union of all nil ideals of V is a nil ideal, called the <u>nil radical</u> and denoted by $\mathrm{Nil}\, V$. Clearly $\mathrm{Nil}\, V$ is the smallest among all ideals I of V such that $\mathrm{Nil}(V/I) = 0$.

Note that $\mathrm{Nil}\, V$ consists of <u>properly nilpotent elements</u> in the following sense: if $x \in \mathrm{Nil}\, V^+$ then (x,y) is nilpotent, for all $y \in V^-$, and if $y \in \mathrm{Nil}\, V^-$ then (x,y) is nilpotent for all $x \in V^+$. Indeed, $Q(y)x$ belongs to $\mathrm{Nil}\, V^-$ and hence $x^{(n,Q(y)x)} = 0$ for some n . By 3.4(a), $0 = Q(y).x^{(n,Q(y)x)} = (Q(y)x)^{(n,x)} = (y^{(2,x)})^{(n,x)} = y^{(2n,x)}$, and by 3.8.1, $Q(x)y^{(2n,x)} = x^{(2n+1,y)} = 0$. It is not known whether $\mathrm{Nil}\, V$ coincides with the set of all properly nilpotent elements.

4.15. PROPOSITION. $\mathrm{Nil}\, V$ <u>and</u> $\mathrm{Rad}\, V$ <u>are semiprime ideals of</u> V <u>and we have</u> $\mathrm{rad}\, V \subset \mathrm{Nil}\, V \subset \mathrm{Rad}\, V$.

<u>Proof</u>. We have noted before (cf. 4.5) that $\mathrm{Rad}\, V$ is a semiprime ideal. To show that $\mathrm{Nil}\, V$ is a semiprime ideal it suffices to show that $T(V)$ is a nil ideal since we then have $T(V/\mathrm{Nil}\, V) = 0$, proving that $\mathrm{Nil}\, V$ is semiprime. If $z \in V^+$ is trivial then $Q(z)y = Q(z)Q(y) = 0$ for all $y \in V^-$ and hence z is a trivial element of all Jordan algebras $k.1 \oplus V_y^+$. It follows that $T^+(V)$ is contained

in the ideal generated by the trivial elements of $k.1 \oplus V_y^+$. Since the latter is a nil ideal (cf. Jacobson[3],III, Th. 5) it follows that $T^+(V)$ consists of properly nilpotent elements. The same arguments apply to $T^-(V)$ and therefore $T(V)$ is a nil ideal of V . To complete the proof, we have $\mathrm{Nil}\, V \subset \mathrm{Rad}\, V$ since $\mathrm{Nil}\, V$ consists of properly nilpotent and therefore properly quasi-invertible elements.

4.16. PROPOSITION. <u>If</u> I <u>is an ideal of</u> V <u>then</u> $\mathrm{Nil}\, I = I \cap \mathrm{Nil}\, V$.

<u>Proof</u>. By 4.15 and 4.9, $\mathrm{Nil}\, I$ is a nil ideal of V and therefore contained in $I \cap \mathrm{Nil}\, V$. The converse is trivial.

4.17. <u>Relations with radicals of Jordan algebras</u>. Let J be a Jordan algebra and let $V = (J,J)$ be the associated Jordan pair. The Jacobson radical, $\mathrm{Rad}\, J$, of J is the set of properly quasi-invertible elements of J (cf. McCrimmon[4]) which shows that

(1) $$\mathrm{Rad}\, V = (\mathrm{Rad}\, J, \mathrm{Rad}\, J).$$

The small radical (=lower radical, cf. McCrimmon[1]) $\mathrm{rad}\, J$ is the smallest ideal of J such that $J/\mathrm{rad}\, J$ has no non-zero trivial elements. This implies that $(\mathrm{rad}\, J, \mathrm{rad}\, J)$ is a semiprime ideal of V and therefore contains $\mathrm{rad}\, V = (S^+,S^-)$. Conversely it is clear that (S^-,S^+) is a semiprime ideal of V which implies $S^- = S^+ = S$. We claim that S is an ideal of J . Since $U(S)J + U(J)S + \{JJS\} \subset S$ by the fact that (S,S) is an ideal of V it remains to be shown that $S^2 + S \circ J \subset S$. If $x \in S$ and $y \in J$ then $U(x^2)J = U(x)^2 J \subset S$ implies $x^2 \in S$, and from the identity $U(x \circ y) = U_x U_y + U_y U_x + U^2_{x,y} - U(x^2, y^2)$ it follows that $U(x \circ y)J \subset S$ which implies $x \circ y \in S$. Thus we have

$$\text{(2)} \qquad \operatorname{rad} V = (\operatorname{rad} J, \operatorname{rad} J).$$

Because of (1) and (2), all the properties of the Jacobson and the small radical which we proved for Jordan pairs carry over to Jordan algebras.

The situation is less satifying for the nil radical. Assume that J is unital and let $\operatorname{Nil} J$ be the nil radical of J, i.e., the largest nil ideal of J. Also let $\operatorname{Nil} V = (N^+, N^-)$ be the nil radical of V. It is easily seen that $N^+ = N^- = N$ is a nil ideal of J and is therefore contained in $\operatorname{Nil} J$. In general, we will not have equality. However, let us call an ideal K of J properly nil if every $x \in K$ is nilpotent in all homotopes $J^{(y)}$ of J. Then it is easy to see that J contains a unique largest properly nil ideal $\operatorname{PN} J$ and that $\operatorname{Nil} J = (\operatorname{PN} J, \operatorname{PN} J)$.

4.18. PROPOSITION. <u>Let</u> V <u>be a Jordan pair</u>. <u>Then</u>

$$\text{(1)} \qquad \operatorname{Rad} V^+ = \{x \in V^+ \mid V^-_x \text{ is radical }\},$$

$$\text{(2)} \qquad \operatorname{Rad} V^+_y = \{x \in V^+ \mid Q(y)x \in \operatorname{Rad} V^-\},$$

$$\text{(3)} \qquad \operatorname{Rad} V^+ = \bigcap_{y \in V^-} \operatorname{Rad} V^+_y .$$

<u>Proof</u>. (1) is obvious from the symmetry principle (3.3) and the definition of the Jacobson radical. For (2) we have $x \in \operatorname{Rad} V^+_y$ if and only if x is quasi-invertible in all homotopes $(V^+_y)^{(z)}$. By 1.9.4 this is equivalent with $(x, Q_y z)$ quasi-invertible, and, by 3.5, with $Q_y x \in \operatorname{Rad} V^-$. Finally, since $\operatorname{Rad} V$ is an ideal it is clear that $\operatorname{Rad} V^+$ is contained in all $\operatorname{Rad} V^+_y$. Conversely, let $x \in \operatorname{Rad} V^+_y$ for all $y \in V^-$. Then by (2), $B(x, Q(y)x)$ is invertible for all $y \in V^-$, and by JP24 it follows that $x \in \operatorname{Rad} V^+$.

4.19. PROPOSITION. <u>For</u> $v \in V^-$ <u>let</u> $J = V^+_v$, <u>and define</u>

$$K = \{x \in J \mid Q(v)x = Q(v)Q(x)v = 0\}.$$

<u>Then</u> K <u>is an ideal of</u> J <u>such that</u> $U_K K = 0$; <u>in particular,</u> $K \subset \operatorname{rad} J$.

<u>Proof</u>. We have to show that $K^2 + K \circ J + U(K)J + U(J)K + V(J,J)K \subset K$, where $x^2 = Q(x)v$, $x \circ y = \{xvy\}$, $U(x) = Q(x)Q(v)$ and $V(x,y) = D(x,Q(v)y)$ (cf. 1.9). First let $K' = \operatorname{Ker} Q(v)$. Then $K \subset K'$, and $U(J)K' = 0$ and $U(K')J + V(J,J)K' \subset K'$ by JP3. Also $Q(v)(J \circ K') = Q(v)\{J,v,K'\} \subset \{v,J,Q(v)K'\} = 0$ by JP1 and hence $J \circ K' \subset K'$.

Clearly K is closed under scalar multiplication. Let $x,y \in K$. Then $Q(v)(x + y) = 0$ and $Q(v)Q(x + y)v = Q(v)(Q(x)v + Q(x,y)v + Q(y)v) = Q(v)\{xvy\} = 0$ since $\{xvy\} = x \circ y \in K'$. This shows that K is a submodule of J . Also, $Q(v)x^2 = Q(v)Q(x)v = 0$ and $Q(v)Q(x^2)v = Q_vQ_xQ_vQ_xv = 0$ which implies $K^2 \subset K$. Now let $x \in K$ and $y \in J$. Then by JP20 we have $Q_vQ(x \circ y)v = Q_vQ_xQ_vQ_yv + Q_vQ_yQ_vQ_xv + Q_vQ(x,y)Q_v(x \circ y) - Q_v(x^2 \circ y^2)$. The first term vanishes since $Q_vQ_xQ_v = Q(Q_vx) = 0$, and the second term by definition of K . Also $Q(v)(x \circ y) = Q(v)(x^2 \circ y^2) = 0$ since $K^2 \subset K$ and $K' \circ J \subset K'$. Hence $K \circ J \subset K$. From the definition of K and JP3 it follows easily that $U(K)J + U(J)K \subset K$. Finally, let $x \in K$ and $y,z \in J$. Then by JP20, $Q(v)Q(V_{x,y}z)v = Q(v)Q(\{x,Q(v)y,z\})v = Q_vQ_xQ(Q_vy)Q_zv + Q_vQ_zQ(Q_vy)Q_xv + Q_vQ(x,z)Q(Q_vy)(x \circ z) - Q_v(Q_xy \circ Q_zy) = 0$, using JP3 and what we already proved. Thus we have shown that K is an ideal of J . Obviously, $U(K)K = Q(K)Q(v)K = 0$. Therefore K is a s-radical ideal of J and hence contained in $\operatorname{rad} J$.

4.20. <u>Remark</u>. If 2 is invertible in k or if V is non-degenerate then we have $K = K' = \operatorname{Ker} Q(v)$. Indeed, $x \in K$ if and only if $v^{(2,x)} = v^{(3,x)} = 0$. If 2 is invertible in k then $v^{(3,x)} = (1/2)v \circ v^{(2,x)}$ in V_x^- . If V is non-degenerate then $Q(Q_vQ_xv) = Q_vQ_xQ_vQ_xQ_v = Q(Q_vx)Q_xQ_v$ and hence $x \in K'$ implies $x \in K$.

4.21. The extreme radical $\mathrm{Extr}(V) = (E^+, E^-)$ of a Jordan pair V is defined as

$$E^\sigma = \{x \in V^\sigma \mid Q(x) = D(x, V^{-\sigma}) = D(V^{-\sigma}, x) = 0\}.$$

It is an easy exercise to verify that $\mathrm{Extr}(V)$ is an ideal of V, obviously contained in $\mathrm{rad}\, V$. If a and b are in the centroid of V then we can sharpen 1.16 to say that the image of $[a,b]$ is contained in the extreme radical. The proof is straightforward and is therefore omitted. By 1.16, the centroid is then a commutative k-algebra, provided the extreme radical is zero (cf. the analogous situation for Jordan algebras in McCrimmon[3]).

§5. Peirce decomposition

5.1. Regular elements and idempotents. Let V be a Jordan pair over k. An element $x \in V^+$ is called (von Neumann) regular if there exists $y \in V^-$ such that $x = Q(x)y$. Since $Q(x)y = x^{(2,y)}$ in the Jordan algebra V_y^+ this means that x is an idempotent of V_y^+. Regular elements of V^- are defined analogously. Note that a non-zero regular element does not belong to $\mathrm{Rad}\, V$. Indeed, if $x = Q(x)y$ then JP23 implies $B(x,y)Q(x) = 0$, and if (x,y) were quasi-invertible then $Q(x) = 0$ and therefore $x = 0$, a contradiction.

A pair $(x,y) \in V$ is called an idempotent if $x = Q(x)y$ and $y = Q(y)x$; in other words, if x is an idempotent in V_y^+ and y is an idempotent in V_x^-. Clearly, if (x,y) is an idempotent of V then (y,x) is an idempotent of V^{op}. It is an elementary but important fact that every regular element can be completed to an idempotent; more precisely:

5.2. LEMMA. *If x is regular, $x = Q_x y$, then $(x, Q_y x)$ is an idempotent.*

Indeed, by JP3 we have $Q(Q_y x)x = Q_y Q_x Q_y x = Q_y Q_x Q_y Q_x y = Q_y Q(Q_x y)y = Q_y Q_x y = Q_y x$ and $Q_x(Q_y x) = Q_x Q_y(Q_x y) = Q(Q_x y)y = Q_x y$.

5.3. LEMMA. *Let $e = (e^+, e^-)$ be an idempotent of V. For any extension R of k let R^* be the group of invertible elements of R, and let $t \in R^*$. Then the pair $(e_R^+, (1-t)e_R^-) \in V_R$ is quasi-invertible with quasi-inverse $t^{-1}e_R^+$, and the formula*

$$\phi(t) = \beta(e_R^+, (1-t)e_R^-)$$

defines a homomorphism $\phi = \phi_e : R^* \to \mathrm{Inn}(V_R)$ from R^* into the inner automorphism group of V_R.
(Here $e_R^+ = e^+ \otimes 1$ in $V_R^+ = V^+ \otimes R$, cf. 0.3)

Proof. Clearly, $e_R = (e_R^+, e_R^-)$ is an idempotent of V_R. To simplify notation, let us write $e^{\pm}$ instead of $e_R^{\pm}$. Then we have

$$B(e^+, (1-t)e^-).t^{-1}e^+ = t^{-1}e^+ - t^{-1}(1-t)\{e^+e^-e^+\} + t^{-1}(1-t)^2 Q(e^+)Q(e^-)e^+$$

$$= t^{-1}(1 - 2(1-t) + (1-t)^2)e^+ = te^+ = e^+ - Q(e^+)(1-t)e^- ,$$

and

$$B(e^+, (1-t)e^-).Q(t^{-1}e^+)(1-t)e^- = (1-t)e^+ = Q(e^+)(1-t)e^- .$$

This proves the first assertion in view of 3.2. By 3.9.2 we have

$$\phi(s)\phi(t) = \beta(e^+, (1-s)e^-)\beta(e^+, (1-t)e^-) = \beta(e^+, (1-s)e^-)\beta(s^{-1}e^+, s(1-t)e^-)$$

$$= \beta(e^+, (1 - s + s(1-t))e^-) = \beta(e^+, (1-st)e^-) = \phi(st) ,$$

and the Lemma is proved.

5.4. THEOREM. (Peirce decomposition) *Let* e *be an idempotent of* V , *and define* $E_i^\sigma \in \mathrm{End}(V^\sigma)$ *by*

$$E_2^\sigma = Q(e^\sigma)Q(e^{-\sigma}), \quad E_1^\sigma = D(e^\sigma, e^{-\sigma}) - 2E_2^\sigma, \quad E_0^\sigma = B(e^\sigma, e^{-\sigma})$$

so that

(1) $$B(e^\sigma, (1-t)e^{-\sigma}) = E_0^\sigma + tE_1^\sigma + t^2E_2^\sigma$$

for all $t \in R$, R *an extension of* k .

(a) *The* E_i^σ *are orthogonal projections whose sum is the identity, and hence*

(2) $$V^\sigma = V_2^\sigma \oplus V_1^\sigma \oplus V_0^\sigma$$

where $V_i^\sigma = E_i^\sigma(V^\sigma)$.

(b) *We have*

(3) $$V_2^\sigma = \mathrm{Im}(Q(e^\sigma)), \quad V_1^\sigma \oplus V_0^\sigma = \mathrm{Ker}(Q(e^{-\sigma})),$$

(4) $$V_1^\sigma = \mathrm{Ker}\ (\mathrm{Id} - D(e^\sigma, e^{-\sigma})),$$

(5) $$V_0^\sigma = \mathrm{Ker}(Q(e^{-\sigma})) \cap \mathrm{Ker}(D(e^\sigma, e^{-\sigma})),$$

(6) $$V_i^\sigma \subset \{x \in V^\sigma \mid \{e^\sigma, e^{-\sigma}, x\} = ix\} .$$

(c) *Set* $V_i^\sigma = 0$ *for* $i \neq 0,1,2$. *Then the following composition rules hold.*

(7) $$Q(V_i^\sigma)V_j^{-\sigma} \subset V_{2i-j}^\sigma ,$$

(8) $$\{V_i^\sigma, V_j^{-\sigma}, V_\ell^\sigma\} \subset V_{i-j+\ell}^\sigma ,$$

(9) $$\{V_2^\sigma, V_0^{-\sigma}, V^\sigma\} = \{V_0^\sigma, V_2^{-\sigma}, V^\sigma\} = 0.$$

Proof. (a) Let R be an extension of k and let

$$\phi_\sigma(t) = B(e^\sigma, (1-t)e^{-\sigma}) ,$$

for all $t \in R^*$. Then $\phi(t) = (\phi_+(t), \phi_-(t)^{-1})$, and by 5.3, $\phi_\sigma \colon R^* \to GL(V_R^\sigma)$ is a homomorphism. In particular, $\mathrm{Id} = \phi_\sigma(1) = E_0^\sigma + E_1^\sigma + E_2^\sigma$. Now let S and T

be indeterminates, and let $R = k[S,T,S^{-1},T^{-1}]$. Thus R is a free k-module with basis S^iT^j , $(i,j) \in \underline{\underline{Z}} \times \underline{\underline{Z}}$, which implies $V_R^\sigma = \bigoplus_{i,j} S^iT^jV^\sigma$ and $\mathrm{End}(V_R^\sigma) = \bigoplus_{i,j} S^iT^j\mathrm{End}(V^\sigma)$. Hence we can compare coefficients at S^iT^j . We have

$$\phi_\sigma(ST) = \sum_{i=0}^{2}(ST)^iE_i^\sigma = \phi_\sigma(S)\phi_\sigma(T) = \sum_{i,j=0}^{2} S^iT^jE_i^\sigma E_j^\sigma$$

and therefore $E_i^\sigma E_j^\sigma = \delta_{ij}E_i^\sigma$.

(b) We have $\mathrm{Im}(Q(e^\sigma)) \supset \mathrm{Im}(E_2^\sigma) = V_2^\sigma \supset \mathrm{Im}(E_2Q(e^\sigma)) = \mathrm{Im}(Q(e^\sigma))$ since $E_2^\sigma Q(e^\sigma) = Q(e^\sigma)Q(e^{-\sigma})Q(e^\sigma) = Q(Q(e^\sigma)e^{-\sigma}) = Q(e^\sigma)$. Similarly, $\mathrm{Ker}(Q(e^{-\sigma})) \subset \mathrm{Ker}(E_2^\sigma) = V_1^\sigma \oplus V_0^\sigma \subset \mathrm{Ker}(Q(e^{-\sigma})E_2^\sigma = \mathrm{Ker}(Q(e^{-\sigma}))$. This proves (3). Now (5) follows from (1) and (3). For (6) let $R = k(\varepsilon)$ be the algebra of dual numbers. If $x_i \in V_i^\sigma$ then we have

$$\phi_\sigma(1-\varepsilon).x_i = B(e^\sigma,\varepsilon e^{-\sigma}).x_i = x_i - \{e^\sigma e^{-\sigma}x_i\}$$
$$= (1-\varepsilon)^ix_i = (1 - i\varepsilon)x_i ,$$

which proves (6). Now let $x = x_2 + x_1 + x_0 \in V^\sigma$ where $x_i \in V_i^\sigma$. Then $\{e^\sigma e^{-\sigma}x\} = 2x_2 + x_1 = ix$ implies $(i-2)x_2 + (i-1)x_1 + ix_0 = 0$. For $i = 1$ this means $x_2 = x_0 = 0$, or $x = x_1$, and hence we have (4).

(c) Let $R = k[T,T^{-1}]$, and let $x \in V_i^\sigma$, $y \in V_j^{-\sigma}$, $z \in V_\ell^\sigma$. Since $\phi(T) = (\phi_+(T),\phi_-(T^{-1}))$ is an automorphism of V_R we have by (1)

$$\phi_\sigma(T)Q(x)y = Q(\phi_\sigma(T)x).\phi_{-\sigma}(T^{-1})y = Q(T^ix)T^{-j}y$$
$$= T^{2i-j}Q(x)y = E_0^\sigma(Q(x)y) + TE_1^\sigma(Q(x)y) + T^2E_2^\sigma(Q(x)y).$$

Comparing the coefficients at powers of T we get $Q(x)y \in V_{2i-j}^\sigma$. Similarly, $\phi_\sigma(T).\{xyz\} = \{\phi_\sigma(T)x,\phi_{-\sigma}(T^{-1})y,\phi_\sigma(T)z\} = \{T^ix,T^{-j}y,T^\ell z\} = T^{i-j+\ell}.\{xyz\}$ implies $\{xyz\} \in V_{i-j+\ell}^\sigma$. This proves (7) and (8). Now let $x \in V_2^\sigma$ and $y \in V_0^{-\sigma}$. Then by JP7, JP8, (3) and (5) we have

$$D(y,e^\sigma) = D(y,Q(e^\sigma)e^{-\sigma}) = D(\{e^{-\sigma}e^\sigma y\},e^\sigma) - D(e^{-\sigma},Q(e^\sigma)y) = 0$$

and hence $D(x,y) = D(Q(e^\sigma)Q(e^{-\sigma})x,y) = D(e^\sigma,\{y,e^\sigma,Q(e^{-\sigma})x\}) - D(Q(e^\sigma)y,Q(e^{-\sigma})x)$ $= 0$. Similarly one proves $D(y,x) = 0$.

5.5. COROLLARY. Let $V_i = (V_i^+,V_i^-)$ for $i = 0,1,2$. Then $V_i = V_i(e)$ is a subpair of V, and $V_2(e)$ contains invertible elements; indeed, $e^+ \in V_2^+$ is invertible in $V_2(e)$ with inverse e^-.

This follows immediately from 5.4. By abuse of notation, we will often write

(1) $$V = V_2(e) \oplus V_1(e) \oplus V_0(e)$$

instead of 5.4.2.

5.6. The case of a Jordan algebra. Let J be a Jordan algebra, and let c be an idempotent of J in the usual sense; i.e., $c^2 = c$. Then (c,c) is an idempotent of the Jordan pair $V = (J,J)$, and we have

$$V_i^\pm = J_{i/2}\,, \quad (i = 0,1,2)$$

if $J = J_1 \oplus J_{1/2} \oplus J_0$ denotes the usual Peirce decomposition of J with respect to c. Indeed, $E_2^\pm = U(c)^2$, $E_1^\pm = V(c,c) - 2U_c$, and $E_0^\pm = \mathrm{Id} - V(c,c) + U_c$ are the familiar Peirce projections (cf. Jacobson[3]). In these notes, Peirce spaces will always be indexed with $0,1,2$ instead of $0,1/2,1$. Thus the Peirce-2-space (resp. Peirce-1-space) in our sense is the Peirce-1-space (resp. Peirce-1/2-space) in the usual sense.

5.7. LEMMA. <u>With the notations of 5.4, let</u> $x \in V_i^+$, $y \in V_j^-$, <u>and assume that</u> $i \neq j$. <u>Then</u> (x,y) <u>is nilpotent; in fact,</u> $x^{(n,y)} = 0$ <u>for all</u> $n \geq 3$, <u>and even for</u> $n \geq 2$ <u>if</u> $i = 2$ <u>or</u> $i = 0$.

Indeed, $x^{(n,y)}$ belongs to $V^+_{(n-1)(i-j)+i}$ by 5.4(c).

5.8. PROPOSITION. <u>Let</u> V_i $(i = 0,1,2)$ <u>be as in 5.5. Then the Jacobson radicals of</u> V <u>and</u> V_i <u>are related by</u>

(1) $$\operatorname{Rad} V_i = V_i \cap \operatorname{Rad} V ,$$

(2) $$\operatorname{Rad} V = \operatorname{Rad} V_2 \oplus \operatorname{Rad} V_1 \oplus \operatorname{Rad} V_0 .$$

<u>Proof</u>. Clearly $V_i \cap \operatorname{Rad} V \subset \operatorname{Rad} V_i$. Conversely let $x_i \in \operatorname{Rad} V_i^+$, and let $y \in V^-$. Decompose $y = y_2 + y_1 + y_0$ where $y_j \in V_j^-$. Then (x_i,y_i) is quasi-invertible in V_i and hence in V , and we have $z_i = (x_i)^{y_i} \in \operatorname{Rad} V_i^+$. For $i = 2$ it follows from 5.7, 3.7.1, and 3.8.2 that (z_2,y_1+y_0) is quasi-invertible, and again by 3.7.1, $(x_2,y_2+y_1+y_0) = (x_2,y)$ is quasi-invertible. Similarly one argues for $i = 0$. In case $i = 1$ note that $Q(y_2+y_0)z_1 = \{y_2z_1y_0\} \in V_1^-$ by 5.4(c). Since $z_1 \in \operatorname{Rad} V_1^+$ we have by JP24 that $B(z_1,Q(y_2+y_0))$ $= B(z_1,y_2+y_0)B(z_1,-y_2-y_0)$ is invertible and hence (z_1,y_2+y_0) is quasi-invertible. By 3.7.1, $(x_1,y_2+y_1+y_0) = (x,y)$ is quasi-invertible. This proves $\operatorname{Rad} V_i^+ = V_i^+ \cap \operatorname{Rad} V^+$. Passing to V^{op} we also get $\operatorname{Rad} V_i^- = V_i^- \cap \operatorname{Rad} V^-$, and therefore (1). Since every ideal of V is the sum of its intersections with the Peirce spaces we have (2).

5.9. LEMMA. <u>Let</u> $z_i \in V_i^{\pm}$ <u>and</u> $z = z_2 + z_1 + z_0$. <u>Then</u> z <u>is a trivial element if and only if</u> $Q(z_2) = Q(z_0) = Q(z_1,z_2) = Q(z_1,z_0) = Q(z_1) + Q(z_2,z_0) = 0$.

The proof is an application of 5.4(c) and is omitted.

5.10. PROPOSITION. Let V_i $(i = 0,1,2)$ be as in 5.5. Then

(1) $$\operatorname{rad} V_i = V_i \cap \operatorname{rad} V \text{, for } i = 0,2.$$

If $V_0 = 0$ then also

(2) $$\operatorname{rad} V_1 = V_1 \cap \operatorname{rad} V .$$

Proof. Let $T_\alpha(V)$ be as in 4.7. We prove by transfinite induction that

(3) $$T_\alpha(V_i) = V_i \cap T_\alpha(V)$$

for $i = 0,2$, and for $i = 1$ in case $V_0 = 0$. For $\alpha = 0$ there is nothing to prove. Assume that (3) holds for all ordinals $\beta < \alpha$. If α is a limit ordinal then (3) holds for α by definition (cf. 4.7.1). Otherwise, let $z \in V_i^+$ such that $Q(z)V_i^- \subset T_{\alpha-1}(V_i^{\ +}) = V_i^+ \cap T_{\alpha-1}(V^+)$, by induction hypothesis. If $i = 2$ (resp. $i = 0$) then it follows from 5.4(c) that $Q(z)(V_1^- + V_0^-) = 0$ (resp. $Q(z)(V_2^- + V_1^-) = 0$). If $i = 1$ and $V_0 = 0$ then $Q(z)V_2^- \subset V_0^+ = 0$. Hence in any case, $Q(z)V^- \subset T_{\alpha-1}(V^+)$ and therefore $z \in T_\alpha(V^+)$. The same arguments apply to V^- and so we have $T_\alpha(V_i) \subset V_i \cap T_\alpha(V)$. For the other inclusion, let $w \in V_i^+ \cap T_\alpha(V)$. Thus w is a finite sum of elements $z \in V^+$ such that $Q(z)V^- \subset T_{\alpha-1}(V^+)$. Denote by z_i the component of z in V_i^+. Since $w \in V_i^+$ we see that w is the sum of the z_i's. Also, since z is a trivial element modulo $T_{\alpha-1}(V)$ so is z_i for $i = 0,2$, and for $i = 1$ in case $V_0 = 0$, by 5.9. In particular, $z_i \in T_\alpha(V_i^+)$ and hence $w \in T_\alpha(V_i^+)$. This proves the other inclusion. Now the Proposition follows from 4.7 by passing to the limit.

5.11. LEMMA. Let e and f be idempotents of V such that $f \in V_0(e)$. Then also $e \in V_0(f)$ and $e + f = (e^+ + f^+, e^- + f^-)$ is an idempotent. Furthermore,

(1) $$D(e^+,f^-) = Q(e^+)Q(f^-) = 0,$$

(2) $$D(e^+,e^-)Q(f^+) = Q(f^+)D(e^-,e^+) = 0,$$

(3) $$Q(e^+)Q(e^-,f^-) = Q(e^+,f^+)Q(e^-) = 0,$$

(4) $$Q(e^+,f^+)Q(e^-,f^-) = D(e^+,e^-)D(f^+,f^-),$$

(5) $$V_2(f) \subset V_0(e),$$

<u>and the same relations hold with</u> e <u>and</u> f <u>and/or</u> + <u>and</u> - <u>interchanged</u>.

<u>Proof</u>. By 5.4(c), $Q(f^{-\sigma})e^{\sigma} = \{f^{\sigma}f^{-\sigma}e^{\sigma}\} = 0$ and hence $e \in V_0(f)$ by 5.4.5. Now (1) - (3) and the fact that $e + f$ is an idempotent follow easily from 5.4. For (4) we substitute $x = f^+$, $y = f^-$, $v = e^-$, $z = e^+$ in JP16 and use (1). Finally, (5) follows from 5.4.3, 5.4.5, (1) and (2). Since the conditions $f \in V_0(e)$ and $e \in V_0(f)$ imply each other (1) - (5) also hold with e and f interchanged. To see that they hold with + and - interchanged one passes to V^{op}.

5.12. DEFINITION. Two non-zero idempotents e and f are called <u>orthogonal</u> if $f \in V_0(e)$. In view of 5.11 this is a symmetric relation. By an <u>orthogonal system of idempotents</u> we mean an ordered set of pairwise orthogonal idempotents. A finite orthogonal system of idempotents is usually denoted by $(e_1,\dots,e_r)$, the order being the one given by the indexing. It follows easily from 5.11 that a finite sum of pairwise orthogonal idempotents is again an idempotent.

A non-zero idempotent e is called <u>primitive</u> if it cannot be written as the sum of two orthogonal idempotents. We say e is <u>local</u> (resp. a <u>division idempotent</u>) if $V_2(e)$ is a local Jordan pair (resp. a Jordan division pair). Clearly, a division idempotent is local, and a local idempotent is primitive. If V is semisimple then a local idempotent is a division idempotent, as follows from 4.4 and 5.8. Even in the semisimple case, however, there may exist primitive idempotents which are not division idempotents. As an example, let k be a field of characteristic two, and let V be the Jordan pair associated with the Jordan algebra of 2×2 symmetric matrices over k . Then one can check that $e = (e^+, e^-)$, where $e^{\pm} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, is a primitive idempotent which is not a division idempotent. For this reason, primitive idempotents are less important than local idempotents in the theory of Jordan pairs.

An orthogonal system E of idempotents is called <u>maximal</u> if it is not properly contained in any larger orthogonal system of idempotents. If $E = (e)$ consists of a single element we also say that e is a maximal idempotent. This obviouly means that $V_0(e)$ contains no non-zero idempotent. If $E = (e_1, \ldots, e_r)$ is finite then E is maximal if and only if $e = e_1 + \ldots + e_r$ is maximal. This follows from the fact that $V_0(e) = \bigcap_{i=1}^{r} V_0(e_i)$, to be proved in 5.14.

5.13. LEMMA. *Let* $(e_1, \ldots, e_r)$ *be an orthogonal system of idempotents. Let* R *be an extension of* k , *and for each* $i = 1, \ldots, r$ *let* $\phi_i = \phi_{e_i} : R^* \to \mathrm{Inn}(V_R)$ *be the homomorphism determined by* e_i *as in 5.3.*

(a) *The* ϕ_i *commute pairwise, and hence*

(1) $$\theta(t_1, \ldots, t_r) = \prod_{i=1}^{r} \phi_i(t_i)$$

defines a homomorphism $\theta: (R^*)^r \to \mathrm{Inn}(V_R)$.

(b) *Let* $e = e_1 + \ldots + e_r$ *and let* $\phi = \phi_e : R^* \to \mathrm{Inn}(V_R)$ *be the homomorphism defined by* e . *Then*

(2) $$\phi(t) = \theta(t, \ldots, t) = \prod_{i=1}^{r} \phi_i(t) .$$

Proof. By induction, it suffices to prove this for $r = 2$. Thus let e and f be orthogonal idempotents. Then $\phi_e(t).f = f$, and hence by 3.9.5,

$$\phi_e(t)\phi_f(s)\phi_e(t)^{-1} = \beta(\phi_e(t).(f^+,(1-s)f^-)) = \beta(f^+,(1-s)f^-) = \phi_f(s),$$

for all $s,t \in R^*$. This proves (a). Next, note that by 5.11.1 we have $\beta(e^+,\lambda f^-) = \beta(f^+,\lambda e^-) = \mathrm{Id}$, for all $\lambda \in R$. Hence we get, using 3.9,

$$\phi_{e+f}(t) = \beta(e^++f^+,((1-t)(e^-+f^-)) = \beta(e^++f^+,(1-t)e^-)\beta((e^++f^+)^{(1-t)e^-},(1-t)f^-)$$

$$= \beta(f^+,((1-t)e^-)^{e'}).\beta(e^+,(1-t)e^-).\beta(f^++t^{-1}e^+,(1-t)f^-)$$

$$= \beta(f^+,t^{-1}(1-t)e^-).\phi_e(t).\beta(t^{-1}e^+,((1-t)f^-)^{f^+}).\beta(f^+,(1-t)f^-)$$

$$= \phi_e(t).\beta(e^+,t^{-2}(1-t)f^-).\phi_f(t) = \phi_e(t).\phi_f(t) .$$

Here we used that $((1-t)e^-)^{e^+} = (1-t)e^- + Q((1-t)e^-)t^{-1}e^+ = t^{-1}(1-t)e^-$, by 3.3 and 5.3, and $(e^++f^+)^{(1-t)e^-} = t^{-1}e^+ + B(e^+,(1-t)e^-)^{-1}.(f^+)^{t^{-1}(1-t)e^-}$ $= t^{-1}e^+ + f^+$, by 3.7, 3.8, and 5.7.

5.14. THEOREM. Let $(e_1,\dots,e_r)$ be an orthogonal system of idempotents of V.

(a) There is a decomposition

(1)
$$V = \bigoplus_{0\le i\le j\le r} V_{ij}$$

of V into subpairs $V_{ij} = (V^+_{ij},V^-_{ij})$ such that (if we set $V_{ij} = V_{ji}$)

(2)
$$V_2(e_i) = V_{ii}, \quad V_1(e_i) = \sum_{j\ne i} V_{ij}, \quad V_0(e_i) = \sum_{j,\ell\ne i} V_{j\ell},$$

and hence

(3)
$$\begin{cases} V_{ij} = V_1(e_i)\cap V_1(e_j) \quad \text{for } i\ne j, \\ V_{i0} = V_1(e_i)\cap \bigcap_{j\ne i} V_0(e_j), \quad V_{00} = \bigcap_{i=1}^{r} V_0(e_i) . \end{cases}$$

(b) More generally, if $I\subset\{1,\dots,r\}$ and $e_I = \sum_{i\in I} e_i$ then the Peirce spaces of e_I are given by

(4) $$V_2(e_I) = \sum_{i,j\in I} V_{ij}, \quad V_1(e_I) = \sum_{\substack{i\in I\\ j\notin I}} V_{ij}, \quad V_0(e_I) = \sum_{i,j\notin I} V_{ij}.$$

In particular, if $e = e_1 + \dots + e_r$ then

(5) $$V_2(e) = \sum_{1\le i\le j\le r} V_{ij}, \quad V_1(e) = \sum_{i=1}^{r} V_{i0}, \quad V_0(e) = V_{00} .$$

(c) Consider triples of unordered pairs of indices of the form (ij,mn,pq) where $i,j,\dots,q \in \{0,1,\dots,r\}$ and (ij,mn,pq) is identified with (pq,mn,ij). Call such a triple connected if it can be written in the form (ij,jm,mn). Then we have the following composition rules.

(6) $$\{V^{\sigma}_{ij}, V^{-\sigma}_{jm}, V^{\sigma}_{mn}\} \subset V^{\sigma}_{in} .$$

If (ij,jm,ij) is connected and $ij = mn$ then

(7) $$Q(V^{\sigma}_{ij})V^{-\sigma}_{jm} \subset V^{\sigma}_{in} .$$

If (ij,mn,pq) resp. (ij,mn,ij) is not connected then

(8) $$\{V^{\sigma}_{ij}, V^{-\sigma}_{mn}, V^{\sigma}_{pq}\} = Q(V^{\sigma}_{ij})V^{-\sigma}_{mn} = 0 .$$

Proof. (a) For an extension R of k and $t \in R^*$ let

(9) $$\phi^{\sigma}_I(t) = B(e^{\sigma}_I, (1-t)e^{-\sigma}_I) = E^{\sigma}_0(e_I) + tE^{\sigma}_1(e_I) + t^2E^{\sigma}_2(e_I)$$

where $E^{\sigma}_j(e_I)$ is the projection onto the Peirce space $V^{\sigma}_j(e_I)$ (cf. 5.4). If $I = \{i\}$ we simply write ϕ^{σ}_i. Then we have (with the notation of 5.13) $\phi_i(t) = (\phi^{+}_i(t), \phi^{-}_i(t)^{-1})$ and $\theta(t_1,\dots,t_r) = (\theta_+(t_1,\dots,t_r), \theta_-(t_1,\dots,t_r)^{-1})$ where

(10) $$\theta_{\sigma}(t_1,\dots,t_r) = \prod_{i=1}^{r} \phi^{\sigma}_i(t_i) .$$

Working in $k[S,T,S^{-1},T^{-1}]$ and comparing coefficients in the equation $\phi_i(S)\phi_j(T) = \phi_j(T)\phi_i(S)$ (cf. 5.13) we see that the projections $E^{\sigma}_j(e_i)$ all commute. From (9) and (10) it follows that

(11) $$\theta_\sigma(t_1,\dots,t_r) = \sum_{\alpha\in\underline{\underline{N}}^r} t_1^{\alpha_1}\dots t_r^{\alpha_r}.E_\alpha^\sigma$$

where the $E_\alpha^\sigma = \prod_{i=1}^r E_{\alpha_i}^\sigma(e_i)$, $\alpha = (\alpha_1,\dots,\alpha_r) \in \underline{\underline{N}}^r$, are orthogonal projections whose sum is the identity (we set $E_j^\sigma(e_i) = 0$ for $j > 2$). Let $e = e_1 + \dots + e_r$, and let E_i^σ be the projection onto the Peirce space $V_i^\sigma(e)$. Also let $R = k[T,T^{-1}]$. Then we have by (10), 5.13.2, and 5.4.1,

(12) $$\theta_\sigma(T,\dots,T) = \sum_{\alpha\in\underline{\underline{N}}^r} T^{\alpha_1+\dots+\alpha_r}.E_\alpha^\sigma = E_0^\sigma + T.E_1^\sigma + T^2.E_2^\sigma .$$

This shows that $E_\alpha^\sigma = 0$ unless $\alpha_1 + \dots + \alpha_r \le 2$. In other words, if we denote by $\varepsilon_i = (0,\dots,1,\dots,0)$ the i-th standard basis vector of $\underline{\underline{Z}}^r$ and set $\varepsilon_0 = (0,\dots,0)$ then α is of the form $\varepsilon_i + \varepsilon_j$, with i,j ranging from 0 to r. Let $E_{ij}^\sigma = E_{ji}^\sigma = E_{\varepsilon_i+\varepsilon_j}^\sigma$. Then the E_{ij}^σ $(0 \le i \le j \le r)$ are orthogonal projections whose sum is the identity, and therefore we have the decomposition (1) with $V_{ij}^\sigma = E_{ij}^\sigma(V^\sigma)$.

Formula (11) reads now

(13) $$\theta_\sigma(t_1,\dots,t_r) = \sum_{0\le i\le j\le r} t_i t_j.E_{ij}^\sigma ,$$

if we set $t_0 = 1$. Now let I be a subset of $\{1,\dots,r\}$ and set $t_i = T$ if $i \in I$ and $t_i = 1$ if $i \notin I$. Then we get by (9), (10), and 5.13.2 that

$$\phi_I^\sigma(T) = \prod_{i\in I} \phi_i^\sigma(T) = T^2(\sum_{\substack{i,j\in I\\ i\le j}} E_{ij}^\sigma) + T(\sum_{\substack{i\in I\\ j\notin I}} E_{ij}^\sigma) + (\sum_{\substack{i,j\notin I\\ i\le j}} E_{ij}^\sigma)$$

$$= T^2.E_2^\sigma(e_I) + T.E_1^\sigma(e_I) + E_0^\sigma(e_I) .$$

Comparing coefficients at powers of T this proves (4) and (2) (in the special case where $I = \{i\}$), and (5) is a special case of (4).

Now we prove (c). Let $R = k[T_1,\dots,T_r;T_1^{-1},\dots,T_r^{-1}]$, and let $x \in V_{ij}^{\sigma}$, $y \in V_{jm}^{-\sigma}$, $z \in V_{mn}^{\sigma}$. Then we have, writing $\underline{T} = (T_1,\dots,T_r)$ for short,

$$\theta_\sigma(\underline{T})\{xyz\} = \{\theta_\sigma(\underline{T})x,\theta_{-\sigma}(\underline{T}^{-1})y,\theta_\sigma(\underline{T})z\}$$
$$= \{T_iT_jx,T_j^{-1}T_m^{-1}y,T_mT_nz\} = T_iT_n\{xyz\} .$$

On the other hand, $\theta_\sigma(\underline{T})\{xyz\} = \sum_{0\le p\le q\le r} T_pT_qE_{pq}^{\sigma}\{xyz\}$ by (13), and hence it follows that $\{xyz\} = E_{in}^{\sigma}\{xyz\} \in V_{in}^{\sigma}$. Similarly,

$$\theta_\sigma(\underline{T})Q(x)y = Q(\theta_\sigma(\underline{T})x)\theta_{-\sigma}(\underline{T}^{-1})y$$
$$= Q(T_iT_jx)T_j^{-1}T_m^{-1}y = T_i^2T_jT_m^{-1}Q(x)y = T_iT_nQ(x)y ,$$

since $ij = mn$ means $T_iT_j = T_mT_n$ and therefore $T_i^2T_jT_m^{-1} = T_iT_n$. This proves (6) and (7).

It is easily seen that the only triples which are not connected are of the form (ij,mn,pq) with $\{i,j\}\cap\{m,n\} = \emptyset$, or (ij,jm,jn) with $m \neq i,j,n$. In the first case, either i and j are different from zero or m and n are different from zero. Let $x \in V_{ij}^{+}$, $y \in V_{mn}^{-}$, $z \in V_{pq}^{+}$. If $i,j \neq 0$ then $x \in V_2^{+}(e_i+e_j)$ and $y \in V_0^{-}(e_i+e_j)$ by (4). If $m,n \neq 0$ then $x \in V_0^{+}(e_m+e_n)$ and $y \in V_2^{-}(e_m+e_n)$. Hence $\{xyz\} = Q(x)y = 0$ by 5.4.9. – In the second case, let $x \in V_{ij}^{+}$, $y \in V_{jm}^{-}$, $z \in V_{jn}^{+}$. Then a similar computation as before shows that $\theta_+(\underline{T})\{xyz\} = T_iT_jT_nT_m^{-1}\{xyz\}$ and $\theta_+(\underline{T})Q(x)y = T_i^2T_jT_m^{-1}Q(x)y$. Since $m \neq i,j,n$ it follows that $T_iT_jT_nT_m^{-1}$ (resp. $T_i^2T_jT_m^{-1}$) is not of the form T_pT_q , and by (13) we must have $\{xyz\} = Q(x)y = 0$. The same argument works with $+$ and $-$ interchanged. This finishes the proof.

NOTES

In [1] and[2], K. Meyberg studied the Koecher-Tits construction of Lie algebras from Jordan algebras in a general setting and was thereby led to introduce what he called "verbundene Paare" (connected pairs), and what we would call linear Jordan pairs. These were pairs (V^+, V^-) of k-modules with trilinear maps $V^\sigma \times V^{-\sigma} \times V^\sigma \to V^\sigma$, $(x,y,z) \mapsto \{xyz\}$, satisfying

(1) $$\{xyz\} = \{zyx\} ,$$

(2) $$\{uv\{xyz\}\} - \{xy\{uvz\}\} = \{\{uvx\}yz\} - \{x\{vuy\}z\} .$$

If 2 is invertible in k and if V has no 3-torsion then $V = (V^+, V^-)$ is a Jordan pair in our sense by setting $Q(x)y = (1/2)\{xyx\}$ (cf. 2.2). He also defined linear Jordan triple systems as modules with a trilinear composition $\{xyz\}$ (whence the name) satisfying (1) and (2). The restriction that there be no 3-torsion can be removed by adding the identity

(3) $$\{xy\{xzx\}\} = \{x\{yxz\}x\} ,$$

which is otherwise a consequence of (1) and (2) (cf. 2.2). This is the point of view taken in Loos[2] and Meyberg[4]. Finally, Meyberg defined quadratic Jordan triple systems over an arbitrary ring of scalars in analogy to McCrimmon's concept of quadratic Jordan algebras, based on the quadratic operators P(x) satisfying (1) - (3) of 1.13 (cf. Meyberg[6]).

The Jordan algebras V_y^+ (1.9) were introduced by Meyberg[1]. The definition of the centroid of a Jordan pair is analogous to the Jordan algebra case (cf. McCrimmon[3]). The fundamental identities (§2) are all due to Meyberg ([4] in the linear case, [6] in the quadratic case). Representations of Jordan triple systems were studied in Loos[2][5].

Most of the material of §3 is, in one form or another, due to Koecher[2-4], McCrimmon[4], Meyberg[4],[6]. The formulas (1) and (2) of 3.7 are important in the construction of algebraic groups from Jordan pairs resp. Jordan algebras, see Koecher[2-3], Loos[7]; in particular, 3.7.2 is a substitute of Hua's formula for Jordan algebras. Meyberg developed the theory of the Jacobson radical for Jordan triple systems in [4] and [6], following McCrimmon's theory for Jordan algebras([1],[4]). The small radical (= lower radical) for Jordan algebras was introduced by McCrimmon[1] and studied by Lewand[1]; 4.9 is due to him for Jordan algebras, and due to Meyberg for Jordan triple systems. Theorem 4.13 was proved by Slin'ko[1] for linear Jordan algebras and by McCrimmon for quadratic Jordan algebras (personal communication). 4.18 is due to Meyberg[6]. It would be interesting to extend this to other radicals. 4.10, due to M. Slater, is a weak form of the Andrunakievich Lemma. There are many open questions concerning radicals; e.g., is the ideal generated by a trivial element nilpotent or solvable in some sense, or does it contain a non-zero ideal consisting of trivial elements? The answer is positive in the finite-dimensional case (§14). It is even unknown if there exist simple s-radical Jordan pairs or Jordan algebras.

The treatment of the Peirce decomposition (§5) is essentially an application of the representation theory of diagonalizable group functors (Demazure& Gabriel[1]) in disguise. A similar approach was used by Springer[1] for finite-dimensional Jordan algebras; it has the advantage of reducing computations to a minimum. 5.8 is due to Meyberg[6]. I don't know if 5.10.2 remains true without the assumption $V_0 = 0$, or if 5.8 holds for the nil radical.

CHAPTER II

ALTERNATIVE PAIRS

§6. Basic properties and relations with alternative algebras

6.1. DEFINITION. Let $A = (A^+, A^-)$ be a pair of k-modules, equipped with trilinear maps $\langle\ \ \rangle: A^\sigma \times A^{-\sigma} \times A^\sigma \to A^\sigma$, $(x,y,z) \mapsto \langle xyz\rangle$. Then A is called an associative pair if the identities

$$\langle uv\langle xyz\rangle\rangle = \langle u\langle yxv\rangle z\rangle = \langle\langle uvx\rangle yz\rangle \tag{1}$$

are satisfied. A is called commutative if $\langle xyz\rangle = \langle zyx\rangle$. We say A is an alternative pair if the following identities hold.

AP1 $$\langle uv\langle xyz\rangle\rangle + \langle xy\langle uvz\rangle\rangle = \langle\langle uvx\rangle yz\rangle + \langle x\langle vuy\rangle z\rangle,$$

AP2 $$\langle uv\langle xyx\rangle\rangle = \langle\langle uvx\rangle yx\rangle,$$

AP3 $$\langle xy\langle xyz\rangle\rangle = \langle\langle xyx\rangle yz\rangle.$$

Since these identities are of degree at most two in each variable they remain valid in every scalar extension, and hence A_K is an alternative pair over K, for every extension K of k. Also note that associative pairs are a fortiori alternative, as follows easily from the definitions. We define left, middle, and right multiplications by

$$L_\sigma(x,y)z = M_\sigma(x,z)y = R_\sigma(z,y)x = \langle xyz\rangle, \tag{2}$$

so that $L_\sigma: A^\sigma \times A^{-\sigma} \to \mathrm{End}(A^\sigma)$, $M_\sigma: A^\sigma \times A^\sigma \to \mathrm{Hom}(A^{-\sigma},A^\sigma)$, and $R_\sigma: A^\sigma \times A^{-\sigma} \to \mathrm{End}(A^\sigma)$ are bilinear maps. To simplify notation, we usually write $L(x,y)$ or $L_{x,y}$ instead of $L_\sigma(x,y)$, and similarly for the middle and right multiplication. This causes no confusion if one remembers that in $L(x,y)$ and $R(x,y)$, x and y have to be in different A^σ's ($x \in A^+$, $y \in A^-$ or conversely) whereas in $M(x,z)$, x and z are in the same A^σ . See also 2.0 for the analogous situation for Jordan pairs.

6.2. A <u>homomorphism</u> $h: A \to B$ of alternative pairs is a pair $h = (h_+,h_-)$ of k-linear maps $h_\sigma: A^\sigma \to B^\sigma$ such that

(1) $$h_\sigma(\langle xyz\rangle) = \langle h_\sigma(x),h_{-\sigma}(y),h_\sigma(z)\rangle,$$

for all $x,z \in A^\sigma$, $y \in A^{-\sigma}$. Isomorphisms and automorphisms are defined as usual. The automorphism group of A is denoted by $\mathrm{Aut}(A)$; it is a subgroup of $GL(A^+) \times GL(A^-)$.

A pair of submodules $B = (B^+,B^-)$ of A is called a <u>subpair</u> (resp. an <u>ideal</u>) if $\langle B^\sigma,B^{-\sigma},B^\sigma\rangle \subset B^\sigma$ (resp. $\langle A^\sigma,A^{-\sigma},B^\sigma\rangle + \langle A^\sigma,B^{-\sigma},A^\sigma\rangle + \langle B^\sigma,A^{-\sigma},A^\sigma\rangle \subset B^\sigma$). If B is an ideal then $A/B = (A^+/B^+,A^-/B^-)$ is an alternative pair in the obvious way. As usual, A is called <u>simple</u> if it has only the trivial ideals 0 and A and if $\langle A^\sigma,A^{-\sigma},A^\sigma\rangle \neq 0$.

A <u>derivation</u> of A is a pair $\Delta = (\Delta_+,\Delta_-) \in \mathrm{End}(A^+) \times \mathrm{End}(A^-)$ such that

(2) $$\Delta_\sigma(\langle xyz\rangle) = \langle \Delta_\sigma(x),y,z\rangle + \langle x,\Delta_{-\sigma}(y),z\rangle + \langle x,y,\Delta_\sigma(z)\rangle$$

for all $x,z \in A^\sigma$, $y \in A^{-\sigma}$. This is equivalent with $\mathrm{Id}+\varepsilon\Delta \in \mathrm{Aut}(A_{k(\varepsilon)})$ where $k(\varepsilon)$ is the algebra of dual numbers. The derivations of A form a Lie subalgebra $\mathrm{Der}(A)$ of $\mathrm{End}(A^+) \times \mathrm{End}(A^-)$.

6.3. As in the Jordan case, the <u>opposite</u> of A is $A^{op} = (A^-,A^+)$ with composition $\langle xyz\rangle^{op} = \langle xyz\rangle$, and an antihomomorphism from A to B is a homomorphism from A to B^{op}. Note that the opposite of A is <u>not</u> defined by reversing the order in the product $\langle xyz\rangle$ but merely by interchanging A^+ and A^-. In fact, our definition of alternative pairs is non-symmetric in the sense that if we set

$$\langle xyz\rangle' = \langle zyx\rangle \tag{1}$$

then $\langle xyz\rangle'$ will no longer satisfy AP1 - AP3. Thus our alternative pairs should perhaps be called left-alternative, and a "right-alternative pair" would satisfy the "dual" identities

$$\langle\langle zyx\rangle vu\rangle + \langle\langle zvu\rangle yx\rangle = \langle zy\langle xvu\rangle\rangle + \langle z\langle yuv\rangle x\rangle,$$

$$\langle\langle xyx\rangle vu\rangle = \langle xy\langle xvu\rangle\rangle,$$

$$\langle\langle zyx\rangle yx\rangle = \langle zy\langle xyx\rangle\rangle.$$

For associative pairs, there is no such distinction, and every associative pair A gives rise to an associative pair A' having the same underlying modules and the reversed multiplication (1). We call A' the <u>reverse</u> of A.

The <u>direct product</u> of alternative pairs A and B is $A \times B = (A^+ \times B^+, A^- \times B^-)$ with componentwise operations. In $A \times A^{op} = (A^+ \times A^-, A^- \times A^+)$ we have the exchange involution given by $(x,y) \mapsto (y,x)$.

6.4. <u>The associative pairs</u> $A(M,R,\Phi)$. Let R be an associative (not necessarily commutative) k-algebra, and let $M = (M^+,M^-)$ be a pair of k-modules such that M^+ is a left and M^- is a right R-module. Consider a k-bilinear map $\Phi: M^+ \times M^- \to R$ which is R-bilinear in the sense that

$$\Phi(ax,yb) = a\Phi(x,y)b$$

for all $a,b \in R$, $x \in M^+$, $y \in M^-$. We say that Φ is <u>non-degenerate</u> if $\Phi(x,M^-) = 0$ implies $x = 0$ and $\Phi(M^+,y) = 0$ implies $y = 0$. In order to describe this situation in a more symmetrical fashion, let $R^+ = R$, $R^- = R^{op}$, and consider M^- as a left R^--module in the obvious way. Also let $a \mapsto \bar{a}$ denote the antiisomorphism $R^\sigma \to R^{-\sigma}$ given by the identity map, and define $\Phi_+ = \Phi$, $\Phi_-: M^- \times M^+ \to R^-$ by $\Phi_-(y,x) = \overline{\Phi(x,y)}$. Then $\Phi_\sigma: M^\sigma \times M^{-\sigma} \to R^\sigma$ is "hermitian" in the sense that $\Phi_\sigma(ax,y) = a\Phi_\sigma(x,y)$ and $\overline{\Phi_\sigma(x,y)} = \Phi_{-\sigma}(y,x)$.

The pair of modules (M^+,M^-) becomes an associative pair over k, denoted by $A = A(M,R,\Phi)$, with

$$\langle xyz\rangle = \Phi_\sigma(x,y)z,$$

for $x,z \in A^\sigma = M^\sigma$, $y \in A^{-\sigma} = M^{-\sigma}$. The simple verification is left to the reader. If R is simple with unity, $M^\sigma \neq 0$, and Φ is non-degenerate, then A is simple. Indeed, $\Phi(M^+,M^-) = \Phi(M)$ is a non-zero ideal of R and therefore is all of R. Let $B = (B^+,B^-)$ be an ideal of A. Then $RB^+ = \Phi(M^+,M^-)B^+ = \langle M^+,M^-,B^+\rangle \subset B^+$ and hence B^+ is a R-submodule of M^+. This implies that $\Phi(B^+,M^-)$ is an ideal of R, and therefore either $B^+ = 0$ or $R = \Phi(B^+,M^-)$. Since $1 \in R$ it follows that $M^+ = \Phi(B^+,M^-)M^+ = \langle B^+,M^-,M^+\rangle \subset B^+$, and a similar proof shows that $B^- = M^-$.

6.5. <u>The alternative pair associated with an alternative algebra</u>. Let R be an alternative algebra over k, and let $x \mapsto \bar{x}$ denote the antiisomorphism $R \to R^{op}$ and $R^{op} \to R$ given by the identity map. Then (R,R^{op}) is an alternative pair with

(1) $$\langle xyz\rangle = (x\bar{y})z,$$

in other words, if $(R,R^{op}) = (A^+,A^-)$ then $\langle xyz\rangle = (xy)z$ and $\langle yxv\rangle = v(xy)$ for $x,z \in A^+$ and $y,v \in A^-$ where xy is the product in R. Indeed, for the proof of AP1 we use the alternative law and get

$$<uv<xyz>> + <xy<uvz>> = (u\bar{v})((x\bar{y})z) + (x\bar{y})((u\bar{v})z)$$
$$= ((u\bar{v})(x\bar{y}) + (x\bar{y})(u\bar{v}))z = (((u\bar{v})x)\bar{y} + x(\bar{y}(u\bar{v})))z$$
$$= (((u\bar{v})x)\bar{y})z + (x(\overline{(v\bar{u})y}))z = <<uvx>yz> + <x<vuy>z>.$$

For AP2 and AP3 we use the Moufang identities:

$$<uv<xyx>> = (u\bar{v})(x\bar{y}x) = (((u\bar{v})x)\bar{y})x = <<uvx>yx>,$$
$$<xy<xyz>> = (x\bar{y})((x\bar{y})z) = ((x\bar{y})(x\bar{y}))z = ((x\bar{y}x)\bar{y})z = <<xyx>yz>.$$

We call (R,R^{op}) the alternative pair associated with R. Obviously, if $f: R \to R'$ is a homomorphism between alternative algebras then $(f,f): (R,R^{op}) \to (R',R'^{op})$ is a homomorphism of alternative pairs. If I is an ideal of R then $(I,\bar{I})$ is an ideal of (R,R^{op}). Conversely, if R has a unit element and (I^+,I^-) is an ideal of (R,R^{op}) then it follows easily that I^+ is an ideal of R and $I^- = \overline{I^+}$. Therefore if R is unital then R is simple if and only if (R,R^{op}) is simple.

6.6. <u>The alternative pairs</u> $A(X,K,\alpha)$. Let K be an extension of k, and let X be a K-module. Suppose that there is given an alternating K-bilinear form $\alpha: X \times X \to K$. Let $A^+ = A^- = X$ as K-modules, and let $J_+: A^- \to A^+$ be an arbitrary K-linear isomorphism. Define $J_-: A^+ \to A^-$ by $J_- = -(J_+)^{-1}$. Also set $\alpha = \alpha_+$ and define $\alpha_-: A^- \times A^- \to K$ by $\alpha_-(x,y) = \alpha_+(J_+(x),J_+(y))$, so that

(1) $$J_\sigma J_{-\sigma} = -\mathrm{Id}_{A^\sigma}, \quad \alpha_\sigma(J_\sigma(x),J_\sigma(y)) = \alpha_{-\sigma}(x,y).$$

Then $A = (A^+,A^-)$ becomes an alternative pair over k (or over K), denoted by $A = A(X,K,\alpha)$, with

(2) $$<xyz> = x\alpha_\sigma(z,J_\sigma(y)) + J_\sigma(y)\alpha_\sigma(x,z),$$

for $x,z \in A^\sigma$, $y \in A^{-\sigma}$. The straightforward verification of AP1 - AP3 is left to the reader. We remark that different choices of J_+ yield isomorphic alternative pairs, and thus $A(X,K,\alpha)$ depends up to isomorphism only on (X,K,α). For instance, we can set $J_+ = \mathrm{Id}_X$. However, it is sometimes convenient to choose J_+ differently.

If K is a field, α is nondegenerate, and $\dim_K X \geq 4$ then A is a simple properly alternative pair (cf. 11.11). Thus, in contrast to the situation for alternative algebras, there exist simple properly alternative pairs of arbitrary (even infinite) dimension.

6.7. Basic identities. We set $x = z$ in AP1 and observe AP2 which implies

AP4 $$\langle xy\langle uvx\rangle\rangle = \langle x\langle vuy\rangle x\rangle.$$

Since the left hand side of AP1 is symmetric in (u,v) and (x,y) so is the right hand side. This yields

AP5 $$\langle\langle xyu\rangle vz\rangle + \langle u\langle yxv\rangle z\rangle = \langle\langle uvx\rangle yz\rangle + \langle x\langle vuy\rangle z\rangle .$$

From AP1 we get

$$\langle xy\langle xvz\rangle\rangle + \langle xv\langle xyz\rangle\rangle = \langle\langle xyx\rangle vz\rangle + \langle x\langle yxv\rangle z\rangle .$$

By linearizing AP3 with respect to y we obtain

$$\langle xy\langle xvz\rangle\rangle + \langle xv\langle xyz\rangle\rangle = \langle\langle xyx\rangle vz\rangle + \langle\langle xvx\rangle yz\rangle ,$$

and a comparison of these two formulas shows

AP6 $$\langle\langle xvx\rangle yz\rangle = \langle x\langle yxv\rangle z\rangle .$$

By linearizing AP3 with respect to x we get

$$\langle uy\langle xyz\rangle\rangle + \langle xy\langle uyz\rangle\rangle = \langle\langle xyu\rangle yz\rangle + \langle\langle uyx\rangle yz\rangle ,$$

and from AP1,

$$\langle uy\langle xyz\rangle\rangle + \langle xy\langle uyz\rangle\rangle = \langle\langle uyx\rangle yz\rangle + \langle x\langle yuy\rangle z\rangle$$

which implies

AP7 $$\langle\langle xyu\rangle yz\rangle = \langle x\langle yuy\rangle z\rangle \ .$$

This and AP3 imply

AP8 $$\langle xy\langle xyz\rangle\rangle = \langle x\langle yxy\rangle z\rangle = \langle\langle xyx\rangle yz\rangle \ .$$

From AP7 and AP2 we get

AP9 $$\langle\langle zyx\rangle yx\rangle = \langle z\langle yxy\rangle x\rangle = \langle zy\langle xyx\rangle\rangle \ ,$$

and from AP7 and AP4,

AP10 $$\langle xy\langle zyx\rangle\rangle = \langle x\langle yzy\rangle x\rangle = \langle\langle xyz\rangle yx\rangle \ .$$

In operator form, AP8 - AP10 read

AP11 $$L(x,y)^2 = L(\langle xyx\rangle,y) = L(x,\langle yxy\rangle) \ ,$$

AP12 $$R(x,y)^2 = R(\langle xyx\rangle,y) = R(x,\langle yxy\rangle) \ ,$$

AP13 $$L(x,y)R(x,y) = R(x,y)L(x,y) = M(x,x)M(y,y) \ .$$

6.8. Let $\mathcal{R}$ be an alternative algebra and let $v \in \mathcal{R}$. Recall that the (left) v-<u>homotope</u> of $\mathcal{R}$ is the alternative algebra $\mathcal{R}^{(v)}$ having the same underlying module as $\mathcal{R}$, and multiplication given by

$$x \underset{v}{\cdot} y = (xv)y$$

(see McCrimmon[5]). If $\mathcal{R}$ has a unit element and v is invertible in $\mathcal{R}$ then $\mathcal{R}^{(v)}$ is called the v-<u>isotope</u> of $\mathcal{R}$. In this case, v^{-1} is the unit element of $\mathcal{R}^{(v)}$. Let $\mathcal{R}$ and $\mathcal{R}'$ be unital alternative algebras. An <u>isotopy</u> from $\mathcal{R}$ to $\mathcal{R}'$ is an isomorphism, say g , from $\mathcal{R}$ onto an isotope $\mathcal{R}'^{(v)}$ of $\mathcal{R}'$. Then necessarily $g(1) = v^{-1}$ since $g(1)$ must be the unit element of $\mathcal{R}'^{(v)}$. Just as in the Jordan case (1.8) we have

6.9. PROPOSITION. Let R and R' be unital alternative algebras with associated alternative pairs A and A'. Then the map $g \to (g, U_{g(1)}^{-1} g)$ (where $U_x y = xyx$) is a bijection between the set of isotopies from R to R' and the set of isomorphisms from A to A'. In particular, A and A' are isomorphic if and only if R and R' are isotopic.

The proof is straightforward and therefore omitted.

6.10. The alternative algebras A_v^+. Let $A = (A^+, A^-)$ be an alternative **pair**, and let $v \in A^-$. Then the k-module A^+ becomes an alternative algebra, denoted by A_v^+, with the multiplication

(1) $$xy = \langle xvy \rangle .$$

Indeed, by AP2 we have $xy^2 = \langle xv\langle yvy\rangle\rangle = \langle\langle xvy\rangle vy\rangle = (xy)y$, and by AP3, $x^2y = \langle\langle xvx\rangle vy\rangle = \langle xv\langle xvy\rangle\rangle = x(xy)$. If $h = (h_+, h_-): A \to B$ is a homomorphism of alternative pairs then

(2) $$h_+ : A_v^+ \to B_{h_-(v)}^+$$

is a homomorphism of alternative algebras. If $u \in A^+$ then

(3) $$(A_v^+)^{(u)} = A_{\langle vuv\rangle}^+ .$$

Indeed, the product in the u-homotope of A_v^+ is $x \underset{u}{\cdot} y = (xu)y = \langle\langle xvu\rangle vy\rangle = \langle x\langle vuv\rangle y\rangle$, by AP7. Similarly, we have alternative algebras A_w^- with product $xy = \langle xwy\rangle$, for all $w \in A^+$.

6.11. Inverses. Let A be an alternative pair. An element $u \in A^\sigma$ is called invertible if there exists an element $v \in A^{-\sigma}$ such that

(1) $$L(u,v) = R(u,v) = \mathrm{Id} \quad \text{and} \quad L(v,u) = R(v,u) = \mathrm{Id} .$$

It follows from AP13 that then $M(u,u)$ is invertible and

(2) $$M(u,u)^{-1} = M(v,v) .$$

Also, since $u = \langle uvu \rangle$ we have

(3) $$v = M(u,u)^{-1}(u)$$

is uniquely determined by u. We call v the inverse of u and denote it by u^{-1}. Clearly, u^{-1} is invertible with inverse u. In general, an alternative pair will have no invertible elements.

6.12. PROPOSITION. *Let $\mathcal{A}$ be an alternative pair, let $v \in \mathcal{A}^-$ be invertible, and let $u = v^{-1} \in \mathcal{A}^+$. Then the alternative algebras $\mathcal{R} = \mathcal{A}^+_v$ and $\mathcal{R}' = \mathcal{A}^-_u$ are unital with unit elements u and v, respectively, and $M(v,v)\colon \mathcal{R} \to \mathcal{R}'$ is an antiisomorphism with inverse $M(u,u)$. Furthermore, $(\mathrm{Id}, M(v,v))\colon (\mathcal{R}, \mathcal{R}^{op}) \to \mathcal{A}$ is an isomorphism of alternative pairs.*

Proof. In $\mathcal{R}$ we have $ux = \langle uvx \rangle = x = \langle xvu \rangle = xu$ and hence u is the unit element of $\mathcal{R}$. Similarly, v is the unit element of $\mathcal{R}'$. Now $M(v,v)(xy)$ $= M(v,v)(\langle xvy \rangle) = \langle v\langle xvy \rangle v \rangle = \langle vy\langle vxv \rangle\rangle$ by AP4, and $(M(v,v)y).(M(v,v)x)$ $= \langle\langle vyv \rangle u \langle vxv \rangle\rangle = \langle\langle\langle vyv \rangle uv \rangle xv \rangle = \langle\langle vyv \rangle xv \rangle = \langle vy\langle vxv \rangle\rangle$, using AP2 twice. Finally we have $\mathrm{Id}((xy)z) = \langle\langle xvy \rangle vz \rangle = \langle x\langle vyv \rangle z \rangle$ by AP7, and $M(v,v)(z(yx))$ $= \langle v\langle zv\langle yvx \rangle\rangle v \rangle = \langle v\langle yvx \rangle\langle vzv \rangle\rangle = \langle\langle vxv \rangle y\langle vzv \rangle\rangle$ by AP4 and AP6 which proves the last statement.

6.13. COROLLARY. *The map $\mathcal{R} \mapsto (\mathcal{R}, \mathcal{R}^{op})$ induces a bijection between isotopy classes of unital alternative algebras and isomorphism classes of alternative*

<u>pairs containing invertible elements</u>. <u>The inverse map is induced by</u> $A \mapsto A_v^+$ <u>where</u> v <u>is any invertible element of</u> A^- .

This follows immediately from 6.12 and 6.9.

6.14. <u>Remark</u>. Let R be a unital alternative algebra. It is easily seen that R is associative if and only if (R,R^{op}) is an associative pair. On the other hand, the v-isotope $R^{(v)}$ of an associative algebra is isomorphic with R ; an isomorphism being left multiplication with v . Hence we have a bijection between isomorphism classes of unital associative algebras and associative pairs containing invertible elements. In the alternative case, it is known that simple isotopic alternative algebras are isomorphic (cf. McCrimmon[5]). Since R is simple if and only (R,R^{op}) is simple (6.5) we get that simple unital alternative algebras and simple alternative pairs containing invertible elements are "essentially the same".

6.15. An <u>alternative triple system</u> is a k-module T with a trilinear composition $T \times T \times T \to T$, $(x,y,z) \to \langle xyz\rangle$, satisfying AP1 - AP3 (cf. Loos[3], Meyberg[6]). A homomorphism between alternative triple systems is a k-linear map f such that $f(\langle xyz\rangle) = \langle f(x),f(y),f(z)\rangle$, for all x,y,z . From the definitions it is clear that every alternative triple system T gives rise to an alternative pair (T,T) , and that the identity maps Id: $A^{\sigma} = T \to T = A^{-\sigma}$ define a canonical involution κ of (T,T) . Conversely, let A be an alternative pair with involution η , let $T = A^+$ as a k-module, and define a trilinear composition on T by

$$\langle xyz\rangle = \langle x,\eta_+(y),z\rangle \ . \tag{4}$$

Then it is easily verified that T is an alternative triple system, and that

$(Id,\eta_+):((T,T);\kappa) \to (A;\eta)$ is an isomorphism of alternative pairs with involution (cf. the corresponding situation for Jordan triple systems in 1.13). Thus we get an equivalence between the category of alternative triple systems and the category of alternative pairs with involution. Under this equivalence, associative pairs with involution correspond to associative triple systems of the second kind (cf. Loos[4]). Finally, we mention that one can define polarized alternative triple systems just as in the Jordan case (1.14).

6.16. The centroid $Z(A)$ of an alternative pair A over k is the set of all $(a_+,a_-) \in \mathrm{End}(A^+) \times \mathrm{End}(A^-)$ satisfying

$$a_\sigma\langle xyz\rangle = \langle a_\sigma x,y,z\rangle = \langle x,a_{-\sigma}y,z\rangle = \langle x,y,a_\sigma z\rangle ,$$

for all $x,z \in A^\sigma$, $y \in A^{-\sigma}$. Clearly, $Z(A)$ is a subalgebra of $\mathrm{End}(A^+) \times \mathrm{End}(A^-)$ containing the unit element $1 = (Id,Id)$. We say A is central if $Z(A) = k.1$.

Let R be a unital alternative algebra with center $Z(R)$. Then the map $z \mapsto (L_z,L_z)$ (where L_z denotes left multiplication with z in R) is an isomorphism between $Z(R)$ and the centroid $Z(R,R^{op})$ of the associated alternative pair. The details are left to the reader.

6.17. PROPOSITION. If A is simple then $Z(A)$ is a field.

Proof. For $a,b \in Z(A)$ we have $a_\sigma b_\sigma\langle xyz\rangle = a_\sigma\langle x,y,b_\sigma z\rangle = \langle a_\sigma x,y,b_\sigma z\rangle = b_\sigma\langle a_\sigma x,y,z\rangle = b_\sigma a_\sigma\langle xyz\rangle$. Since $A^\sigma = \langle A^\sigma,A^{-\sigma},A^\sigma\rangle$ by simplicity of A we see that $Z(A)$ is commutative. Also, it is clear that both $\mathrm{Ker}(a) = (\mathrm{Ker}(a_+),\mathrm{Ker}(a_-))$ and $\mathrm{Im}(a) = (a_+(A^+),a_-(A^-))$ are ideals of A . Since A is simple it follows that $Z(A)$ is a field.

§7. The Jordan pair associated with an alternative pair

7.1. THEOREM. *Let* $\mathcal{A} = (A^+, A^-)$ *be an alternative pair over* k . *Then* $\mathcal{A}$ *becomes a Jordan pair over* k , *denoted by* $\mathcal{A}^J$, *with quadratic maps* $Q : A^\sigma \to \mathrm{Hom}(A^{-\sigma}, A^\sigma)$ *given by*

(1) $$Q(x)y = \langle xyx\rangle = M(x,x)y \ .$$

Proof. We have to verify JP1 – JP3 in every scalar extension $\mathcal{A}_K$ of $\mathcal{A}$. Since $\mathcal{A}_K$ is an alternative pair over K (cf. 6.1) we may assume that K = k . By (1) and 1.1 we have

(2) $$\{xyz\} = \langle xyz\rangle + \langle zyx\rangle \ ,$$

so that

(3) $$D(x,y) = L(x,y) + R(x,y) \ .$$

Now $\{x,y,Q(x)z\} = \langle xy\langle xzx\rangle\rangle + \langle\langle xzx\rangle yx\rangle = \langle x\langle zxy\rangle x\rangle + \langle x\langle yxz\rangle x\rangle = Q(x)\{yxz\}$ by AP4 and AP6. Also $\{Q(x)y,y,z\} = \langle\langle xyx\rangle yz\rangle + \langle zy\langle xyx\rangle\rangle = \langle x\langle yxy\rangle z\rangle + \langle z\langle yxy\rangle x\rangle$ $= \{x,Q(y)x,z\}$ by AP8 and AP9, and finally $Q(x)Q(y)Q(x)z = Q(x)L(y,x)R(y,x)z$ $= \langle x\langle yx\langle zxy\rangle\rangle x\rangle = \langle\langle x\langle zxy\rangle x\rangle yx\rangle = \langle\langle\langle xyx\rangle zx\rangle yx\rangle = \langle\langle xyx\rangle z\langle xyx\rangle\rangle = Q(Q(x)y)z$, using AP13, twice AP6, and AP2.

We remark that a homomorphism between alternative pairs is also a homomorphism of the associated Jordan pairs.

7.2. Compatibility with previous constructions. (i) Recall that with every alternative algebra, $\mathcal{R}$ we can associate a Jordan algebra $\mathcal{R}^J$ having the same underlying module, squaring given by the square in $\mathcal{R}$, and quadratic operators $U_x y$

$= xyx$. Clearly we have $R^J = (R^{op})^J$. On the other hand, we have the alternative pair (R, R^{op}) associated with R (6.5) and the Jordan pair (R^J, R^J) associated with R^J (1.6). From the definitions it follows immediately that

(1) $$(R^J, R^J) = (R, R^{op})^J .$$

(ii) Let A be an alternative pair, and let $v \in A^-$. Then the Jordan algebra associated with the alternative algebra A_v^+ (cf. 6.10) is the Jordan algebra V_v^+ (cf. 1.9) where $V = A^J$, i.e.,

(2) $$(A_v^+)^J = (A^J)_v^+ .$$

Indeed, $x^2 = \langle xvx\rangle = Q(x)v$, and $U_x y = xyx = \langle\langle xvy\rangle vx\rangle = \langle x\langle vyv\rangle x\rangle = Q(x)Q(v)y$.

(iii) If v is invertible in A then by 6.11 and 1.10, v is invertible in the Jordan pair A^J . Conversely, assume that v is invertible in A^J , i.e., $Q(v) = M(v,v)$ is invertible, and let $u = v^{-1}$. Then it follows from AP11 - AP13 that u and v are inverses of each other in the alternative sense as well, and hence invertibility in A and A^J are equivalent.

(iv) With every alternative triple system T we can associate a Jordan triple system T^J by setting $Q(x)y = \langle xyx\rangle$ (cf. Loos[3], Meyberg[6]). From 1.13 and 6.15 it is clear that

(3) $$(T,T)^J = (T^J, T^J) .$$

(v) The centroids of A and A^J (cf. 1.15 and 6.16) are related by

(4) $$Z(A) \subset Z(A^J) .$$

(vi) Let A be alternative. We say that a pair $(x,y) \in A$ is <u>quasi-invertible</u> if x is quasi-invertible in the alternative algebra A_y^+ ; i.e., if $1 - x$ is invertible in the unital alternative algebra $k.1 \oplus A_y^+$. By (2), this is equivalent with x being quasi-invertible in $(A^J)^+_v$ or (x,y) being quasi-invertible in the Jordan pair A^J. From standard facts on inverses in alternative algebras we get easily

7.3. PROPOSITION. *For* $(x,y) \in A$, *the following conditions are equivalent.*

(i) (x,y) *is quasi-invertible;*

(ii) *there exists* $z \in A^+$ *such that*

(1) $$z - x = \langle xyz\rangle = \langle zyx\rangle .$$

(iii) $\mathrm{Id} - L(x,y)$ *is invertible;*

(iv) $\mathrm{Id} - R(x,y)$ *is invertible.*

If these conditions are satisfied then

(2) $$z = x^y = (\mathrm{Id} - L(x,y))^{-1}x = (\mathrm{Id} - R(x,y))^{-1}x .$$

7.4. *More identities.* From 7.1.3 and AP13 we get

AP14 $$B(x,y) = (\mathrm{Id} - L(x,y))(\mathrm{Id} - R(x,y)) = (\mathrm{Id} - R(x,y))(\mathrm{Id} - L(x,y)) .$$

By AP2 and AP4 we have

$$\langle\langle xyz\rangle uz\rangle + \langle zu\langle xyz\rangle\rangle = \langle xy\langle zuz\rangle\rangle + \langle z\langle yxu\rangle z\rangle$$

which implies

AP15 $$Q(z,\langle xyz\rangle) = L(x,y)Q(z) + Q(z)L(y,x) .$$

By AP4, $\langle xy\langle z\langle yxu\rangle z\rangle\rangle = \langle xy\langle zu\langle xyz\rangle\rangle\rangle$. Using AP1, AP4, and AP8 we obtain

$$\langle xy\langle zu\langle xyz\rangle\rangle\rangle - \langle\langle xyz\rangle u\langle xyz\rangle\rangle = \langle z\langle yxu\rangle\langle xyz\rangle\rangle - \langle zu\langle xy\langle xyz\rangle\rangle\rangle$$
$$= \langle z\langle yx\langle yxu\rangle\rangle z\rangle - \langle zu\langle x\langle yxy\rangle z\rangle\rangle = \langle z\langle\langle yxy\rangle xu\rangle z\rangle - \langle z\langle\langle yxy\rangle xu\rangle z\rangle = 0 .$$

This shows

AP16 $$Q(L(x,y)z) = L(x,y)Q(z)L(y,x) ,$$

and together with AP15 we get

AP17 $$Q((\mathrm{Id} - L(x,y))z) = (\mathrm{Id} - L(x,y))Q(z)(\mathrm{Id} - L(y,x)) .$$

To derive a similar formula for $R(x,y)$, let $k(\varepsilon) = k[T]/(T^3)$ so that $\varepsilon^3 = 0$. Then $(\varepsilon x,y)$ is quasi-invertible (even nilpotent) in $A_{k(\varepsilon)}$, and hence $\mathrm{Id} - \varepsilon L(x,y)$, $\mathrm{Id} - \varepsilon R(x,y)$, and $B(\varepsilon x,y)$ are all invertible, and so are $\mathrm{Id} - \varepsilon L(y,x)$, $\mathrm{Id} - \varepsilon R(y,x)$, and $B(y,\varepsilon x)$. By JP26, AP14, and AP17 we have $B(\varepsilon x,y)Q(z)B(y,\varepsilon x) = (\mathrm{Id} - \varepsilon L_{x,y})(\mathrm{Id} - \varepsilon R_{x,y})Q(z)(\mathrm{Id} - \varepsilon R_{y,x})(\mathrm{Id} - \varepsilon L_{y,x})$ $= Q(B(\varepsilon x,y)z) = Q((\mathrm{Id}-\varepsilon L_{x,y})(\mathrm{Id}-\varepsilon R_{x,y})z) = (\mathrm{Id}-\varepsilon L_{x,y})Q((\mathrm{Id}-\varepsilon R_{x,y})z)(\mathrm{Id}-\varepsilon L_{y,x})$, and cancelling gives $Q((\mathrm{Id}-\varepsilon R_{x,y})z) = (\mathrm{Id} - \varepsilon R_{x,y})Q(z)(\mathrm{Id} - \varepsilon R_{y,x})$. If we expand this and compare the terms at ε and ε^2 the result is

AP18 $$Q(z,\langle zyx\rangle) = R(x,y)Q(z) + Q(z)R(y,x) ,$$

AP19 $$Q(R(x,y)z) = R(x,y)Q(z)R(y,x) .$$

Consequently,

AP20 $$Q((\mathrm{Id} - R(x,y))z) = (\mathrm{Id} - R(x,y))Q(z)(\mathrm{Id} - R(y,x)) .$$

Finally, it follows from AP4, AP6, and AP19 that

$$\langle x\langle z\langle yux\rangle z\rangle y\rangle = \langle x\langle zx\langle uyz\rangle\rangle y\rangle = \langle\langle x\langle uyz\rangle x\rangle zy\rangle = \langle\langle xzy\rangle u\langle xzy\rangle\rangle$$

and therefore

AP21 $$Q(M(x,y)z) = M(x,y)Q(z)M(y,x) .$$

7.5. PROPOSITION. Let A be an alternative pair.

(a) If $(x,y) \in A$ is quasi-invertible then $\lambda(x,y) = (\mathrm{Id} - L_{x,y}, (\mathrm{Id} - L_{y,x})^{-1})$ and $\rho(x,y) = (\mathrm{Id} - R_{x,y}, (\mathrm{Id} - R_{y,x})^{-1})$ are commuting automorphisms of the associated Jordan pair A^J, and we have $\beta(x,y) = \lambda(x,y)\rho(x,y)$ where β is as in 3.9.

(b) *Let* $(x,y) \in A$. *Then* $(L(x,y),-L(y,x))$ *and* $(R(x,y),-R(y,x))$ *are derivations of* A^J *whose sum is the inner derivation* $\delta(x,y) = (D(x,y),-D(y,x))$ *of* A^J *(cf. 3.11)*.

This follows immediately from the identities derived above.

7.6. LEMMA. *Let* $u \in A^+$ *and* $v,w \in A^-$. *Then*

$$L(u,v) : A^+_{<vuw>} \to A^+_w ,$$

$$M(w,v) : A^+_{<vuw>} \to A^-_u ,$$

$$R(u,v) : A^+_{<wuv>} \to A^+_w ,$$

are homomorphisms of Jordan algebras.

The proof follows by an easy computation using AP16, AP19, an AP21.

7.7. PROPOSITION. (Shifting principle) *Let* $x,z \in A^+$, $y,v \in A^-$. *Then quasi-invertibility of* $(x,<yzv>)$, $(<zyx>,v)$, $(<vxy>,z)$, *and* $(<xvz>,y)$ *are all equivalent, and we have*

(1) $$<z,y,x^{<yzv>}> = <zyx>^v ,$$

(2) $$<v,x^{<yzv>},y> = <vxy>^z ,$$

(3) $$<x^{<yzv>},v,z> = <xvz>^y .$$

The proof is similar to the one of 3.5, using 7.6 and the symmetry principle.

7.8. PROPOSITION. *If* B *and* C *are ideals of* A *then* $T(B,C)$ *(defined as in 4.6) is an ideal of* A .

This follows immediately from AP16, AP19, and AP21.

7.9. Radicals. (i) The Jacobson radical of an alternative pair A is defined as $\mathrm{Rad}\, A = (\mathrm{Rad}\, A^+, \mathrm{Rad}\, A^-)$ where $\mathrm{Rad}\, A^\pm$ is the set of properly quasi-invertible elements of $A^\pm$ (cf. 4.1). Hence we have by 7.2(v) that

$$\mathrm{Rad}\, A = \mathrm{Rad}\, A^J \, . \tag{1}$$

(ii) An element $z \in A^\pm$ is called trivial if $Q(z) = M(z,z) = 0$. An ideal I of A is called semiprime if A/I is non-degenerate in the sense that it has no non-zero trivial elements. The small radical of A is the smallest semiprime ideal of A, denoted by $\mathrm{rad}\, A$. The existence of such an ideal is proved as in the Jordan case (4.5).

(iii) For $(x,y) \in A$ we denote by $x^{(n,y)}$ the n-th power of x in the alternative algebra A^+_y. Since powers in an alternative algebra and in the associated Jordan algebra coincide, the powers in A and A^J are the same by 7.2.2. Just as for Jordan pairs (cf. 4.14) there is a largest nil ideal $\mathrm{Nil}(A)$ of A, called the nil radical.

7.10. LEMMA. Every semiprime ideal of A^J is an ideal of A.

Proof. Let $I = (I^+, I^-)$ be a semiprime ideal of A^J. We have to show that I is invariant under left, right, and middle multiplications. Let $z \in I^+$, $x \in A^+$, $y,u \in A^-$. Then by AP15 and AP16,

$$Q(\langle xyz\rangle)u = L(x,y)Q(z)L(y,x)u = -Q(z)\langle yx\langle yxu\rangle\rangle + \{z,\langle yxu\rangle,\langle xyz\rangle\} \, , \tag{1}$$

and this belongs to I^+ since I is a Jordan ideal. By semiprimeness of I, we have $\langle xyz\rangle \in I^+$. Hence also $\langle zyx\rangle = \{xyz\} - \langle xyz\rangle \in I^+$. Finally, let $y \in I^-$

and $x,z \in A^+$ and $u \in A^-$. Applying what we just proved to A^{op} it follows that $\langle yxu\rangle$ and $\langle yx\langle yxu\rangle\rangle$ belong to I^-, and from (1) it follows that $Q(\langle xyz\rangle)u \in I^+$, again since I is a Jordan ideal. This implies $\langle xyz\rangle \in I^+$ by semiprimeness of I. Since the same proof works with + and − interchanged I is an ideal of A.

7.11. THEOREM. Let A be an alternative pair, and let r stand for rad, Nil, or Rad. Then $r(A) = r(A^J)$, and if I is an ideal of A then we have $r(I) = I \cap r(A)$.

Indeed, by 7.10, A and A^J have the same semiprime ideals. Now the theorem follows from 4.2, 4.13, 4.15, and 4.16.

§8. Imbedding into Jordan pairs

8.0. The purpose of this section is to show that alternative pairs are essentially Peirce-1-spaces of Jordan pairs with respect to idempotents whose Peirce-0-spaces are zero (8.2, 8.11). Let e be an idempotent of a Jordan pair V and let $V^\sigma = V_2^\sigma \oplus V_1^\sigma \oplus V_0^\sigma$ be the corresponding Peirce decomposition (cf. 5.4). In the Jordan algebras $V^\sigma_{e^{-\sigma}}$ (cf. 1.9) we have

$$U_x y = Q(x)Q(e^{-\sigma})y, \quad x \circ y = \{x,e^{-\sigma},y\}, \quad x^2 = Q(x)e^{-\sigma}.$$

Also, V_2^σ is a subalgebra of $V^\sigma_{e^-}$ with unit element e^σ, since e^σ is an invertible element of the Jordan pair $V_2(e)$ with inverse $e^{-\sigma}$ (cf. 5.5 and 1.11).

For $x \in V_1^\sigma$ we have $e^\sigma \circ x = x$, by 5.4. Let $a \in V_2^\sigma$, and define $\bar{a} \in V_2^{-\sigma}$ by $\bar{a} = Q(e^{-\sigma})a$. Then by 1.11, $a \to \bar{a}$ is an isomorphism of Jordan algebras, $V_2^\sigma \to V_2^{-\sigma}$, and we have $\bar{\bar{a}} = Q(e^\sigma)Q(e^{-\sigma})a = a$. Note also that $U_a b = Q(a)\bar{b}$, for $a,b \in V_2^\sigma$.

8.1. LEMMA. <u>With the above notations, assume that</u> $V_0 = 0$, <u>and let</u> $a,b \in V_2^\sigma$, $x \in V_1^\sigma$, $y \in V_1^{-\sigma}$. <u>Then the following formulas hold.</u>

(1) $$a \circ (b \circ x) = \{a\bar{b}x\},$$

(2) $$(U_a b) \circ x = (Q(a)\bar{b}) \circ x = a \circ (b \circ (a \circ x)),$$

(3) $$a^2 \circ x = a \circ (a \circ x),$$

(4) $$(a \circ b) \circ x = a \circ (b \circ x) + b \circ (a \circ x),$$

(5) $$\{xya\} = \{x, \bar{a} \circ y, e^\sigma\},$$

(6) $$a \circ \{xye^\sigma\} = \{a \circ x, y, e^\sigma\} + \{x, \bar{a} \circ y, e^\sigma\},$$

(7) $$\overline{\{xye^\sigma\}} = \{yxe^{-\sigma}\},$$

(8) $$\{xye^\sigma\}^2 = \{Q_x y, y, e^\sigma\}.$$

<u>Proof.</u> By JP9 and 5.4.3 we have $a \circ (b \circ x) = D(a,e^{-\sigma})D(b,e^{-\sigma})x = \{b, Q(e^{-\sigma})x, a\} + D(a, Q(e^{-\sigma})b)x = \{a\bar{b}x\}$. This implies by JP11, $a \circ (b \circ (a \circ x)) = \{a\bar{b}\{ae^{-\sigma}x\}\}$ $= D(a,\bar{b})Q(a,x)e^{-\sigma} = \{Q(a)\bar{b}, e^{-\sigma}, x\} + Q(a)\{\bar{b}xe^{-\sigma}\} = (Q(a)\bar{b}) \circ x$ since $\{\bar{b}xe^{-\sigma}\} = 0$ by 5.4(c). For $b = e^\sigma$ we get (3), and (4) follows from (3) by linearizing. By JP10 we have $\{xya\} = \{xy\bar{\bar{a}}\} = D(x,y)Q(e^{-\sigma})\bar{a} = \{x\{ye^\sigma\bar{a}\}e\} - \{Q(e^\sigma)y, \bar{a}, x\}$ $= \{x, \bar{a} \circ y, e^\sigma\}$ since $Q(e^\sigma)y = 0$ by 5.4.3. For (6) we use JP14 and (1) and (5): $a \circ \{xye^\sigma\} = \{a \circ x, y, e^\sigma\} - \{x\{e^{-\sigma}ay\}e\} + \{x, y, a \circ e^\sigma\} = \{a \circ x, y, e^\sigma\} - \{x, \bar{a} \circ y, e^\sigma\}$ $+ 2\{xya\} = \{a \circ x, y, e^\sigma\} + \{x, \bar{a} \circ y, e^\sigma\}$. By JP12 we have $\{yxe^{-\sigma}\} = \{y, x, Q(e^{-\sigma})e^\sigma\}$ $= -Q(e^{-\sigma})\{xye^\sigma\} + \{\{yxe^{-\sigma}\}e^\sigma e^{-\sigma}\} = -\overline{\{xye^\sigma\}} + 2\{yxe^{-\sigma}\}$ since $\{yxe^{-\sigma}\} \in V_2^{-\sigma}$. Finally, by JP20, $\{xye^\sigma\}^2 = Q(\{xye^\sigma\})e^{-\sigma} = Q_x Q_y e^\sigma + Q(e^\sigma)Q_y Q_x e^{-\sigma}$ $+ Q(x,e^\sigma)Q_y\{e^\sigma e^{-\sigma} x\} - \{Q_x y, e^{-\sigma}, Q(e^\sigma)y\} = \{x, Q_y x, e^\sigma\} = \{Q_x y, y, e^\sigma\}$ since $Q_x e^{-\sigma}$

$\in V_0^\sigma = 0$ by 5.4(c).

Note that by (2), V_1^σ is a special Jordan module for the Jordan algebra V_2^σ . Also, (1) - (7) hold without the assumption that $V_0(e) = 0$.

8.2. THEOREM. *Let* e *be an idempotent of the Jordan pair* V *and assume that* $V_0(e) = 0$. *Then the pair of modules* $V_1(e) = (V_1^+, V_1^-)$ *becomes an alternative pair, denoted* A , *with*

(1) $$\langle xyz\rangle = \{\{xye^\sigma\}e^{-\sigma}z\},$$

for $x,z \in A^\sigma = V_1^\sigma$, $y \in A^{-\sigma} = V_1^{-\sigma}$. *The Jordan pair* A^J *associated with* A *is* $V_1(e)$ *(considered as a subpair of* V , *cf. 5.5); i.e., we have*

(2) $$\langle xyx\rangle = \{\{xye^\sigma\}e^{-\sigma}x\} = Q(x)y .$$

Proof. Let $u \in A^\sigma$, $v \in A^{-\sigma}$, and set $a = \{uve^\sigma\}$, and $b = \{xye^\sigma\}$. Then $\langle uv\langle xyz\rangle\rangle + \langle xy\langle uvz\rangle\rangle = a\circ(b\circ z) + b\circ(a\circ z) = (a\circ b)\circ z = (a\circ\{xye^\sigma\})\circ z = \{a\circ x,y,e^\sigma\}\circ z + \{x,\bar a\circ y,e^\sigma\}\circ z = \langle\langle uvx\rangle yz\rangle + \langle x\langle vuy\rangle z\rangle$, by (4),(6) and (7) of 8.1. This proves AP1. Next we prove (2), using JP11 and the fact that $Q(x)e^{-\sigma} \in V_0^\sigma = 0$: $Q(x)y = Q(x)\{e^{-\sigma}e^\sigma y\} = D(x,e^{-\sigma})Q(x,e^\sigma)y - \{Q(x)e^{-\sigma},y,e^\sigma\} = \{xe^{-\sigma}\{xye^\sigma\}\} = \langle xyx\rangle$. Now we have, with a as before, $\langle x\langle vuy\rangle x\rangle = Q(x)(\bar a\circ y) = Q(x)D(y,e^\sigma)\bar a = \{x\bar a\{e^\sigma yx\}\} - \{e^\sigma,y,Q(x)\bar a\} = \{\{xye^\sigma\}\bar a x\} = \{xye^\sigma\}\circ(a\circ x) = \langle xy\langle uvx\rangle\rangle$, by JP12, 8.1.1, 8.1.6, and $Q(x)\bar a \in V_0^\sigma = 0$. This proves AP4, and together with AP1 implies AP2. Finally, AP3 follows from 8.1.3 and 8.1.8: $\langle xy\langle xyz\rangle\rangle = b\circ(b\circ z) = b^2\circ z = \{Q(x)y,y,e^\sigma\}\circ z = \langle\langle xyx\rangle yz\rangle$.

8.3. We shall show in 8.11 that every alternative pair can be obtained in the way described above. As a first step in this direction, we construct a Jordan algebra J as follows (the Jordan pair (J,J) will later play the role of $V_2(e)$, cf.

8.10). Let A be an alternative pair, and let $E = E(A)$ be the associative algebra $\mathrm{End}(A^+) \times \mathrm{End}(A^-)^{op}$ with componentwise operations; i.e.,

$$ab = (a_+b_+, b_-a_-)$$

for $a = (a_+, a_-)$ and $b = (b_+, b_-)$ in E. As usual, $[a,b] = ab - ba$ is the commutator in E. Let E^J be the Jordan algebra associated with E so that $U_a b = aba$, $a \circ b = ab + ba$ in E^J. For $(x,y) \in A$ we set

$$ax = a_+(x), \quad ay = a_-(y) \ .$$

Note that we have $(ab)x = a(bx)$ but $(ab)y = b(ay)$.

Let $J^* = J^*(A)$ be the set of all $a \in E$ such that

(1) $$a\langle xyz\rangle + \langle x,y,az\rangle = \langle ax,y,z\rangle + \langle x,ay,z\rangle \ ,$$

(2) $$a\langle xyx\rangle = \langle ax,y,x\rangle \ ,$$

for all $x,z \in A^\sigma$, $y \in A^{-\sigma}$. Note that (1) and (2) imply (by setting $x = z$)

(3) $$\langle x,y,ax\rangle = \langle x,ay,x\rangle \ .$$

Now we define

(4) $$J = J(A) = \{a \in J^* \mid a^2 \in J^*\} \ .$$

(The fact that J is indeed a subalgebra of E^J is by no means trivial; it will be proved in 8.8).

For $(x,y) \in A$ we define $\ell(x,y)$ and $r(x,y)$ in E by

$$\ell(x,y) = (L(x,y), L(y,x)), \quad r(x,y) = (R(x,y), R(y,x)) \ .$$

8.4. LEMMA. Let $a \in J^*$ and $(x,y), (u,v) \in A$. Then $\ell(x,y) \in J$, and we have the following formulas.

(1) $$\ell(x,y)^2 = \ell(\langle xyx\rangle, y),$$

(2) $$\ell(\langle xvx\rangle,y) = \ell(x,\langle yxv\rangle),$$

(3) $$\ell(\langle xyu\rangle,y) = \ell(x,\langle yuy\rangle),$$

(4) $$a\circ\ell(x,y) = \ell(ax,y) + \ell(x,ay),$$

(5) $$[a,r(x,y)] = r(x,ay) - r(ax,y),$$

(6) $$a\ell(x,y) = \ell(ax,y) - [a,r(x,y)],$$

(7) $$\ell(x,y)a = \ell(x,ay) + [a,r(x,y)].$$

Proof. By AP1 and AP2 we have $\ell(x,y) \in J^*$, and from AP11 we get (1) which shows that $\ell(x,y) \in J$. Formulas (2) and (3) follow from AP6 and AP7, and (4) and (5) are just 8.3.1 in operator form. If we linearize 8.3.2 and 8.3.3 the result is

(8) $$a\langle xyz\rangle + a\langle zyx\rangle = \langle ax,y,z\rangle + \langle az,y,x\rangle,$$

(9) $$\langle x,y,az\rangle + \langle z,y,ax\rangle = \langle x,ay,z\rangle + \langle z,ay,x\rangle,$$

which together with (5) implies (6), and (7) follows from (4) and (6).

8.5. LEMMA. Let $a,b \in J^*$. Then $a\circ b \in J^*$, and for all $(x,y) \in A$ we have

(1) $$ab\ell(x,y) + \ell(x,y)ba = \ell(a(bx),y) + \ell(x,a(by)),$$

(2) $$a\ell(x,y)b + b\ell(x,y)a = \ell(ax,by) + \ell(bx,ay),$$

(3) $$\ell(x,y)a\ell(x,y) = \ell(\langle x,y,ax\rangle,y) = \ell(x,\langle y,x,ay\rangle).$$

Proof. We first prove (1). By (5) - (7) of 8.4 we have

$$\begin{aligned}
&ab\ell(x,y) + \ell(x,y)ba \\
&= a(\ell(bx,y) - [b,r(x,y)]) + (\ell(x,by) + [b,r(x,y)])a \\
&= \ell(a(bx),y) - [a,r(bx,y)] - a[b,r(x,y)] \\
&+ \ell(x,a(by)) + [a,r(x,by)] + [b,r(x,y)]a \\
&= \ell(a(bx,y) + \ell(x,a(by)) + [a,[b,r(x,y)]] - [a,[b,r(x,y)]]
\end{aligned}$$

$$= \ell(a(bx),y) + \ell(x,a(by)).$$

If we add (1) and the identity obtained from it by interchanging a and b we see that $a \circ b$ satisfies 8.4.4, and hence 8.3.1. To show that it also satisfies 8.3.2, replace x by bx in 8.4.8 and set $z = x$. This yields

$$a\langle bx,y,x\rangle + a\langle x,y,bx\rangle = \langle a(bx),y,x\rangle + \langle ax,y,bx\rangle. \tag{4}$$

From 8.3.1 we get

$$b\langle ax,y,x\rangle + \langle ax,y,bx\rangle = \langle b(ax),y,x\rangle + \langle ax,by,x\rangle. \tag{5}$$

Now add (4) and (5) and observe that $a\langle x,y,bx\rangle = a\langle x,by,x\rangle = \langle ax,by,x\rangle$ by 8.3.2 and 8.3.3.Then it follows that $(a\circ b)\langle xyx\rangle = \langle (a\circ b)x,y,x\rangle$, and therefore $a\circ b$ belongs to $\mathcal{J}^*$.

For (2) we again use (5) - (7) of 8.4:

$$a\ell(x,y)b + b\ell(x,y)$$
$$= (\ell(ax,y) - [a,r(x,y)])b + b(\ell(x,ay) + [a,r(x,y)])$$
$$= \ell(ax,by) + [b,r(ax,y)] - [a,r(x,y)]b$$
$$+ \ell(bx,ay) - [b,r(x,ay)] + b[a,r(x,y)]$$
$$= \ell(ax,by) + \ell(bx,ay).$$

Finally, we have similarly, since $\ell(x,y) \in \mathcal{J}$ by 8.4:

$$\ell(\langle x,y,ax\rangle,y) = \ell(\ell(x,y)ax,y) = \ell(x,y)\ell(ax,y) + [\ell(x,y),r(ax,y)]$$
$$=\ell(x,y)a\ell(x,y) + \ell(x,y)[a,r(x,y)] + [\ell(x,y),r(ax,y)]$$
$$= \ell(x,y)a\ell(x,y) + \ell(x,y)r(x,ay) - r(ax,y)\ell(x,y).$$

Thus we have to show that $\ell(x,y)r(x,ay) = r(ax,y)\ell(x,y)$, or, equivalently, $\langle xy\langle z,ay,x\rangle\rangle = \langle\langle xyz\rangle,y,ax\rangle$. Now $\langle xy\langle z,ay,x\rangle\rangle = \langle x\langle ay,z,y\rangle x\rangle = \langle x,a\langle yzy\rangle,x\rangle$ $= \langle x,\langle yzy\rangle,ax\rangle = \langle\langle xyz\rangle,y,ax\rangle$, using AP4 and AP7. This proves the first equality of (3), and the second one follows from $\ell(\langle x,y,ax\rangle,y) = \ell(\langle x,ay,x\rangle,y)$ $= \ell(x,\langle y,x,ay\rangle)$, by 8.4.2.

8.6. COROLLARY. (a) $J(A)$ is a k-submodule of E. For every extension R of k, there is a natural homomorphism $\phi: J(A)\otimes R \to J(A\otimes R)$. If R is free (more generally, projective) as a k-module then ϕ is injective.

(b) If $1/2 \in k$ then $J = J^*$ is a (linear) Jordan algebra.

Proof. (a) Obviously, J is closed under scalar multiples. If $a,b \in J$ then $(a+b)^2 = a^2 + a\circ b + b^2$. By definition, a^2 and b^2 belong to J^*, and by 8.5, so does $a\circ b$. Hence $J = J(A)$ is a submodule of $E = E(A)$. Since J^* is defined by linear equations we clearly have a natural homomorphism $J^*(A)\otimes R \to J^*(A\otimes R)$. Let $a_i \in J(A)$ and $\lambda_i \in R$. Then $\phi(\Sigma\, a_i\otimes\lambda_i) = \Sigma\lambda_i(a_i)_R$ where $(a_i)_R$ is the R-linear endomorphism of $A\otimes R$ induced by a_i. It follows that $\phi(\Sigma\, a_i\otimes\lambda_i)^2 = \Sigma\lambda_i^2(a_i^2)_R + \Sigma\, \lambda_i\lambda_j(a_i\circ a_j)_R \in J^*(A\otimes R)$ since $a_i\circ a_j \in J^*(A)$ by 8.5. Hence ϕ maps $J(A)\otimes R$ into $J(A\otimes R)$. Finally, ϕ may be factored in the obvious way as

$$J(A)\otimes R \to E(A)\otimes R \to E(A\otimes R),$$

and both these maps are injective if R is a projective k-module.

(b) This is obvious.

8.7. LEMMA. Let $a \in J$ and $b \in J^*$. Then

(1) $$a\ell(x,y)a = \ell(ax,ay),$$

(2) $$Q(ax)y = a(Q(x)(ay)),$$

and $U_a b = aba$ belongs to J^*.

Proof. By 8.3.1 and (6) and (7) of 8.4 we have

$$\begin{aligned}\ell(ax,ay) &= a\ell(ax,y) + \ell(ax,y)a - \ell(a^2x,y)\\ &= a^2\ell(x,y) + a[a,r(x,y)] + a\ell(x,y)a + [a,r(x,y)]a\\ &\quad - a^2\ell(x,y) - [a^2,r(x,y)] = a\ell(x,y)a\,.\end{aligned}$$

The second formula follows from 8.3.1 - 8.3.3:

$$aQ(x)ay = a\langle x,ay,x\rangle = a\langle x,y,ax\rangle = \langle ax,y,ax\rangle + \langle x,ay,ax\rangle$$
$$- \langle x,y,a^2x\rangle = \langle ax,y,ax\rangle = Q(ax)y .$$

To show that $U_a b \in J^*$ we first interchange a and b in 8.5.1 and multiply this on the left (resp. on the right) with a and use (1) to obtain

$$aba\ell(x,y) + \ell(ax,ay)b = a\ell(b(ax),y) + a\ell(x,b(ay)),$$
$$b\ell(ax,ay) + \ell(x,y)aba = \ell(b(ax),y)a + \ell(x,b(ay))a.$$

Addition gives

$$(aba)\circ\ell(x,y) + b\circ\ell(ax,ay) = a\circ\ell(b(ax),y) + a\circ\ell(x,b(ay)),$$

and by 8.4.4 we get

$$(aba)\circ\ell(x,y) = \ell((aba)x,y) + \ell(b(ax),ay) + \ell(ax,b(ay))$$
$$+ \ell(x,(aba)y) - \ell(b(ax),y) - \ell(ax,b(ay))$$
$$= \ell((aba)x,y) + \ell(x,(aba)y),$$

which shows that aba satisfies 8.4.4 and hence 8.3.1. Furthermore,

$$aba\langle xyx\rangle = ab\langle ax,y,x\rangle = \langle abax,y,x\rangle + \langle ax,aby,x\rangle - \langle ax,y,bax\rangle$$

by 8.3.2 and 8.5.1, and

$$\langle ax,y,bax\rangle = \langle ax,by,ax\rangle = a\langle x,aby,x\rangle = \langle ax,aby,x\rangle,$$

by (2), 8.3.3, and 8.3.2. This proves $aba \in J^*$.

8.8. THEOREM. Let A be an alternative pair over k, and let $J = J(A) \subset E$ be defined as in 8.3. Then J is a (quadratic) Jordan subalgebra of E containing the unit element of E, and the k-submodule F of E spanned by all $\ell(x,y)$, $(x,y) \in A$, is an ideal of J.

We call J the Jordan structure algebra of A and F the inner structure algebra.

Proof. Obviously $1 = (\mathrm{Id},\mathrm{Id}) \in J$, and by 8.6, J is a submodule of E. Moreover, if a and b are in J then aba and ba^2b are in J^* by 8.7. Hence also $(aba)^2 = a(ba^2b)a$ belong to J^* and therefore $aba \in J$. This shows that J is a Jordan subalgebra of E. By 8.4 we have $F \subset J$, and from 8.5.3 and 8.7.1 it follows that F is an ideal of J.

8.9. Examples. (a) Let $A = A(M,R,\Phi)$ be as in 6.4. For $a \in R$ let $(a_+,a_-) \in \mathrm{End}(M^+) \times \mathrm{End}(M^-)$ be defined by $a_+(x) = ax$, $a_-(y) = ya$. Then it is immediate from the definitions that the map $a \to (a_+,a_-)$ is a homomorphism from the Jordan algebra R^J associated with R into $J = J(A)$, and that $\Phi(x,y)$ is mapped into $\ell(x,y)$ so that the ideal $\Phi(M)$ of R is mapped onto F. If R is simple with unity and $\Phi(M) \neq 0$ it follows that $F = J \cong R^J$.

(b) Let R be an alternative algebra over k with unit element 1, and let (R,R^{op}) be the corresponding alternative pair (cf. 6.5). For an element $a = (a_+, a_-) \in J^*(R,R^{op})$ let $u = a_+(1) \in R$. Then it follows easily from the definitions that $a = (L_u,R_u)$ where L_u (resp. R_u) is the left (resp. right) multiplication with u in R, and we have $J^* = J = F \cong R^J$, the Jordan algebra associated with R.

(c) Let K be an extension field of k, and consider an alternative pair $A = A(X,K,\alpha)$ as in 6.6. Assume that the alternating bilinear form α is not zero. Then it can be seen that $a = (a_+,a_-) \in J = J^*$ if and only if

(1) $$\alpha(a_+(x),x) = 0 \quad \text{for all} \quad x \in X,$$

and

(2) $$J_+a_- = a_+J_+ .$$

Thus J is isomorphic with the Jordan algebra of symplectic symmetric linear transformations of X with respect to α (i.e., those linear transformations of X which satisfy (1)). If α is non-degenerate then F turns out to be the set of transformations of finite rank in J . In particular, if X is finite-dimensional over K then we have $F = J$. The details are left to the reader as an exercise.

8.10. PROPOSITION. Let $A = V_1(e)$ be as in 8.2. For $a \in V_2^+$ define $a' = (a'_+, a'_-) \in E = \mathrm{End}(A^+) \times \mathrm{End}(A^-)$ by $a'_+(x) = a \circ x$, $a'_-(y) = \bar{a} \circ y$ (notation as in 8.0). Then $f: a \mapsto a'$ is a homomorphism from V_2^+ (considered as a subalgebra of $V^+_{e^-}$ as in 8.0) into the structure Jordan algebra J of A . For $(x,y) \in A$ we have $f(\{xye^+\}) = \ell(x,y)$ and hence f maps $\{V_1^+, V_1^-, e^+\}$ onto the inner structure algebra of A .

Proof. By 8.1.2, $f: V_2^+ \to J$ is a homomorphism of Jordan algebras. By 8.2.1 and 8.1.7, we have, for $(x,y), (z,w) \in A$,

$$\{xye^+\} \circ z = \{\{xye^+\}e^- z\} = \langle xyz\rangle = L(x,y)z,$$

$$\overline{\{xye^+\}} \circ w = \{\{yxe^-\}e^+ w\} = \langle yxw\rangle = L(y,x)w,$$

which proves $f(\{xye^+\}) = \ell(x,y)$. Applying f to 8.1.6 we get $a' \circ \ell(x,y) = \ell(a'x,y) + \ell(x,a'y)$, and hence $a' = f(a)$ satisfies 8.3.1. Also, $\langle x,\bar{a}\circ y,x\rangle = Q(x)(\bar{a}\circ y) = Q(x)D(y-e^+)\bar{a} = \{x\bar{a}\{e^+yx\}\} - \{e^+,y,Q(x)\bar{a}\} = \{\{xye^+\}\bar{a}x\} = \{xye^+\}\circ(a\circ x) = \langle x,y,a\circ x\rangle$ by JP12, 8.1.1, and 8.1.6 since $Q(x)\bar{a} \in V_0^+ = 0$. Similarly, $\langle y,a\circ x,y\rangle = \langle y,x,\bar{a}\circ y\rangle$, and hence a' satisfies 8.3.3. Together with 8.3.1 this implies 8.3.2, and therefore $a' \in J^*$. Since $(a^2)' = (a')^2$ we also have $(a')^2 \in J^*$ and therefore $a' \in J$.

8.11. We keep the notations of 8.8. With the Jordan structure algebra J of A we can associate a Jordan pair (J,J) as usual (1.6). For our purposes, it will be convenient to realize this Jordan pair as follows. Let $E^+ = E = \mathrm{End}(A^+) \times$

$\mathrm{End}(A^-)^{op}$ as in 8.3, and let $E^- = \mathrm{End}(A^-) \times \mathrm{End}(A^+)^{op}$. Then the exchange map $a = (a_\sigma, a_{-\sigma}) \mapsto \bar{a} = (a_{-\sigma}, a_\sigma)$ is an antiisomorphism from E^+ to E^- and conversely, and we have $\bar{\bar{a}} = a$. Now let $J^+ = J$, and set $J^- = \overline{J^+} = \{(a_-, a_+) \in E^- \mid (a_+, a_-) \in J\}$. Then J^- is a Jordan subalgebra of E^-, and $a \mapsto \bar{a}$ induces an isomorphism $J^\sigma \to J^{-\sigma}$ of Jordan algebras. Expressed in a different way, J^- is the structure Jordan algebra of A^{op}, and the isomorphism $J^+ \to J^-$ is induced by the canonical antiisomorphism $A \to A^{op}$. Now (J^+, J^-) is a Jordan pair, isomorphic with (J, J), by defining quadratic operators by

(1) $$Q(a)b = a\bar{b}a,$$

for $a \in J^\sigma$, $b \in J^{-\sigma}$. The unit element e^σ of J^σ is $e^\sigma = (\mathrm{Id}_{A^\sigma}, \mathrm{Id}_{A^{-\sigma}})$ and hence

(2) $$e = (e^+, e^-)$$

is an idempotent of the Jordan pair (J^+, J^-). We let J^σ act on A^σ by projection onto the first factor; i.e., for $a = (a_\sigma, a_{-\sigma}) \in J^\sigma$ and $x \in A^\sigma$ we set

(3) $$ax = a_\sigma(x).$$

If $y \in A^{-\sigma}$ we therefore have $\bar{a}y = a_{-\sigma}(y)$. Furthermore, define

(4) $$\ell_\sigma(x,y) = (L_\sigma(x,y), L_{-\sigma}(y,x))$$

for $(x,y) \in A^\sigma \times A^{-\sigma}$, and let $F^+ = F$, $F^- = \overline{F^+}$ so that F^σ is spanned by all $\ell_\sigma(x,y)$ and (F^+, F^-) is an ideal of the Jordan pair (J^+, J^-). By (4), we have

(5) $$\ell_\sigma(x,y)z = \langle xyz\rangle, \quad \overline{\ell_\sigma(x,y)} = \ell_{-\sigma}(y,x),$$

which implies

(6) $$\langle A^\sigma, A^{-\sigma}, A^\sigma\rangle = F^\sigma.A^\sigma.$$

8.12. THEOREM. (Standard imbedding) *Let* $\mathcal{A} = (A^+, A^-)$ *be an alternative pair over* k . *With the above notations, let* $\mathcal{W}^\sigma = \mathcal{J}^\sigma \oplus A^\sigma$ *(direct sum of* k-*modules), and define quadratic maps* $Q_\sigma\colon \mathcal{W}^\sigma \to \mathrm{Hom}(\mathcal{W}^{-\sigma}, \mathcal{W}^\sigma)$ *by*

(1) $$Q_\sigma(a\oplus x)(b\oplus y) = (a\bar{b}a + \ell_\sigma(x,\bar{a}y)) \oplus (a(\bar{b}x) + \langle xyx\rangle),$$

where $a \in \mathcal{J}^\sigma$, $b \in \mathcal{J}^{-\sigma}$, $x \in A^\sigma$, $y \in A^{-\sigma}$. *Then* $\mathcal{W} = (\mathcal{W}^+, \mathcal{W}^-)$ *is a Jordan pair, called the standard imbedding of* $\mathcal{A}$, *with the following properties.*

(a) $e = (e^+, e^-)$ *(as in 8.11.2) is an idempotent of* $\mathcal{W}$, *and the Peirce spaces of* $\mathcal{W}$ *with respect to* e *are*

(2) $$\mathcal{W}_2^\sigma(e) = \mathcal{J}^\sigma, \quad \mathcal{W}_1^\sigma(e) = A^\sigma, \quad \mathcal{W}_0^\sigma(e) = 0.$$

For $x,z \in A^\sigma$, $y \in A^{-\sigma}$ *we have*

(3) $$\langle xyz\rangle = \{\{xye^\sigma\}e^{-\sigma}z\},$$

i.e., $\mathcal{A} = \mathcal{W}_1(e)$ *as an alternative pair as in 8.2.*

(b) *Conversely, let* $V = V_2 \oplus V_1$ *be a Jordan pair and* $\mathcal{A} = V_1$ *the associated alternative pair as in 8.2. Define* $f_\sigma\colon V_2^\sigma \to \mathcal{J}^\sigma$ *by* $f_+(a) = f(a)$ *as in 8.10,* $f_-(a) = \overline{f(\bar{a})}$, *and extend* f_σ *to a map* $h_\sigma\colon V^\sigma \to \mathcal{W}^\sigma$ *by* $h_\sigma(a \oplus x) = f_\sigma(a) \oplus x$ $(a \oplus x \in V_2 \oplus V_1)$. *Then* $h = (h_+, h_-)$ *is a homomorphism of Jordan pairs which is the identity on* V_1 . *In general,* h *is neither injective nor surjective.*

(c) *If* $\mathcal{A}$ *is non-degenerate (resp. semisimple) then so are the standard imbedding and the Jordan structure algebra of* $\mathcal{A}$.

Proof. We begin by showing that $\mathcal{W}$ is a Jordan pair. Since the proof consists of a brute force computation which isn't very illuminating the reader may be well advised to skip it at first reading. Throughout, let a,b,c,d denote elements of

$J^{\pm}$ and x,y,z,u,v elements of $A^{\pm}$. From (1) we get the formulas

(4) $$Q(x)a = Q(a)x = 0,$$

(5) $$\{xya\} = \{ayx\} = \ell_\sigma(x,\bar{a}y),$$

(6) $$\{abx\} = \{xba\} = a(\bar{b}x).$$

Also it is clear that (J^+,J^-) and (A^+,A^-) are subpairs of $\mathcal{W}$ which by 8.10 and 7.1 are Jordan pairs. We have to verify JP1 - JP3 in all scalar extensions of k . By 1.2, it suffices to do this for extensions R of k which are free as k-modules. Using the fact that A_R is again an alternative pair as well as 8.6(a) we see that we may assume $R = k$. For JP1 we have to show

$$\{a\oplus x, b\oplus y, Q(a\oplus x)(c\oplus z)\} = Q(a\oplus x)\{b\oplus y, a\oplus x, c\oplus z\}.$$

Expanding this and using (1), (4), and 8.11 one sees that the following identities remain to be checked.

(7) $$\{ab\{xza\}\} = Q(a)\{bxz\} + \{x\{baz\}a\},$$

(8) $$\{xb\{xza\}\} + \{ab\langle xzx\rangle\} = \{a\{bxz\}x\} + \langle x\{baz\}x\rangle,$$

(9) $$\{ay\{acx\}\} + \{x,y,Q(a)c\} = Q(a)\{yxc\} + \{x\{yac\}a\},$$

(10) $$\{ay\langle xzx\rangle\} + \{xy\{xza\}\} = \{x\{yxz\}a\},$$

(11) $$\{xy\{acx\}\} = \{a\{yxc\}x\} + \langle x\{yac\}x\rangle.$$

By definition, (7) is equivalent with

$$a\bar{b}\ell_\sigma(x,\bar{a}z) + \ell_\sigma(x,\bar{a}z)\bar{b}a = a\overline{\ell_{-\sigma}(z,\bar{b}x)}a + \ell_\sigma(x,\bar{a}\bar{b}\bar{a}z).$$

Using 8.5.2, 8.7.1, and 8.11 we see that this holds. Similarly, (8) follows from 8.3.2 and 8.3.3, (9) follows from 8.7.1, (10) from 8.4.2 and a linearization of 8.3.2. Finally, (11) is equivalent with

$$\langle x,y,acx\rangle + \langle acx,y,x\rangle = a\langle cx,y,x\rangle + \langle x,\bar{c}\bar{a}y,x\rangle.$$

Using 8.3.1 on the first term and 8.3.3 on the second term of the right hand side and setting $u = cx$ this is equivalent with

$$\langle x,y,au\rangle + \langle u,y,ax\rangle = \langle u,\bar{a}y,x\rangle + \langle x,\bar{a}y,u\rangle$$

which is a linearization of 8.3.3.

Next we prove JP2:

$$\{Q(a\oplus x)(b\oplus y), b\oplus y, c\oplus z\} = \{a\oplus x, Q(b\oplus y)(a\oplus x), c\oplus z\}.$$

Proceeding as before, we end up with the following cases:

$$\{\{xya\}bc\} + \{\{abx\}yc\} = \{a\{yxb\}c\} + \{x\{bay\}c\}, \tag{12}$$

$$\{\{xya\}bz\} + \{\{abx\}yz\} = \{a\{yxb\}z\} + \{x\{bay\}z\}, \tag{13}$$

$$\{\{xya\}yz\} = \{a\langle yxy\rangle z\}, \tag{14}$$

$$\{\langle xyx\rangle bc\} = \{x\{yxb\}c\}, \tag{15}$$

$$\{\langle xyx\rangle yc\} = \{x\langle yxy\rangle c\}. \tag{16}$$

By definition, (12) is equivalent with

$$\ell_\sigma(x,\bar{a}y)\bar{b}c + c\bar{b}\ell_\sigma(x,\bar{a}y) + \ell_\sigma(a\bar{b}x,\bar{c}y)$$
$$= a\ell_\sigma(\bar{b}x,y)c + c\ell_\sigma(\bar{b}x,y)a + \ell_\sigma(x,\bar{c}b\bar{a}y),$$

which follows easily from 8.5.1 and 8.5.2. Similarly, (13) is equivalent with

$$\langle x,\bar{a}y,bz\rangle + \langle abx,y,z\rangle + \langle z,y,abx\rangle$$
$$= a\langle bx,y,z\rangle + \langle x,\bar{b}\bar{a}y,z\rangle + \langle z,\bar{b}\bar{a}y,x\rangle.$$

We use 8.3.1 on the first term and the linearization of 8.3.3 on the second and third term of the right hand side to transform this into the linearization of 8.3.3. Finally, (14) - (16) follow immediately from the definitions and 8.3.2 and 8.4.2.

The verification of JP3 is considerably simplified by the following

8.13. LEMMA. <u>Let</u> $V = (V^+, V^-)$ <u>be a pair of</u> k-<u>modules with quadratic maps</u> $Q : V^\sigma \to \mathrm{Hom}(V^{-\sigma}, V^\sigma)$ <u>satisfying JP1, JP2, and</u>

JP3' $$Q(Q(x)y)y = Q(x)Q(y)Q(x)y$$

<u>in all scalar extensions</u>. <u>Then</u> V <u>is a Jordan pair</u>.

<u>Proof</u>. We linearize JP3' and get

$$Q(Q_x y)z \; + \; Q(Q_x y, Q_x z)y \; = \; Q_x Q_y Q_x z + Q_x Q(y,z) Q_x y.$$

Now $Q(Q_x y, Q_x z)y = \{Q_x y, y, Q_x z\} = \{x, Q_y x, Q_x z\} = Q_x\{Q_y x, x, z\} = Q_x\{y, Q_x y, z\}$ $= Q_x Q(y,z) Q_x y$ by JP2 and JP1, and the Lemma follows.

We continue the proof of 8.12 and verify JP3. By the Lemma, we only have to show:

$$Q(Q(a\oplus x)(b\oplus y))(b\oplus y) = Q(a\oplus x)Q(b\oplus y)Q(a\oplus x)(b\oplus y).$$

Expanding this in the usual way and using the definitions, we end up with

(17) $$a\bar{b}a\bar{b}\ell_\sigma(x,\bar{a}y) + \ell_\sigma(x,\bar{a}y)\bar{b}a\bar{b}a + \ell_\sigma(a\bar{b}x,\bar{a}b\bar{a}y)$$
$$= a\ell_\sigma(\bar{b}a\bar{b}x,y)a + \ell_\sigma(x,\bar{a}b\bar{a}b\bar{a}y) + a\bar{b}\ell_\sigma(x,\bar{a}y)\bar{b}a,$$

(18) $$\ell_\sigma(x,\bar{a}y)\bar{b}\ell_\sigma(x,\bar{a}y) + \ell_\sigma(a\bar{b}x,\langle\bar{a}y,x,y\rangle) + \ell_\sigma(\langle xyx\rangle,\bar{a}b\bar{a}y)$$
$$= \ell_\sigma(x,\bar{a}\langle y,a\bar{b}x,y\rangle) + a\ell_\sigma(\bar{b}\langle xyx\rangle,y)a + \ell_\sigma(x,\bar{a}b\langle\bar{a}y,x,y\rangle),$$

(19) $$\ell_\sigma(\langle xyx\rangle,\langle\bar{a}y,x,y\rangle) = \ell_\sigma(x,\bar{a}\langle y\langle xzx\rangle y\rangle),$$

(20) $$a\bar{b}a\bar{b}\langle xyx\rangle + \langle x,\bar{a}y,\bar{b}a\bar{b}x\rangle + \langle a\bar{b}x,y,a\bar{b}x\rangle$$
$$= a\langle\bar{b}a\bar{b}x,y,x\rangle + \langle x,b\bar{a}b\bar{a}y,x\rangle + a\bar{b}\langle x,\bar{a}y,\bar{b}x\rangle,$$

(21) $$\langle x,\bar{a}y,\bar{b}\langle xyx\rangle\rangle + \langle a\bar{b}x,y,\langle xyx\rangle\rangle + \langle\langle xyx\rangle,y,a\bar{b}x\rangle$$
$$= \langle x\langle y,a\bar{b}x,y\rangle x\rangle + a\langle\bar{b}\langle xyx\rangle,y,x\rangle + \langle x,b\langle\bar{a}y,x,y\rangle,x\rangle.$$

The first four identities are fairly straightforward consequences of 8.3 – 8.7, so we omit the details. For the first term of (21) we use 8.3.2, AP4, and 8.3.3:

$$\langle x,\bar{a}y,\bar{b}\langle xyx\rangle\rangle = \langle x,\bar{a}y,\langle\bar{b}x,y,x\rangle\rangle = \langle x\langle y,\bar{b}x,\bar{a}y\rangle x\rangle = \langle x\langle y,a\bar{b}x,y\rangle x\rangle.$$

If we set $u = \bar{b}x$, $v = \langle yxy\rangle$ and use 8.3.2, 8.3.3, AP2 and AP7 then (21) reduces to

$$\langle au,v,x\rangle + \langle x,v,au\rangle = a\langle uvx\rangle + \langle x,\bar{a}v,u\rangle.$$

By 8.3.1, $a\langle uvx\rangle = \langle au,v,x\rangle + \langle u,\bar{a}v,x\rangle - \langle u,v,ax\rangle$ so we are left with

$$\langle x,v,au\rangle + \langle u,v,ax\rangle = \langle x,\bar{a}v,u\rangle + \langle u,\bar{a}v,x\rangle$$

which is a consequence of 8.3.3. Thus we have verified that $\mathcal{W}$ is a Jordan pair.

From the definitions it is obvious that e is an idempotent and that (2) holds. By (5) and (6) and 8.11.5,

$$\{\{xye^{\sigma}\}e^{-\sigma}z\} = \{\ell_{\sigma}(x,y),e^{-\sigma},z\} = \ell_{\sigma}(x,y)z = \langle xyz\rangle,$$

which proves (3). Now (b) follows immediately from the definitions and 8.8.

Finally, let r stand for Rad or rad . By 5.8 and 5.10, it suffices to show that $r(W_2) = 0$ provided $r(A) = 0$. If $a = (a_{\sigma}, a_{-\sigma}) \in r(W_2^{\sigma})$ then $\{ae^{-\sigma}x\} = ax = a_{\sigma}(x) \in r(A^{\sigma})$ for all $x \in A^{\sigma}$ since $r(W)$ is an ideal of W . Hence $a_{\sigma} = 0$, and similarly $\{\bar{a}e^{\sigma}y\} = \bar{a}y = a_{-\sigma}(y) = 0$ for all $y \in A^{-\sigma}$ implies $a_{-\sigma} = 0$. Therefore $a = 0$ and we have $r(W_2) = 0$. This completes the proof.

For relations between ideals of A and W see §12. Here we compute the standard imbeddings for the three types of alternative pairs considered in 6.6 - 6.8.

8.14. Let $A = A(M,R,\Phi)$ be as in 6.4, and assume that R is simple with unity and $\Phi(M) \neq 0$ so that we can identify R^J and J (cf. 8.9(a)). Then one verifies immediately that the standard imbedding of A is the Jordan pair $A(N,R,\Psi)^J$ associated with the associative pair $A(N,R,\Psi)$ constructed from the R-modules $N^+ = R \oplus M^+$ and $N^- = R \oplus M^-$ (where R is considered as a left and right R-module in the canonical way) and the bilinear form $\Psi\colon N^+ \times N^- \to R$ given by $\Psi(a\oplus x, b\oplus y) = ab + \Phi(x,y)$.

8.15. Let $(R,R^{op}) = (R^+,R^-)$ be the alternative pair associated with a unital alternative algebra R , let $L_{\sigma}(x)$ be left multiplication with x in R^{σ} , and denote by $x \mapsto \bar{x}$ the canonical anti-isomorphism $R^+ \to R^-$ and conversely. By 8.9 (b) and 8.11 we have $a_{\sigma} = (L_{\sigma}(u), L_{-\sigma}(\bar{u}))$ for $a \in J^{\sigma}$ where $u = a_{\sigma}(1)$. Also $\ell_{\sigma}(x,y) = (L_{\sigma}(x\bar{y}), L_{-\sigma}(y\bar{x}))$. The map $a \mapsto a_{\sigma}(1)$ is an isomorphism between J^{σ} and $(R^{\sigma})^J$, the Jordan algebra associated with R^{σ} , by means of which we will identify these two algebras. Then we get for the standard imbedding by 8.12.1:

(1) $$Q(u+x)(v+y) = (u\bar{v}u + x(\bar{y}u)) + (u(\bar{v}x) + x\bar{y}x).$$

If we set $X = (u,x)$ and $Y = (v,y)$ then we may write this in matrix notation as

(2) $$Q(X)Y = X(Y^*X)$$

where $Y^* = {}^t\bar{Y}$ (cf. 0.5). Thus the standard imbedding is $\mathcal{W} = (M_{1,2}(\mathcal{R}), M_{1,2}(\mathcal{R}^{op}))$ with product given by (2).

8.16. Let K be an extension field of k and consider an alternative pair $\mathcal{A} = \mathcal{A}(\mathcal{X},K,\alpha)$ as in 6.6. Assume that $\mathcal{X}$ is finite-dimensional over K and α is non-degenerate. Then we can choose a basis $b_1,\dots,b_{2n}$ of $\mathcal{X}$ over K such that the matrix of α with respect to this basis is the standard symplectic matrix

$$S = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$$

where 1_n is the $n \times n$ unit matrix. If we identify $\mathcal{X}$ with K^{2n} (whose elements we write as column vectors) then

(1) $$\alpha(x,y) = {}^txSy,$$

where ${}^tx$ is the transpose of x. Let now $\mathcal{A}^+ = \mathcal{A}^- = \mathcal{X}$, and let $J_\sigma\colon \mathcal{A}^{-\sigma} \to \mathcal{A}^\sigma$ and $\alpha_\sigma\colon \mathcal{A}^\sigma \times \mathcal{A}^\sigma \to K$ be as in 6.6. Consider the basis $c_1,\dots c_{2n}$ of $\mathcal{A}^-$ given by $c_i = -J_-(b_{n+i})$, $c_{n+i} = J_-(b_i)$, $i = 1,\dots,n$. Then the matrix of α_- with respect to $c_1,\dots,c_{2n}$ is S, and if we identify $\mathcal{A}^\sigma$ with K^{2n} by means of these bases then $J_\sigma\colon \mathcal{A}^{-\sigma} \to \mathcal{A}^\sigma$ is given by matrix multiplication with $-S$ on the left. For $x,z \in \mathcal{A}^\sigma$, $y \in \mathcal{A}^{-\sigma}$ we obtain, in matrix notation,

$$\langle xyz\rangle = x({}^tzS(-Sy)) - (Sy)({}^txSz) = x\,{}^tyz + Sy({}^tx)S^{-1}z,$$

so that

(2) $$L(x,y) = x({}^t y) + Sy({}^t x)S^{-1}.$$

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a $2n \times 2n$ matrix with coefficients in K, divided into four blocks of $n \times n$ matrices. Then it is easily verified that A is symplectic symmetric, i.e., satisfies $\alpha(Ax,x) = 0$ for all $x \in K^{2n}$, if and only if b and c are alternating and $d = {}^t a$. This implies

(3) $$SAS^{-1} = {}^t A .$$

(If $\mathrm{char}(K) \neq 2$ then (3) is sufficient for A to be symplectic symmetric). According to 8.9(c), (A_+,A_-) is in the structure Jordan algebra $\mathcal{J}$ of $\mathcal{A}$ if and only if $A = A_+$ is symplectic symmetric and $A_- = J_+^{-1}A_+J_+ = (-S)^{-1}A(-S) = {}^t A$. If we identify $\mathcal{J}^{\pm}$ with the symplectic symmetric matrices and $\mathcal{A}^{\pm}$ with K^{2n} then, by 8.12, we have for the standard imbedding $\mathcal{W}$ of $\mathcal{A}$:

(4) $$Q(A\oplus x)(B\oplus y) = (A{}^t BA + x{}^t yA + A{}^t Ay{}^t xS^{-1}) \oplus (A{}^t Bx + x{}^t yx) .$$

Now if $A = \begin{pmatrix} a & b \\ c & {}^t a \end{pmatrix}$ is symplectic symmetric then $SA = \begin{pmatrix} c & {}^t a \\ -a & -b \end{pmatrix}$ is an alternating $2n \times 2n$ matrix. Consider the map $\phi\colon A \oplus x \to \begin{pmatrix} SA & Sx \\ {}^t xS & 0 \end{pmatrix}$ from $\mathcal{W}^\sigma$ into the space $A_{2n+1}(K)$ of alternating $(2n+1) \times (2n+1)$ matrices. Clearly, ϕ is a vector space isomorphism, and one checks that (ϕ,ϕ) is actually an isomorphism of Jordan pairs, where the pair $(A_{2n+1}(K),A_{2n+1}(K))$ is considered as a Jordan pair with

(5) $$Q(X)Y = X{}^t YX.$$

§9. Peirce decomposition

9.1. Let R be an alternative algebra, and let e be an idempotent of R. Recall that we have the Peirce decomposition

$$R = R_{11} \oplus R_{10} \oplus R_{01} \oplus R_{00}$$

where

$$R_{ij} = \{x \in R \mid ex = ix, \quad xe = jx\}.$$

The multiplication rules for the Peirce spaces are

(1) $$R_{ij}R_{jm} \subset R_{im},$$

(2) $$R_{ij}R_{ij} \subset R_{ji}, \quad \text{for } i \neq j,$$

while all other products are zero. For the proof, see, e.g., Schafer[1].

Now let $A = (A^+, A^-)$ be an alternative pair. An idempotent of A is a pair $e = (e^+, e^-) \in A$ such that

(3) $$\langle e^\sigma, e^{-\sigma}, e^\sigma \rangle = e^\sigma.$$

Thus e is also an idempotent of the Jordan pair A^J associated with A.

9.2. PROPOSITION. *Let e be an idempotent of the alternative pair A.*

(a) *There is a Peirce decomposition*

$$A = A_{11} \oplus A_{10} \oplus A_{01} \oplus A_{00}$$

where $A_{ij} = (A^+_{ij}, A^-_{ij})$ *and*

(1) $$A^\sigma_{ij} = \{x \in A^\sigma \mid \langle e^\sigma e^{-\sigma} x \rangle = ix, \quad \langle x e^{-\sigma} e^\sigma \rangle = jx\}.$$

(b) <u>The Peirce decomposition of the Jordan pair</u> A^J <u>is given by</u>

$$A_2^J = A_{11}, \quad A_1^J = A_{10} \oplus A_{01}, \quad A_0^J = A_{00},$$

<u>i.e.,</u>

$$A_n^J = \sum_{i+j=n} A_{ij} .$$

(c) <u>Let</u> $\mathcal{B}^\sigma$ <u>denote the alternative algebra</u> $A^\sigma_{e^{-\sigma}}$ <u>with product</u> $xy = \langle xe^{-\sigma}y\rangle$ <u>(cf. 6.10). Then</u> e^σ <u>is an idempotent of</u> $\mathcal{B}^\sigma$, <u>and the Peirce spaces of</u> $\mathcal{B}^\sigma$ <u>with respect to</u> e^σ <u>are</u> $\mathcal{B}^\sigma_{ij} = A^\sigma_{ij}$.

(d) <u>The map</u> $a \mapsto \bar{a} = \langle e^{-\sigma}ae^{-\sigma}\rangle$ <u>is an antiisomorphism between</u> $\mathcal{B}^\sigma_{11}$ <u>and</u> $\mathcal{B}^{-\sigma}_{11}$ <u>such that</u> $\bar{\bar{a}} = a$. <u>For</u> $x,y \in A^\sigma$, $a \in A^{-\sigma}_{11}$ <u>we have</u>

(2) $$\langle xay\rangle = (x\bar{a})y.$$

<u>Proof</u>. Let $E_\sigma = L(e^\sigma,e^{-\sigma})$ and $F_\sigma = R(e^\sigma,e^{-\sigma})$. By AP11 - AP13 and 9.1.3 we have $E_\sigma^2 = E_\sigma$, $F_\sigma^2 = F_\sigma$, $E_\sigma F_\sigma = F_\sigma E_\sigma$. Thus E_σ and F_σ are commuting idempotents of $\mathrm{End}(A^\sigma)$. This proves (a). The projections onto A^σ_{11}, $A^\sigma_{10} + A^\sigma_{01}$, A^σ_{00} are given by $E_\sigma F_\sigma$, $E_\sigma + F_\sigma - 2E_\sigma F_\sigma$, $\mathrm{Id} - E_\sigma - F_\sigma + E_\sigma F_\sigma$. From AP13 and 7.1.3 we get $E_\sigma F_\sigma = Q(e^\sigma)Q(e^{-\sigma})$, $E_\sigma + F_\sigma - 2E_\sigma F_\sigma = D(e^\sigma,e^{-\sigma}) - 2Q(e^\sigma)Q(e^{-\sigma})$, and $\mathrm{Id} - E_\sigma - F_\sigma + E_\sigma F_\sigma = B(e^\sigma,e^{-\sigma})$ which proves (b) in view of 5.4. From the definitions and 9.1 it is obvious that (c) holds. Also, e^σ is an invertible element of the Jordan pair A_2^J with inverse $e^{-\sigma}$ by 5.5. Hence $a \mapsto \bar{a}$ maps A^σ_{11} onto $A^{-\sigma}_{11}$ and we have $\bar{\bar{a}} = a$, by 1.11. Next we prove (2):

$$\langle xay\rangle = \langle x\bar{\bar{a}}y\rangle = \langle x\langle e^{-\sigma}\bar{a}e^{-\sigma}\rangle y\rangle = \langle xe^{-\sigma}\bar{a}\rangle e^{-\sigma}y\rangle = (x\bar{a})y,$$

using AP7. Now we have by AP4:

$$\overline{ab} = \langle e^{-\sigma}\langle ae^{-\sigma}b\rangle e^{-\sigma}\rangle = \langle e^{-\sigma}b\langle e^{-\sigma}ae^{-\sigma}\rangle\rangle = \langle e^{-\sigma}b\bar{a}\rangle = (e^{-\sigma}\bar{b})\bar{a} = \bar{b}\bar{a}.$$

This proves that $a \mapsto \bar{a}$ is an antiisomorphism.

9.3. THEOREM. *Let* $A = A_{11} \oplus A_{10} \oplus A_{01} \oplus A_{00}$ *be the Peirce decomposition of* A *with respect to* e . *Then we have the composition rules*

(1) $$\langle A^{\sigma}_{ij}, A^{-\sigma}_{mj}, A^{\sigma}_{mn}\rangle \subset A^{\sigma}_{in},$$

(2) $$\langle A^{\sigma}_{ij}, A^{-\sigma}_{mj}, A^{\sigma}_{im}\rangle \subset A^{\sigma}_{mi}, \quad \text{for } i \neq m,$$

(3) $$\langle A^{\sigma}_{ij}, A^{-\sigma}_{ji}, A^{\sigma}_{im}\rangle \subset A^{\sigma}_{jm}, \quad \text{for } i \neq j,$$

(4) $$\langle A^{\sigma}_{ij}, A^{-\sigma}_{ji}, A^{\sigma}_{ji}\rangle \subset A^{\sigma}_{ij}, \quad \text{for } i \neq j.$$

Products with index combinations other than (1) - (4) are zero.

Proof. Let $x \in A^{\sigma}_{ij}$, $y \in A^{-\sigma}_{mn}$, $z \in A^{\sigma}_{pq}$. Then we get from AP1 and 9.2.1

(5) $$\langle e^{\sigma} e^{-\sigma} \langle xyz\rangle\rangle = (i+m-p)\langle xyz\rangle .$$

From the linearizations of AP2 and AP7 we see that

$$\langle\langle xyz\rangle e^{-\sigma} e^{\sigma}\rangle + \langle\langle xye^{\sigma}\rangle e^{-\sigma} z\rangle = (p+q)\langle xyz\rangle,$$

$$\langle\langle xye^{\sigma}\rangle e^{-\sigma} z\rangle + j\langle xyz\rangle = (m+n)\langle xyz\rangle$$

which implies

(6) $$\langle\langle xyz\rangle e^{-\sigma} e^{\sigma}\rangle = (p+q-m-n+j)\langle xyz\rangle .$$

Now (1) - (4) follow immediately from (5) and (6). Next we have to show that all other index combinations (ij,mn,pq) give zero. An enumeration of all possibilities results in the following list.

(7) $(i0,11,mn)$, $(i1,11,0j)$ for $i \neq j$, $(mn,11,00)$,

(8) $(11,i0,mn)$,

(9) $(00,01,ij)$,

(10) $(11,01,11)$,

(11) $(01,00,ij)$,

(12) $(01,10,11)$,

(13) $(0i, 0j, 1m)$,

(14) $(i0, 10, 0j)$, for $(i,j) \neq (0,1)$,

(15) $(10, 01, 00)$.

From 9.2, (c) and (d), and the multiplication rules for the Peirce spaces in an alternative algebra (cf. 9.1) it follows that all products of type (7) are zero. Now let $x \in A^{\sigma}_{11}$, $y \in A^{-\sigma}_{i0}$, $z \in A^{\sigma}$. Then by AP6, $\langle xyz\rangle = \langle\langle e^{\sigma}\bar{x}e^{\sigma}\rangle yz\rangle = \langle e^{\sigma}\langle ye^{\sigma}\bar{x}\rangle z\rangle = 0$ since $\langle ye^{\sigma}\bar{x}\rangle = 0$ by (7). This proves (8). For (9), let $x \in A^{\sigma}_{00}$ and $y \in A^{-\sigma}_{01}$. Then by AP1, $\langle xyz\rangle = \langle x\langle ye^{\sigma}e^{-\sigma}\rangle z\rangle = \langle e^{\sigma}y\langle xe^{-\sigma}z\rangle\rangle + \langle xe^{-\sigma}\langle e^{\sigma}yz\rangle\rangle - \langle\langle e^{\sigma}yx\rangle e^{-\sigma}z\rangle$. The first two terms vanish by (7), and since $\langle e^{\sigma}yx\rangle \in A^{\sigma}_{10}$ by (1) the last term also vanishes by (7). Now let $x,z \in A^{\sigma}_{11}$, $y \in A^{-\sigma}_{01}$. Then $\langle xyz\rangle = \langle xy\langle e^{\sigma}\bar{z}e^{\sigma}\rangle\rangle = \langle\langle xye^{\sigma}\rangle\bar{z}e^{\sigma}\rangle$, and therefore it suffices to prove $\langle xye^{\sigma}\rangle = 0$. But $\langle xye^{\sigma}\rangle = \langle\langle e^{\sigma}\bar{x}e^{\sigma}\rangle ye^{\sigma}\rangle = \langle e^{\sigma}\bar{x}\langle e^{\sigma}ye^{\sigma}\rangle\rangle$, and $\langle e^{\sigma}ye^{\sigma}\rangle = \langle e^{\sigma}y\langle e^{\sigma}e^{-\sigma}e^{\sigma}\rangle\rangle = \langle e^{\sigma}\langle e^{-\sigma}e^{\sigma}y\rangle e^{\sigma}\rangle = 0$. This proves (10). For (11) we use AP6:

(16) $$\langle xyz\rangle = \langle\langle xe^{-\sigma}e^{\sigma}\rangle yz\rangle = -\langle\langle e^{\sigma}e^{-\sigma}x\rangle yz\rangle + \langle x\langle ye^{\sigma}e^{-\sigma}\rangle z\rangle + \langle e^{\sigma}\langle yxe^{-\sigma}\rangle z\rangle.$$

The first two terms on the right hand side vanish since $x \in A^{\sigma}_{01}$ and $y \in A^{-\sigma}_{00}$, and the last one is zero by (9). Similarly, (12) follows from (16) since in this case $\langle yxe^{-\sigma}\rangle \in A^{-\sigma}_{01}$ by (3), and hence $\langle e^{\sigma}\langle yxe^{-\sigma}\rangle z\rangle = 0$ by (10). Next we prove (13), using AP4: $\langle xyz\rangle = \langle xy\langle e^{\sigma}e^{-\sigma}z\rangle\rangle = -\langle zy\langle e^{\sigma}e^{-\sigma}x\rangle\rangle + \langle z\langle e^{-\sigma}e^{\sigma}y\rangle x\rangle + \langle x\langle e^{-\sigma}e^{\sigma}y\rangle z\rangle = 0$. For (14) we use AP7: $\langle xyz\rangle = \langle x\langle e^{-\sigma}e^{\sigma}y\rangle z\rangle = -\langle x\langle ye^{\sigma}e^{-\sigma}\rangle z\rangle + \langle\langle xe^{-\sigma}e^{\sigma}\rangle yz\rangle + \langle\langle xye^{\sigma}\rangle e^{-\sigma}z\rangle$. The first two terms are zero since $x \in A^{\sigma}_{i0}$ and $y \in A^{-\sigma}_{10}$, and the last one vanishes by (7) since $\langle xye^{\sigma}\rangle \in A^{\sigma}_{i1}$. From the same formula we get (15).

9.4. COROLLARY. $A_{ij} = (A^{+}_{ij}, A^{-}_{ij})$ *is a subpair of* A.

This follows immediately from 9.3.

9.5. Remark. Theorem 9.3 shows that, with respect to the Peirce decomposition, the product $\langle xyz \rangle$ in an alternative pair behaves exactly like $(x\bar{y})z$ in an alternative algebra with involution. This is helpful for memorizing the Peirce multiplication rules.

9.6. LEMMA. With the notations of 9.2, let $a,b \in A_{11}^{\pm}$, $f,g,h \in A_{10}^{\pm}$, $u,v,w \in A_{01}^{\pm}$. Then we have the following formulas.

(1) $$a(bf) = (ab)f,$$

(2) $$(ua)b = u(ab),$$

(3) $$\overline{\langle fge^{\sigma} \rangle} = \langle gfe^{-\sigma} \rangle,$$

(4) $$a\langle fgb \rangle = \langle af,g,b \rangle,$$

(5) $$\langle fgb \rangle a = \langle f,g,ba \rangle,$$

(6) $$\overline{\langle e^{\sigma}vu \rangle} = \langle e^{-\sigma},u,v \rangle,$$

(7) $$\langle bvu \rangle a = \langle b,v,ua \rangle,$$

(8) $$a\langle bvu \rangle = \langle ab,v,u \rangle,$$

(9) $$\langle fva \rangle = -\langle avf \rangle,$$

(10) $$\langle fva \rangle b = \langle f,v,ab \rangle = \langle af,v,b \rangle = \langle f,v\bar{a},b \rangle,$$

(11) $$uu = 0;\quad a(uv) = (ua)v = \langle u\bar{a}v \rangle,$$

(12) $$\langle fgh \rangle = \langle fge^{\sigma} \rangle h,$$

(13) $$\langle uvw \rangle = u\langle e^{\sigma}vw \rangle + \langle e^{\sigma},v,uw \rangle,$$

(14) $$\langle ugv \rangle = -\langle uv,g,e \rangle,$$

(15) $$\langle ugf \rangle = u\langle fge^{\sigma} \rangle,$$

(16) $$\langle fvu \rangle = \langle fve^{\sigma} \rangle u..$$

Proof. By the alternative law in the alternative algebra $A^{\sigma}_{e^{-\sigma}}$ we have $a(bf) - (ab)f = b(fa) - (bf)a$, and $fa = (bf)a = 0$ by 9.2. Similarly, one proves (2). By AP4,

$$\overline{\langle fge^{\sigma}\rangle} = \langle e^{-\sigma}\langle fge^{\sigma}\rangle e^{-\sigma}\rangle = \langle e^{-\sigma}e^{\sigma}\langle gfe^{-\sigma}\rangle\rangle = \langle gfe^{-\sigma}\rangle .$$

The linearized form of AP2 gives

$$a\langle fgb\rangle + a\langle bgf\rangle = \langle af,g,b\rangle + \langle ab,g,f\rangle ,$$

and $\langle bgf\rangle = \langle ab,g,f\rangle = 0$ by 9.3. Also, by AP6,

$$\langle fgb\rangle a + \langle bgf\rangle a = \langle f\langle e^{-\sigma}bg\rangle a\rangle + \langle b\langle e^{-\sigma}fg\rangle a\rangle ,$$

which shows $\langle fgb\rangle a = \langle f\langle e^{-\sigma}bg\rangle a\rangle = \langle f,\bar{b}g,a\rangle$ by 9.2.2. Now we get by (1):

$$\langle fgb\rangle a = \langle f,\bar{b}g,e^{\sigma}\rangle a = \langle f,\bar{a}(\bar{b}g),e^{\sigma}\rangle$$

$$= \langle f,(\bar{a}\bar{b})g,e^{\sigma}\rangle = \langle f,(\overline{ba})g,e^{\sigma}\rangle = \langle f,g,ba\rangle ,$$

which proves (5). Next, (6) follows from AP4, and for (7) we have by AP2:

$$\langle bvu\rangle a + \langle bva\rangle u = \langle b,v,ua\rangle + \langle b,v,au\rangle ,$$

and $au = \langle bva\rangle = 0$ by 9.3. Also, by AP2,

$$a\langle bvu\rangle + a\langle uvb\rangle = \langle ab,v,u\rangle + \langle au,v,b\rangle$$

where $\langle uvb\rangle = \langle au,v,b\rangle = 0$ by 9.3. This proves (8).

By 9.3, $\langle fva\rangle$ and $\langle avf\rangle$ belong to A_{01}^{σ}. Hence we get from AP6 that
$\langle fva\rangle + \langle avf\rangle = \langle\langle fva\rangle e^{-\sigma}e^{\sigma}\rangle + \langle\langle avf\rangle e^{-\sigma}e^{\sigma}\rangle = \langle f\langle e^{-\sigma}av\rangle e^{\sigma}\rangle + \langle a\langle e^{-\sigma}fv\rangle e^{\sigma}\rangle = 0$,
since $\langle e^{-\sigma}av\rangle = \langle e^{-\sigma}fv\rangle = 0$ by 9.3. Now we prove (10). By AP7 and AP1, we have

$$\langle fva\rangle b + \langle fa,v,b\rangle = \langle f\langle vae^{-\sigma}\rangle b\rangle + \langle f\langle e^{-\sigma}av\rangle b\rangle ,$$

$$a\langle fvb\rangle + \langle f,v,ab\rangle = \langle af,v,b\rangle + \langle f\langle e^{-\sigma}av\rangle b\rangle ,$$

and moreover $fa = \langle e^{-\sigma}av\rangle = a\langle fvb\rangle = 0$. This shows $\langle fva\rangle b = \langle f,v\bar{a},b\rangle$, and $\langle f,v,ab\rangle = \langle af,v,b\rangle$. Now $\langle fva\rangle b = (\langle fve^{\sigma}\rangle a)b = \langle fve^{\sigma}\rangle(ab) = \langle f,v,ab\rangle$ by (2). Next, we have $uu = \langle ue^{-\sigma}u\rangle = \langle e^{\sigma}e^{-\sigma}\langle ue^{-\sigma}u\rangle\rangle = \langle\langle e^{\sigma}e^{-\sigma}u\rangle e^{-\sigma}u\rangle = 0$ by AP2. By the alternative law, $a(uv) - (au)v = -u(av) + (ua)v$, and $au = av = 0$. Finally, $(ua)v = \langle u\bar{a}v\rangle$ by 9.2.2.

For (12) we use AP2:

$\langle fgh\rangle = \langle f,g,\langle e^{\sigma}e^{-\sigma}h\rangle\rangle = -\langle fg\langle he^{-\sigma}e^{\sigma}\rangle\rangle + \langle fge^{\sigma}\rangle h + \langle\langle fgh\rangle e^{-\sigma}e^{\sigma}\rangle = \langle fge^{\sigma}\rangle h$, since h and $\langle fgh\rangle$ are in A_{10}^{σ}. Similarly, (13) follows from AP2: $\langle uvw\rangle$ $= \langle\langle ue^{-\sigma}e^{\sigma}\rangle vw\rangle = \langle ue^{-\sigma}\langle e^{\sigma}vw\rangle\rangle + \langle ue^{-\sigma}\langle wve^{\sigma}\rangle\rangle - \langle\langle ue^{-\sigma}w\rangle ve^{\sigma}\rangle = u\langle e^{\sigma}vw\rangle + \langle e^{\sigma},v,uw\rangle$ by (9) and $\langle wve^{\sigma}\rangle = 0$. By AP7 and (9) we have $\langle ugv\rangle + \langle uv,g,e^{\sigma}\rangle = \langle\langle ugv\rangle e^{-\sigma}e^{\sigma}\rangle$ $+ \langle\langle ue^{-\sigma}v\rangle ge^{\sigma}\rangle = \langle u,\langle gve^{-\sigma}\rangle + \langle e^{-\sigma}vg\rangle,e^{\sigma}\rangle = 0$. From AP2 we get $\langle ugf\rangle$ $= \langle\langle ue^{-\sigma}e^{\sigma}\rangle gf\rangle = -\langle\langle ue^{-\sigma}f\rangle ge^{\sigma}\rangle + \langle ue^{-\sigma}\langle e^{\sigma}gf\rangle\rangle + \langle ue^{-\sigma}\langle fge^{\sigma}\rangle\rangle = u\langle fge^{\sigma}\rangle$ since $\langle ue^{-\sigma}f\rangle = \langle e^{\sigma}gf\rangle = 0$. And finally, by AP4 and (11), $\langle fvu\rangle - \langle fve^{\sigma}\rangle u$ $= \langle e^{\sigma}e^{-\sigma}\langle fvu\rangle\rangle + \langle ue^{-\sigma}\langle fve\rangle\rangle = \langle e^{\sigma}\langle vfe^{-\sigma}\rangle u\rangle + \langle u\langle vfe^{-\sigma}\rangle e\rangle = 0$.

9.7. LEMMA. With the notations of 9.2, we have

(a) A_{10} is an associative pair;

(b) if $A_{01}^{\sigma}\cdot A_{01}^{\sigma} = 0$ then A_{01} is an associative pair.

Proof. (a) For $c,d,f,g,h \in A_{10}^{\pm}$ we have by 9.6.12, 9.6.1, and 9.6.4,

$$\begin{aligned}\langle cd\langle fgh\rangle\rangle &= \langle cde^{\sigma}\rangle(\langle fge^{\sigma}\rangle h) = (\langle cde^{\sigma}\rangle\langle fge^{\sigma}\rangle)h \\ &= \langle\langle cde^{\sigma}\rangle f,g,e^{\sigma}\rangle h = \langle\langle cdf\rangle,g,e^{\sigma}\rangle h = \langle\langle cdf\rangle gh\rangle .\end{aligned}$$

From this and AP1 it follows that A_{10} is associative.

(b) Let $x,y,u,v,w \in A_{01}^{\pm}$. Then by (13), (7) and (2) of 9.6 we have

$$\begin{aligned}\langle xy\langle uvw\rangle\rangle &= x\langle e^{\sigma}y\langle uvw\rangle\rangle = x\langle e^{\sigma},y,u\langle e^{\sigma}vw\rangle\rangle \\ &= x(\langle e^{\sigma}yu\rangle\langle e^{\sigma}vw\rangle) = (x\langle e^{\sigma}yu\rangle)\langle e^{\sigma}vw\rangle = \langle\langle xyu\rangle vw\rangle ,\end{aligned}$$

and hence A_{01} is associative.

9.8. PROPOSITION. Let $A = \bigoplus A_{ij}$ be the Peirce decomposition of A with respect to e. Then A is an associative pair if and only if A_{ii} $(i = 0,1)$ is an associative pair, and all products 9.3.2, 9.3.3, 9.3.4 vanish.

Proof. If A is associative then so is A_{ii}. Also, for $x \in A_{ij}^{\sigma}$, $y \in A_{mn}^{-\sigma}$, and

$z \in A^{\sigma}_{pq}$ we have $\langle e^{\sigma}e^{-\sigma}\langle xyz\rangle\rangle = \langle\langle e^{\sigma}e^{-\sigma}x\rangle yz\rangle = i\langle xyz\rangle$ and $\langle\langle xyz\rangle e^{-\sigma}e^{\sigma}\rangle =$ $\langle xy\langle ze^{-\sigma}e^{\sigma}\rangle\rangle = q\langle xyz\rangle$ which proves $\langle xyz\rangle \in A^{\sigma}_{iq}$. Hence all products 9.3.2–9.3.4 are zero. For the converse, it suffices to show that $\langle\langle xyz\rangle uv\rangle = \langle x\langle uzy\rangle v\rangle$, since $\langle xy\langle zuv\rangle\rangle = \langle\langle xyz\rangle uv\rangle$ then follows from AP1. Thus let $x_{ij} \in A^{\pm}_{ij}$, $y_{k\ell} \in A^{\pm}_{k\ell}$, etc.. Then we have $\langle\langle x_{ij}y_{k\ell}z_{mn}\rangle u_{pq}v_{rs}\rangle = 0$ if $j \neq \ell$ or $k \neq m$ or $n \neq q$ or $p \neq r$. But in each of these cases, $\langle x_{ij}\langle u_{pq}z_{mn}y_{k\ell}\rangle v_{rs}\rangle$ is zero as well. Hence we only have to consider the case $\langle\langle x_{ij}y_{mj}z_{mn}\rangle u_{pn}v_{ps}\rangle$. There are the following possibilities.

1°. $i \neq m$ or $n \neq j$. Then we have by AP6, omitting indices for easier notation,

$$\langle\langle xyz\rangle uv\rangle + \langle\langle zyx\rangle uv\rangle = \langle x\langle uzy\rangle v\rangle + \langle z\langle uxy\rangle v\rangle,$$

and the second terms on each side vanish.

2°. $p \neq m$. By AP 7 we get

$$\langle\langle xyz\rangle uv\rangle + \langle\langle xuz\rangle yv\rangle = \langle x\langle uzy\rangle v\rangle + \langle x\langle yzu\rangle v\rangle,$$

and again the second terms on each side are zero.

3°. $i = m = p$, $n = j$, $s \neq j$. From AP2 it follows that

$$\langle\langle xyz\rangle uv\rangle + \langle\langle xyv\rangle uz\rangle = \langle xy\langle zuv\rangle\rangle + \langle xy\langle vuz\rangle\rangle,$$

and the second terms on each side vanish. Since x,y,z,u are all in $A^{\pm}_{ij}$ it follows from AP1 that we also have $\langle\langle xyz\rangle uv\rangle = \langle x\langle uzy\rangle v\rangle$.

4°. $i = m = p$, $j = n = s$, $i \neq j$. For $i = 1$ this follows from 9.7(a). For $i = 0$ we have that $A^{\sigma}_{01}\cdot A^{\sigma}_{01} = \langle A^{\sigma}_{01}e^{-\sigma}A^{\sigma}_{01}\rangle$ is a product of type 9.3.2 and hence is zero by hypothesis. Thus the assertion follows from 9.7(b).

5°. All indices are equal. Then the assertion follows from our assumption about the A_{ii}.

9.9. PROPOSITION. <u>The Jacobson radicals of</u> A <u>and</u> A_{ij} <u>are related by</u>

(1) $$\operatorname{Rad} A_{ij} = A_{ij} \cap \operatorname{Rad} A ,$$

(2) $$\operatorname{Rad} A = \sum_{i,j} \operatorname{Rad} A_{ij} .$$

<u>Proof</u>. Since every ideal of A is the sum of its intersections with the Peirce spaces it suffices to prove (1). For $i = j$ this follows from 9.2(b) and 5.8. Now let $i \neq j$ and let $x \in A_{ij}^{\sigma}$, $y \in A_{ji}^{-\sigma}$. Then by 9.3.3, $Q(x)y = \langle xyx\rangle \in A_{jj}^{\sigma}$. On the other hand, by 9.2(b) and 5.4(c), $Q(x)y \in A_{10}^{\sigma} \oplus A_{01}^{\sigma}$. Hence we have $Q(x)y = 0$, and therefore $x^y = x$ by 3.8. If now $x \in \operatorname{Rad} A_{ij}^{\sigma}$ then by 3.7(a) it follows that $(x,y+z)$ is quasi-invertible, for all $y+z \in A_{ji}^{\sigma} \oplus A_{ij}^{\sigma}$ which shows that x belongs to the radical of A_1^J. By 5.8, this is contained in $\operatorname{Rad} A^J = \operatorname{Rad} A$. Hence we have $\operatorname{Rad} A_{ij} \subset \operatorname{Rad} A$. The converse is trivial.

9.10. Let $(e_1,\ldots,e_r)$ be a system of orthogonal idempotents of A^J (cf. 5.12). Then it is not hard to see that

(1) $$L(e_i^{\sigma},e_j^{-\sigma}) = R(e_i^{\sigma},e_j^{-\sigma}) = 0, \quad \text{for} \quad i \neq j,$$

and that we have the Peirce decomposition

(2) $$A = \bigoplus_{i,j=0}^{r} A_{ij}$$

where $A_{ij} = (A_{ij}^{+},A_{ij}^{-})$ and

$$A_{ij}^{\sigma} = \{x \in A^{\sigma} \mid \langle e_{\ell}^{\sigma}e_{\ell}^{-\sigma}x\rangle = \delta_{i\ell}x \quad \text{and} \quad \langle xe_{\ell}^{-\sigma}e_{\ell}^{\sigma}\rangle = \delta_{j\ell}x, \quad \text{for} \quad \ell = 1,\ldots,r\}.$$

The composition rules are the same as in case of a single idempotent (see 9.3). The Peirce decomposition of the associated Jordan pair A^J is related to (2) by

(3) $$(A^J)_{ii} = A_{ii}, \quad (A^J)_{ij} = A_{ij} \oplus A_{ji}, \quad \text{for} \quad i \neq j.$$

Also, the exact analogue holds for (2). Since these results will not be needed in the sequel we leave the proofs as an exercise.

NOTES

For associative triple systems (ternary algebras) see Hestenes[1], Lister [1], Loos[4], Stephenson[1]. Associative pairs are really "abstract off-diagonal Peirce spaces" of associative algebras in the following sense: Let $\mathcal{R}$ be a unital associative algebra, and consider the Peirce decomposition $\mathcal{R} = \mathcal{R}_{11} \oplus \mathcal{R}_{12} \oplus \mathcal{R}_{21} \oplus \mathcal{R}_{22}$ of $\mathcal{R}$ with respect to an idempotent e. Then $(\mathcal{R}_{12},\mathcal{R}_{21})$ is an associative pair if we define $\langle xyz\rangle = xyz$ and $\langle yzw\rangle = wzy$, for $x,z \in \mathcal{R}_{12}$ and $y,w \in \mathcal{R}_{21}$. Conversely, every associative pair $A = (A^+,A^-)$ can be obtained in this way. Indeed, let $\mathcal{R}_{11}$ be the subalgebra of $E = \mathrm{End}(A^+) \times \mathrm{End}(A^-)^{op}$ generated by $e_1 = (\mathrm{Id},\mathrm{Id})$ and all $\ell(x,y) = (L(x,y),L(y,x))$, and $\mathcal{R}_{22}$ the subalgebra of E^{op} generated by $e_2 = (\mathrm{Id},\mathrm{Id})$ and all $r((y,x)) = (R(x,y),R(y,x))$ where $(x,y) \in A$. For $a = (a_+,a_-) \in \mathcal{R}_{11}$, $b = (b_+,b_-) \in \mathcal{R}_{22}$ and $(x,y) \in A$ we set

$$ax = a_+(x), \quad ya = a_-(y), \quad xb = b_+(x), \quad by = b_-(y).$$

Then A^+ is a $\mathcal{R}_{11}$-left and $\mathcal{R}_{22}$-right bi-module, and A^- is a $\mathcal{R}_{22}$-left and $\mathcal{R}_{11}$-right bi-module. Let $\mathcal{R}_{12} = A^+$, $\mathcal{R}_{21} = A^-$, and write the elements of $\mathcal{R} = \mathcal{R}_{11} \oplus \mathcal{R}_{12} \oplus \mathcal{R}_{21} \oplus \mathcal{R}_{22}$ in matrix form:

$$\begin{pmatrix} a & x \\ y & b \end{pmatrix}, \quad a \in \mathcal{R}_{11}, \quad (x,y) \in A, \ b \in \mathcal{R}_{22}.$$

Then $\mathcal{R}$ becomes an associative algebra with the desired properties by defining

$$\begin{pmatrix} a & x \\ y & b \end{pmatrix}\begin{pmatrix} a' & x' \\ y' & b' \end{pmatrix} = \begin{pmatrix} aa' + \ell(x,y'), & ax' + xb' \\ ya' + by' & , r(y,x') + bb' \end{pmatrix}.$$

This construction is analogous to the one given in Loos[4]. We haven't used this approach here since it does not work for alternative pairs, and also is not suitable for the structure theory with chain conditions on principal inner ideals (see chapter III). In fact, even if A is semisimple and has acc and dcc on principal inner ideals the algebra R constructed above is not necessarily Artinian. This is so, however, if A has dcc on all inner ideals. It seems possible to develop a structure theory for associative pairs with chain conditions on left and right ideals.

Alternative triple systems were introduced in Loos[3], over rings of scalars containing 1/2. This restriction was later removed by Meyberg([6]).

CHAPTER III

ALTERNATIVE AND JORDAN PAIRS WITH CHAIN CONDITIONS

§10. Inner ideals and chain conditions

10.1. Inner ideals. Let $V = (V^+, V^-)$ be a Jordan pair over the ring k . A submodule $m \subset V^\sigma$ is called an inner ideal if $Q(m)V^{-\sigma} \subset m$. By an inner ideal of an alternative pair we mean an inner ideal of the associated Jordan pair. We now give some examples of inner ideals.

(a) If $m \subset V^\sigma$ is an inner ideal then it follows from JP3 (resp. JP26) that $Q(x)m$ (resp. $B(x,y)m$) is an inner ideal, for all $x \in V^{-\sigma}$ (resp. $(x,y) \in V^\sigma \times V^{-\sigma}$). In particular, $Q(x)V^\sigma$ is an inner ideal, called the principal inner ideal generated by x . Note that $x \in Q(x)V^\sigma$ if and only if x is (von Neumann) regular (cf. 5.1).

(b) Let e be an idempotent of V . From 5.4 it follows that the Peirce spaces $V_2^\sigma(e)$ and $V_0^\sigma(e)$ are inner ideals; in fact, $V_2^\sigma(e)$ is the principal inner ideal generated by e^σ , and $V_0^\sigma(e) = B(e^\sigma, e^{-\sigma})V^\sigma$. If $V_0(e) = 0$ then $V_1^\sigma(e)$ is also an inner ideal.

(c) If $x \in V^\sigma$ is a trivial element (cf. 4.5) then obviously k.x is an inner ideal, called a trivial inner ideal.

10.2. LEMMA. (a) Inverse images of inner ideals under homomorphisms are inner ideals.

(b) Images of inner ideals under surjective homomorphisms are inner ideals.

(c) Let e be an idempotent of V. Then the inner ideals of $V_2(e)$ (resp. $V_0(e)$) are precisely the inner ideals of V contained in $V_2(e)$ (resp. $V_0(e)$), and every principal inner ideal of $V_2(e)$ (resp. $V_0(e)$) is also a principal inner ideal of V. If $V_0(e) = 0$ then the same statements are true for $V_1(e)$.

This is immediate from the definitions and the composition rules for the Peirce decomposition (5.4(c)).

10.3. PROPOSITION. Let $X \subset V^-$ be a subset, and let $\mathrm{Ann}(X)$ be the set of all $a \in V^+$ such that, for all $x \in X$,

$$Q(a)x = Q(x)a = 0, \tag{1}$$

$$Q(a)Q(x) = Q(x)Q(a) = 0, \tag{2}$$

$$D(a,x) = D(x,a) = 0. \tag{3}$$

Then $\mathrm{Ann}(X)$ is an inner ideal, called the annihilator of X. We have

$$X \subset \mathrm{Ann}(\mathrm{Ann}(X)). \tag{4}$$

If $X = \{x\}$ consists of a single element then

$$\mathrm{Ann}(\{x\}) \subset \mathrm{Ann}(Q(x)V^+), \tag{5}$$

and equality holds in case x is regular. If e is an idempotent then

$$\mathrm{Ann}(\{e^-\}) = \mathrm{Ann}(V_2^-(e)) = V_0^+(e). \tag{6}$$

Analogous statements hold for a subset X of V^+.

Proof. Clearly, $m = \mathrm{Ann}(X)$ is closed under scalar multiples. Let $a,b \in m$. Then, for all $x \in X$, $Q(a+b)x = \{axb\} = 0$, $Q(a+b)Q(x) = Q(a,b)Q(x) = D(a,x).$ $.D(b,x) - D(a,Q(x)b) = 0$ by JP9, and $Q(x)Q(a+b) = Q(x)Q(a,b) = D(x,a)D(x,b) - D(Q(x)a,b) = 0$ by JP13. Hence $a + b \in m$, and m is a submodule of V^+. Next let $a \in m$ and $y \in V^-$. Then it follows from JP3 that $Q(a)y$ satisfies (1) and (2). Furthermore, $D(Q(a)y,x) = D(a,\{yax\}) - D(Q(a)x,y) = 0$ by JP8, and $D(x,Q(a)y) = D(\{yax\},a) - D(y,Q(a)x) = 0$ by JP7. Hence $Q(a)y \in m$, and m is an inner ideal. Since (1) - (3) is symmetric in a and x we have (4). If a belongs to the annihilator of a single element x then it follows from JP3, JP7, and JP8 that it satisfies (1) - (3) with x replaced by $Q(x)y$, for an arbitrary $y \in V^+$. This proves (5). Finally, (6) follows from 5.4 and the definitions.

10.4. PROPOSITION. A Jordan pair is a division pair if and only if it has non-trivial multiplication and no proper inner ideals.

Proof. Let V be a division pair, let $m \subset V^+$ be an inner ideal, and let x be a non-zero element of m. Then $Q(x)$ is invertible, and hence $V^+ = Q(x)V^- \subset m$. In the same way one shows that V^- has no proper inner ideals.

Conversely, assume that V has no proper inner ideals, and let x be a non-zero element of V^+. Then the principal inner ideal generated by x is either zero or all of V^+. In the first case, x is a trivial element, and hence $k.x$ is an inner ideal (10.1(c)) which must be all of V^+. In particular, $Q(V^+)V^- = 0$. If $Q(V^-)V^+ \neq 0$ then there exists $y \in V^-$ such that $Q(y)V^+ \neq 0$, and since V has no proper inner ideals, $V^- = Q(y)V^+$ which implies $y = \lambda Q(y)x$ for some $\lambda \in k$. But then $Q(y) = 0$, a contradiction. Thus we also have $Q(V^-) = 0$, and V has trivial multiplication which is ruled out by hypothesis. It follows that $Q(x)V^- = V^+$, and hence $x = Q(x)z$ for some $z \in V^-$. By JP3, $Q(x) = Q(x)Q(z)Q(x)$ which implies $Q(x)Q(z) = \mathrm{Id}$ by surjectivity of $Q(x)$. But

$Q(z)$ is surjective as well since we can apply everything we proved so far to V^{op}. Hence x is invertible, and the Proposition follows.

10.5. THEOREM. _Let_ M _be a set of inner ideals of a Jordan pair_ V, _ordered by inclusion. Assume that for all_ $m \in M$ _and for all non-trivial elements_ $x \in m$, _the principal inner ideal generated by_ x _belongs to_ M. _Let_ $m \neq 0$, $m \subset V^+$, _be minimal in_ M. _Then either_ m _contains a non-zero trivial element of_ V, _or_ $m = V_2^+(e)$ _where_ e _is a division idempotent. In the latter case,_ m _is minimal among all non-zero inner ideals of_ V.

Proof. Assume that m contains no non-zero trivial element, and let x be any non-zero element of m. Then the principal inner ideal generated by x is non-zero, contained in m, and belongs to M. By minimality of m we have that $m = Q(x)V^-$. In particular, x is regular, and hence by 5.2 there exists $y \in V^-$ such that $e = (e^+,e^-) = (x,y)$ is an idempotent of V. By 5.4, we have then $m = V_2^+(e)$. Let $n \subset V_2^+(e)$ be an inner ideal of $V_2(e)$. By 10.2, n is an inner ideal of V. If u is a non-zero element of n then $0 \neq Q(u)V^- \subset n \subset m$, and $Q(u)V^- \in M$ which implies $n = m$. Hence $V_2^+(e)$ contains no proper inner ideals. If $p \subset V_2^-(e)$ is an inner ideal then $Q(e^+)p \subset V_2^+(e)$ is an inner ideal, and $p = Q(e^-)Q(e^+)p$ by 5.4. This shows that $V_2^-(e)$ contains no proper inner ideals, and $V_2(e)$ is a division pair by 10.4.

10.6. _Chain conditions._ Let M be a set of inner ideals of V. We say that V satisfies the _descending chain condition on_ M if every descending chain $m_1 \supset m_2 \supset \dots$ of inner ideals $m_i \in M$ becomes stationary. Similarly, V satisfies the _ascending chain condition on_ M if every ascending chain of inner ideals in M becomes stationary. Descending (resp. ascending) chain condition are usually abbreviated to dcc (resp. acc). We say V satisfies the _chain condition on idem-_

potents (= cci) if there exist no infinite sets of pairwise orthogonal idempotents. If V satisfies the cci then there obviously exist maximal idempotents; i.e., idempotents e such that $V_0(e)$ contains no non-zero idempotent (5.12).

10.7. The chain condition on idempotents is implied by either the acc on principal inner ideals or the dcc on all inner ideals. Indeed, if $E = \{e_1, e_2, \ldots\}$ is an orthogonal set of idempotents then $V_2^\sigma(e_1) \subset V_2^\sigma(e_1+e_2) \subset \cdots$ is an ascending chain of principal inner ideals, and $V_0^\sigma(e_1) \supset V_0^\sigma(e_1+e_2) \supset \cdots$ is a descending chain of inner ideals. If one of these chains becomes stationary then E is finite. Conversely, it will follow from the classification in §§11 and 12 that for a semisimple alternative or Jordan pair with dcc on principal inner ideals the acc on principal inner ideals, the cci, and the existence of a maximal idempotent are all equivalent. It would be desirable to have a direct proof of this fact.

Let J be a Jordan algebra and $V = (J,J)$ the associated Jordan pair. Then it is clear from the definitions that V satisfies the dcc or acc on (principal) inner ideals if and only if J does. Indeed, (principal) inner ideals of J and $V^\pm$ are the same.

10.8. THEOREM. Let V be a Jordan pair with dcc on principal inner ideals. Then the Jacobson radical, the set of properly nilpotent elements, the nil radical, and the small radical all coincide.

Proof. Since all these sets are contained in each other (4.14, 4.15) it suffices to show that $\mathrm{rad}\, V = 0$ implies $\mathrm{Rad}\, V = 0$. Assume this is not the case, and let M be the set of all principal inner ideals generated by non-zero elements $x \in \mathrm{Rad}\, V^\pm$. Let m be a minimal element of M, say $m \subset V^+$. Then by 10.5, $m = V_2^+(e)$ where e is a division idempotent; in particular, it contains a nonzero regular element. This is a contradiction since $\mathrm{Rad}\, V$ contains no such elements.

10.9. LEMMA. Let N be a nil ideal of a Jordan algebra J (resp. a Jordan pair V), and let $x \mapsto \bar{x}$ denote the canonical map $J \to J/N$ (resp. $V \to V/N$). If $\{d_1, d_2, \ldots\}$ is an at most countable orthogonal set of idempotents of J/N (resp. V/N) then there exists an orthogonal set $\{e_1, e_2, \ldots\}$ of idempotents of J (resp. V) such that $\bar{e}_i = d_i$, $i = 1, 2, \ldots$.

Proof. Consider first the case of a single idempotent d of $\bar{J} = J/N$, and let $x \in J$ be such that $\bar{x} = d$. Then $x^2 - x \in N$ and therefore $(x^2 - x)^n = 0$ for some n. In the polynomial algebra $k[T]$ there exist polynomials $f(T)$ and $g(T)$ such that

$$f(T)T^{2n+1} - g(T)(T-1)^{2n+1} = 1. \tag{1}$$

Let $k[T]_+$ be the ideal of $k[T]$ generated by T. Then there exists a unique homomorphism $k[T]_+ \to J$ of Jordan algebras sending T into x which we denote as usual by $h(T) \mapsto h(x)$. Let in particular $h(T) = f(T)T^{2n+1}$, and let $e = h(x) \in J$. If $h(T) = \Sigma\, \alpha_i T^i$ then it follows from (1) that $h(1) = \Sigma\, \alpha_i = 1$. Hence $\bar{e} = \overline{h(x)} = \Sigma\, \alpha_i \bar{x}^i = \Sigma\, \alpha_i d^i = (\Sigma\, \alpha_i)d = d$. Also by (1), $h(T) - 1 = g(T)(T-1)^{2n+1}$, and multiplying this with $h(T)$ we get

$$h(T)^2 - h(T) = f(T)g(T)T^{2n+1}(T-1)^{2n+1} = (T^2-T)^n p(T)(T^2-T)^n = U^n_{T^2-T}.p(T),$$

where $p(T) = f(T)g(T)(T^2-T)$. Since $T \to x$ is a Jordan homomorphism it follows that $e^2 - e = U^n_{x^2-x}.p(x) = U((x^2-x)^n).p(x) = 0$.

Now assume that $e_1, \ldots, e_r$ are orthogonal idempotents of J such that $\bar{e}_i = d_i$. Let $e = e_1 + \ldots + e_r$, and let $J = J_2 \oplus J_1 \oplus J_0$ be the Peirce decomposition with respect to e (cf. 5.6). Then the Peirce decomposition of $\bar{J}$ with respect to $\bar{e} = d_1 + \ldots + d_r$ is $\bar{J} = \bar{J}_2 \oplus \bar{J}_1 \oplus \bar{J}_0$, and $d_{r+1} \in \bar{J}_0$. If we apply what we first proved to $J_0 \to \bar{J}_0$ we find an idempotent $e_{r+1} \in J_0$ such that $\bar{e}_{r+1} = d_{r+1}$, and e_{r+1} is orthogonal to $e_1, \ldots, e_r$. By continuing in this way we obtain the desired set of idempotents.

Next consider the case of a Jordan pair V, and let $d = (d^+, d^-)$ be an idempotent of $\overline{V} = V/N$. If $y \in V^-$ is such that $\bar{y} = d^-$ then the canonical map $V \to \overline{V}$ induces a Jordan algebra homomorphism $J = V_y^+ \to \overline{V}_{d^-}^+ = \overline{J}$ with kernel N^+ and this is a nil ideal of V_y^+. Moreover, d^+ is an idempotent of $\overline{J}$. By what we proved before, there exists an idempotent $e^+ \in J$ such that $\overline{e^+} = d^+$. Thus we have $(e^+)^2 = Q(e^+)y = e^+$, i.e., e^+ is a regular element. By 5.2, (e^+, e^-) where $e^- = Q(y)e^+$ is an idempotent of V, and we have $\overline{e^-} = Q(\bar{y})\overline{e^+} = Q(d^-)d^+ = d^-$. This shows that we can lift idempotents from V/N to V. The proof for orthogonal sets of idempotents is verbatim the same as in the Jordan algebra case.

10.10. THEOREM. <u>If a Jordan pair</u> V <u>satisfies the dcc on principal inner ideals and if</u> $\operatorname{Rad} V \neq V$ <u>then</u> V <u>contains a local idempotent</u>.

<u>Proof</u>. By 10.2, $\overline{V} = V/\operatorname{Rad} V$ has dcc on principal inner ideals, and by 10.5, $\overline{V}$ contains a division idempotent d. By 10.8, the radical is a nil ideal, and by 10.9, d lifts to an idempotent e of V. We have $\operatorname{Rad} V_2(e) = V_2(e) \cap \operatorname{Rad} V$ by 5.8, and $V_2(e)/\operatorname{Rad} V_2(e) = \overline{V}_2(d)$ is a division pair. Hence by 4.4(b), $V_2(e)$ is local, and e is a local idempotent.

10.11. COROLLARY. <u>Let</u> V <u>have dcc on principal inner ideals, and let</u> e <u>be an idempotent of</u> V. <u>Then</u> e <u>is maximal if and only if</u> $V_0(e) \subset \operatorname{Rad} V$.

<u>Proof</u>. If e is maximal then $V_0(e)$ contains no non-zero idempotents, and by 10.2 has dcc on principal inner ideals. Hence by 10.10, $V_0(e)$ is radical, and by 5.8 it is contained in the radical of V. Conversely, if $V_0(e)$ is radical then it contains no non-zero idempotents by 5.1, and therefore e is maximal.

10.12. Frames. Let V be a Jordan pair satisfying the dcc on principal inner ideals and the chain condition on idempotents. A frame of V is a maximal orthogonal system of local idempotents of V ; i.e., an orthogonal system of local idempotents not properly contained in any other orthogonal system of local idempotents. Frames obviously exist and, by the cci, are finite but possibly empty. Let $E = (e_1,\dots,e_r)$ be an orthogonal system of local idempotents and let $e = e_1 + \dots + e_r$. Then E is a frame if and only if e is a maximal idempotent. This follows from 10.10 and 10.11. In particular, the empty set is a frame if and only if $e = 0$, which means $V = \mathrm{Rad}\, V$ by 10.11. Also, a frame is maximal among orthogonal systems of (not necessarily local) idempotents.

It is not true that every maximal idempotent e is of the form $e_1 + \dots + e_r$ where $(e_1,\dots,e_r)$ is a frame; cf. the example given in 5.12.

10.13. LEMMA. *Let V be a Jordan pair, and let e be an idempotent of V such that $V_0(e) = 0$. Assume that V is non-degenerate and that $V_2(e)$ is simple. Then V is simple.*

Proof. Let B be an ideal of V . Then $B = B_2 \oplus B_1$ where $B_i = B \cap V_i(e)$ is an ideal of $V_i(e)$. Since $V_2(e)$ is simple we have $B_2 = V_2(e)$ or $B_2 = 0$. In the first case, $e \in B_2$ and hence $B_1^\sigma \supset \{e^\sigma e^{-\sigma} V_1^\sigma\} = V_1^\sigma$ which shows $B = V$. In the second case, $\{xye^\sigma\} \in V_2^\sigma = 0$ for all $x \in V_1^\sigma$, $y \in B_1^{-\sigma}$, and by 8.2.2 we get $Q(x)y = \{\{xye^\sigma\}e^{-\sigma}x\} = 0$. Thus x is a trivial element of V_1 . By 5.10, $V_1(e)$ is nondegenerate, and hence $x = 0$.

10.14. THEOREM. (a) *Let V be a semisimple Jordan pair with dcc on principal inner ideals and containing a maximal idempotent e . Then*

$$V = V^{(1)} \oplus \dots \oplus V^{(n)} \tag{1}$$

is a finite direct sum of simple ideals with the same properties. The decomposition (1) is unique up to order, and every ideal of V *is the sum of some of the* $V^{(i)}$. *The map* $I \mapsto I \cap V_2(e)$ *is a bijection between the ideals of* V *and the ideals of* $V_2(e)$. *In particular,* V *is simple if and only if* $V_2(e)$ *is simple.*

(b) *Conversely, a finite direct product of simple and semisimple Jordan pairs with dcc on principal inner ideals and maximal idempotent is semisimple and has dcc on principal inner ideals and a maximal idempotent.*

(c) *Let* A *be a semisimple alternative pair with dcc on principal inner ideals and maximal idempotent. Then* A *and* A^J *have the same ideals, and hence* (a) *and* (b) *hold (mutatis mutandis) for* A *as well.*

Proof. (a) By 10.11 we have $V_0(e) = 0$. By 5.5, e^+ is invertible in $V_2(e)$ with inverse e^- . Let J be the unital Jordan algebra $(V_2^+)_{e^-}$. Then we may identify $V_2(e)$ and (J,J) by 1.11. Also J is semisimple since $V_2(e)$ is and has dcc on principal inner ideals (10.7). By the first structure theorem for such Jordan algebras (Jacobson[3]), $J = J_1 \oplus \dots \oplus J_n$ is a direct sum of simple ideals with the same properties. Hence $V_2(e) = (J_1,J_1) + \dots + (J_n,J_n)$ is a direct sum of simple ideals. Let c_i be the unit element of J_i , and let $e_i = (c_i,c_i) \in V_2(e)$. Then the e_i are orthogonal idempotents whose sum is e . For the corresponding Peirce decomposition $V = \bigoplus_{0\le i\le j\le n} V_{ij}$ (cf. 5.14) we have $V_2(e_i) = V_{ii} = (J_i,J_i)$, $V_{ij} = 0$ for $0 < i < j < n$, $V_{00} = V_0(e) = 0$. From the composition rules for the Peirce spaces it follows easily that $V^{(i)} = V_{ii} \oplus V_{i0}$ ($i = 1,\dots, n$) is an ideal of V , and hence we have (1). Also, $V^{(i)}$ is semisimple, has dcc on principal inner ideals, a maximal idempotent (namely e_i), and is simple by 10.13. A standard argument shows that the decomposition (1) is unique up to order, and that every ideal of V is a sum of some of the $V^{(i)}$. This also implies the correspondence between the ideals of V and $V_2(e)$.

(b) This is obvious.

(c) Clearly the Jordan pair A^J satisfies the hypotheses of (a). In particular, every ideal of A^J is semiprime and is therefore an ideal of A (7.10). Also, $A_{11}(e)$ is the Peirce-2-space of A^J by 9.2. This implies (c).

It is unknown whether a simple Jordan pair with dcc on principal inner ideals is semisimple, except in the finite-dimensional case (see 14.12).

Our next aim is to prove that semisimple Jordan pairs with dcc on principal inner ideals are von Neumann regular.

10.15. LEMMA. (McCoy's Lemma) *Let* $(x,y) \in V$ *and assume that* $x - Q(x)y$ *is von Neumann regular*. *Then* x *is von Neumann regular*.

Proof. By assumption, there exists an element $z \in V^-$ such that

$$x - Q(x)y = Q(x-Q(x)y)z \ .$$

This implies by JP23 that

$$\begin{aligned} x &= Q(x)y + Q(x-Q(x)y)z \\ &= Q(x)y + Q(x)B(y,x)z \\ &= Q(x)(y + B(y,x)z) \ , \end{aligned}$$

and hence x is von Neumann regular.

10.16. LEMMA. $(x,y) \in V$ *is quasi-invertible if and only if the principal inner ideals generated by* x *and by* $x - Q(x)y$ *coincide*; *i.e.*,

$$Q(x)V^- = Q(x-Q(x)y)V^- \ .$$

Proof. If (x,y) is quasi-invertible then $B(x,y)$ and $B(y,x)$ are invertible (3.2 and 3.3), and since $Q(x-Q(x)y) = Q(x)B(y,x)$ by JP23 we have that $Q(x)V^-$ $= Q(x-Q(x)y)V^-$. Conversely, assume this to be the case. By 3.2(v), it suffices to show that $2x - Q(x)y$ belongs to the image of $B(x,y)$. Now $Q(x)y \in Q(x)V^-$, and by assumption there exists an element $z \in V^-$ such that $Q(x)y = Q(x-Q(x)y)z$. By JP23 it follows that

$$Q(x)y = B(x,y)Q(x)z \in \operatorname{Im} B(x,y) .$$

Furthermore,

$$\begin{aligned} B(x,y)x &= x - \{xyx\} + Q(x)Q(y)x \\ &= x - 2Q(x)y + Q(x)Q(y)x \\ &= x + Q(x)(-2y + Q(y)x) , \end{aligned}$$

which implies

$$x = B(x,y)x + Q(x)(2y - Q(y)x) .$$

Now we have

$$\begin{aligned} &Q(x)(2y - Q(y)x) \in \operatorname{Im} Q(x) = \operatorname{Im} Q(x-Q(x)y) \\ &= \operatorname{Im} B(x,y)Q(x) \subset \operatorname{Im} B(x,y) , \end{aligned}$$

which shows that x belongs to the image of $B(x,y)$. This completes the proof.

10.17. THEOREM. *The following conditions on a Jordan pair* V *with dcc on principal inner ideals are equivalent.*

(i) V *is von Neumann regular;*

(ii) V *is semisimple;*

(iii) V <u>is non-degenerate</u>.

<u>Proof</u>. The implication (i) $\to$ (ii) follows from 5.1, and the equivalence of (ii) and (iii) is a consequence of 10.8. Thus assume V to be semisimple, and let x be a non-zero element of V^+. By semisimplicity, there exists $y \in V^-$ such that (x,y) is not quasi-invertible. By 10.16 and JP23 this implies that $Q(x-Q(x)y)V^-$ is properly contained in $Q(x)V^-$. If $x_1 = x - Q(x)y$ is not zero we can find y_1 such that (x_1,y_1) is not quasi-invertible, and therefore $Q(x_1)V^- \supsetneq Q(x_2)V^-$ where $x_2 = x_1 - Q(x_1)y_1$. In this way, we obtain a strictly decreasing chain

$$Q(x)V^- \supsetneq Q(x_1)V^- \supsetneq Q(x_2)V^- \supsetneq \cdots$$

of principal inner ideals such that $x_{i+1} = x_i - Q(x_i)y_i$. By the dcc, we have $Q(x_n)V^- = 0$ for some n, and since V is non-degenerate this implies $x_n = 0$. This means that $x_{n-1} = Q(x_{n-1})y_{n-1}$ is von Neumann regular. Since $x_{n-1} = x_{n-2} - Q(x_{n-2})y_{n-2}$ it follows from 10.15 that x_{n-2} is von Neumann regular, and continuing in this way, we see that x is von Neumann regular. Hence V^+ consists of von Neumann regular elements. By passing to V^{op} the same follows for V^-.

§11. Classification of alternative pairs

11.0. The purpose of this section is to classify semisimple alternative pairs A (over a ring k) with dcc on principal inner ideals and containing a maximal idempotent e. By 10.11, we have $A_0^J(e) = A_{00}(e) = 0$. By 10.14, we may assume that A is simple. It turns out that a classification is possible under the weaker assumption that A be simple and contain an idempotent e with $A_{00}(e) = 0$, but does not necessarily satisfy any chain conditions. This is carried out in 11.1 - 11.11. Then we study inner ideals and obtain structure theorems under various chain conditions.

11.1. Let A be an alternative pair, and let e be an idempotent of A with $A_{00}(e) = 0$ (this assumption will be in force throughout). Then we have the Peirce decomposition

$$A = A_{11} \oplus A_{10} \oplus A_{01} ,$$

and from 9.3 it follows that of the 27 possible products $\langle A^\sigma_{ij}, A^{-\sigma}_{mn}, A^\sigma_{pq}\rangle$ at most the following 13 are non-zero (we use $\langle ij,mn,pq\rangle \subset rs$ as an abbreviation for $\langle A^\sigma_{ij}, A^{-\sigma}_{mn}, A^\sigma_{pq}\rangle \subset A^\sigma_{rs}$).

(1)	$\langle 11,11,11\rangle \subset 11$,	(8)	$\langle 01,11,01\rangle \subset 10$,
(2)	$\langle 11,11,10\rangle \subset 10$,	(9)	$\langle 11,01,10\rangle \subset 01$,
(3)	$\langle 01,11,11\rangle \subset 01$,	(10)	$\langle 10,01,11\rangle \subset 01$,
(4)	$\langle 10,10,11\rangle \subset 11$,	(11)	$\langle 01,10,01\rangle \subset 11$,
(5)	$\langle 11,01,01\rangle \subset 11$,	(12)	$\langle 01,10,10\rangle \subset 01$,
(6)	$\langle 10,10,10\rangle \subset 10$,	(13)	$\langle 10,01,01\rangle \subset 10$.
(7)	$\langle 01,01,01\rangle \subset 01$,		

Here (1) - (7) are of type 9.3.1 whereas the others are of type 9.3.2 - 9.3.4. We shall denote by $xy = \langle xe^{-\sigma}y\rangle$ the product in the alternative algebra $A^{\sigma}_{e^{-\sigma}}$. Let $R = R^+$ be the subalgebra A^+_{11} of $A^+_{e^-}$ with unit element e^+ and similarly let $R^- = A^-_{11}$. Recall hat

$$a \mapsto \bar{a} = \langle e^{-\sigma}ae^{-\sigma}\rangle$$

is an antiisomorphism between R^{σ} and $R^{-\sigma}$ with $\bar{\bar{a}} = a$, and that

$$\langle xay\rangle = (x\bar{a})y$$

for all $x,y \in A^{\sigma}$, $a \in R^{-\sigma}$ (cf. 9.2). We will often identify R^{op} and R^-. Then alternative pair $A_{11} = (A_{11},A_{11})$ is isomorphic with (R,R^{op}), and A_{11} is associative (resp. simple) if and only if R has these properties (cf. 6.12).

11.2. With the above notations, let $Z(R)$ be the center of the alternative algebra R. For $c \in Z(R)$ define $\tilde{c} = (c_+,c_-) \in \mathrm{End}(A^+) \times \mathrm{End}(A^-)$ by $c_+(x_{11} + x_{10} + x_{01}) = cx_{11} + cx_{10} + x_{01}c$ and $c_-(y_{11} + y_{10} + y_{01}) = \bar{c}y_{11} + \bar{c}y_{10} + y_{01}\bar{c}$ where $(x_{ij},y_{ij}) \in A_{ij}$. Then it follows from 9.6 that $\tilde{c}$ belongs to the centroid $Z(A)$ of A. Conversely, if $a = (a_+,a_-) \in Z(A)$ then $a_+(e^+) \in Z(R)$, and the map $c \mapsto \tilde{c}$ is an isomorphism from $Z(R)$ onto $Z(A)$ whose inverse is given by $a \mapsto a_+(e^+)$.

11.3. With A as above, let $B_{ij} \subset A_{ij}$ be pairs of submodules, and let $B = B_{11} \oplus B_{10} \oplus B_{01}$. Then B is an ideal of A if and only if B_{11} is an ideal of A_{11}, and

(1) $$B^{\sigma}_{11}\cdot A^{\sigma}_{10} + A^{\sigma}_{11}\cdot B^{\sigma}_{10} \subset B^{\sigma}_{10},$$

(2) $$A^{\sigma}_{01}\cdot B^{\sigma}_{11} + B^{\sigma}_{01}\cdot A^{\sigma}_{11} \subset B^{\sigma}_{01},$$

(3) $$\langle B^{\sigma}_{10},A^{-\sigma}_{10},e^{\sigma}\rangle + \langle e^{\sigma},A^{-\sigma}_{01},B^{\sigma}_{01}\rangle \subset B^{\sigma}_{11},$$

(4) $$\langle B^{\sigma}_{10},A^{-\sigma}_{01},e^{\sigma}\rangle + \langle A^{\sigma}_{10},B^{-\sigma}_{01},e^{\sigma}\rangle \subset B^{\sigma}_{01},$$

(5) $$A_{01}^{\sigma}\cdot B_{01}^{\sigma}\subset B_{10}^{\sigma}\ .$$

Obviously, these conditions are necessary. The sufficiency follows by checking that $\langle A_{ij}^{\sigma},A_{mn}^{-\sigma},B_{pq}^{\sigma}\rangle + \langle A_{ij}^{\sigma},B_{mn}^{-\sigma},A_{pq}^{\sigma}\rangle + \langle B_{ij}^{\sigma},A_{mn}^{-\sigma},A_{pq}^{\sigma}\rangle \subset B_{rs}^{\sigma}$, for all possible cases (1)-(13) of 11.1, using 9.6 extensively. We omit the details.

11.4. LEMMA. Let B_{11} be an ideal of A_{11}, and let $B_{10}^{\pm} = B_{11}^{\pm}\cdot A_{10}^{\pm}$, and $B_{01}^{\pm} = A_{01}^{\pm}\cdot B_{11}^{\pm}$. Then $B = B_{11}\oplus B_{10}\oplus B_{01}$ is an ideal of A. In particular, if A is simple then so are A_{11} and R.

The proof consists in a straightforward verification of (1) - (5) of 11.3, using 9.6 and the definition of B.

11.5. LEMMA. If A is simple then A and the A_{ij} are semisimple.

Indeed, since A contains a non-zero idempotent, Rad A is a proper ideal and therefore zero. Now the assertion follows from 9.9.

11.6. Assume that A_{11} and therefore R are associative. Then by 9.6.1, A_{10}^{+} is a left R-module, and A_{10}^{-} is a right R-module by setting $ya = \bar{a}y$, for $a \in R = A_{11}^{+}$, $y \in A_{10}^{-}$. Define $\Phi : A_{10}^{+}\times A_{10}^{-}\to R$ by

$$\Phi(x,y) = \langle xye^{+}\rangle.$$

Then $\Phi(ax,y) = a\Phi(x,y)$ and $\Phi(x,ya) = \langle x,\bar{a}y,e^{+}\rangle = \overline{\langle \bar{a}y,x,e^{-}\rangle} = \overline{\bar{a}\langle y,x,e^{-}\rangle}$ $= \Phi(x,y)a$ by 9.6.3 and 9.6.4. Thus Φ is a R-bilinear form on the pair of modules A_{10} as in 6.4. Usually it will be more convenient to consider A_{10}^{σ} as a left-R^{σ}-module (remember that $R^{+} = R$, $R^{-} = R^{op}$) and define $\Phi_{\sigma}: A_{10}^{\sigma}\times A_{10}^{-\sigma}\to R^{\sigma}$ by $\Phi_{\sigma}(x,y) = \langle xye^{\sigma}\rangle$ so that $\Phi_{-\sigma}(y,x) = \overline{\Phi_{\sigma}(x,y)}$, $\Phi_{\sigma}(ax,y) = a\Phi_{\sigma}(x,y)$ and $\Phi_{\sigma}(x,ay) = \bar{a}\Phi_{\sigma}(x,y)$; i.e., Φ_{σ} is "hermitian".

Similarly, A_{01}^{σ} is a right R^{σ}-module by 9.6.2, and we have "hermitian" forms $\Psi_{\sigma}: A_{01}^{\sigma} \times A_{01}^{-\sigma} \to R^{\sigma}$ given by $\Psi_{\sigma}(x,y) = \langle e^{\sigma}yx\rangle$, which by 9.6.6 and 9.6.7 satisfy $\overline{\Psi_{\sigma}(y,x)} = \Psi_{-\sigma}(x,y)$, $\Psi_{\sigma}(xa,y) = \Psi_{\sigma}(x,y)a$, $\Psi_{\sigma}(x,ya) = \bar{a}\Psi_{\sigma}(x,y)$. We set $\Psi = \Psi_{+}$. Non-degeneracy of Φ and Ψ is defined as in 6.4.

11.7. LEMMA. Let A be simple. If $A_{10} \neq 0$ or $A_{01} \neq 0$ then R (and therefore A_{11}) is associative. The bilinear forms Φ and Ψ are non-degenerate.

Proof. By 9.6.1, the map $F: R \to \mathrm{End}(A_{10}^{+})$ defined by $F(a)(x) = ax$ is an algebra homomorphism, and by definition of the Peirce spaces, $F(1)$ is the identity on A_{10}^{+}. Thus if $A_{10}^{+} \neq 0$ then the kernel of F is a proper ideal of R. By 11.4, R is simple and hence F is injective. Therefore R, being isomorphic with a subalgebra of the associative algebra $\mathrm{End}(A_{10}^{+})$, is associative. A similar proof applies if A_{10}^{-} or $A_{01}^{\pm}$ is non-zero. Now let $x \in A_{10}^{\sigma}$ and assume that $\Phi_{\sigma}(x,A_{10}^{-\sigma}) = 0$. Then by 9.6.12, $Q(x)(A_{10}^{-\sigma}) = 0$, i.e., x is a trivial element of A_{10}. By 11.5, this implies $x = 0$. Hence Φ is non-degenerate. Similarly, if $\Psi_{\sigma}(u,A_{01}^{-\sigma}) = 0$ then it follows from (11) and (13) of 9.6 that $Q(u)(A_{01}^{-\sigma}) = 0$ which implies $u = 0$ as before.

11.8. LEMMA. Let A be simple. Then A is associative if and only if A_{11} is associative and either $A_{10} = 0$ or $A_{01} = 0$.

Proof. If A is associative then so is A_{11}. Also, $\Phi(A_{10})$ and $\Psi(A_{01})$ are ideals of R which annihilate each other. Indeed, for all $(x,y) \in A_{10}$, $(u,v) \in A_{01}$ we have $\Phi(x,y)\Psi(u,v) = \langle xye^{+}\rangle\langle e^{+}vu\rangle = \langle\langle xye^{+}\rangle vu\rangle = \langle x\langle ve^{+}y\rangle u\rangle$, using 9.6.8 and associativity, and $\langle ve^{+}y\rangle \in A_{00}^{-} = 0$. Since A is simple by 11.4 we have $\Phi(A_{10}) = 0$ or $\Psi(A_{01}) = 0$, and by 11.7 this implies $A_{10} = 0$ or $A_{01} = 0$. Conversely, if $A_{10} = 0$ or $A_{01} = 0$ then all products (8) - (13) of 11.1 vanish, and by 9.8, $\mathcal{A}$ is associative provided $\mathcal{A}_{11}$ is.

11.9. LEMMA. *Let* $\mathcal{A}$ *be simple and let* $\mathcal{A}_{10}$ *and* $\mathcal{A}_{01}$ *be both non-zero. Then* $\mathcal{R}$ *is commutative and, being associative and simple, is therefore an extension field of the base ring* k .

Proof. Let $\mathcal{U}^{\sigma} = \mathcal{A}_{01}^{\sigma}\cdot\mathcal{A}_{01}^{\sigma} \subset \mathcal{A}_{10}^{\sigma}$. We claim that $\mathcal{U}^{+}$ and $\mathcal{U}^{-}$ cannot both be zero. Indeed, if $\mathcal{U}^{+} = \mathcal{U}^{-} = 0$ then we get by (11), (14), and (16) of 9.6 that the products (8), (11), and (13) of 11.1 vanish. For $a \in \mathcal{A}_{11}^{\sigma}$, $u,v \in \mathcal{A}_{01}^{-\sigma}$, $f \in \mathcal{A}_{10}^{\sigma}$ we have by AP16,

$$Q(\langle auf\rangle)v = \langle\langle auf\rangle v\langle auf\rangle\rangle = \langle au\langle f\langle uav\rangle f\rangle\rangle = 0$$

since $\langle uav\rangle = 0$. Hence by 11.5, $\langle auf\rangle = 0$, and by 9.6.9, the products (9) and (10) of 11.1 vanish. Similarly, if $g \in \mathcal{A}_{10}^{-\sigma}$ then $Q(\langle ufg\rangle) = \langle uf\langle g\langle fuv\rangle g\rangle\rangle = 0$ since $\langle fuv\rangle = 0$ which implies $\langle ufg\rangle = 0$. Thus also the products (13) of 11.1 are zero. By 11.7, $\mathcal{A}_{11}$ is associative, and by 9.8, it follows that $\mathcal{A}$ is associative, contradicting 11.8.

We therefore have $\mathcal{U}^{+}$ or $\mathcal{U}^{-}$ different from zero, and, possibly after passing to $\mathcal{A}^{op}$, we may assume that $\mathcal{A}^{+} \neq 0$. By 9.6.11, $\mathcal{U}^{+}$ is a $\mathcal{R}$-submodule of $\mathcal{A}_{10}^{+}$, and for $a,b \in \mathcal{R}$, $u,v \in \mathcal{A}_{01}^{+}$ we have

$$(ab)(uv) = a(b(uv)) = a((ub)v) = ((ub)a)v = (u(ba))v = (ba)(uv).$$

Since $\mathcal{R}$ is simple this implies that $\mathcal{R}$ is commutative, and therefore an extension field of the base ring k .

In the above situation, we will *identify* $\mathcal{R}$ and $\mathcal{R}^{op} = \mathcal{R}^{-}$ via $a \mapsto \bar{a}$, and write K instead of $\mathcal{R}$. Clearly, $\mathcal{A}_{10}^{\pm}$ and $\mathcal{A}_{01}^{\pm}$ are vector spaces over K and Φ and Ψ (cf. 11.6) are non-degenerate bilinear forms on $\mathcal{A}_{10}^{+} \times \mathcal{A}_{10}^{-}$ and $\mathcal{A}_{01}^{+} \times \mathcal{A}_{01}^{-}$ with values in K . Also, $\Phi_{-}(y,x) = \Phi_{+}(x,y) = \Phi(x,y)$, and similarly for Ψ .

11.10. LEMMA. Let A be as in 11.9, and for any $f \in A_{10}^{\sigma}$ define $J_f: A_{01}^{-\sigma} \to A_{01}^{\sigma}$ by $J_f(u) = \langle e^{\sigma} u f\rangle$. Let $(f^+, f^-) \in A_{10}$. Then

(1) $$J_{f^+} \circ J_{f^-} = -\Phi(f^+,f^-).\mathrm{Id}_{A_{01}^+} \quad \text{and} \quad J_{f^-} \circ J_{f^+} = -\Phi(f^+,f^-).\mathrm{Id}_{A_{01}^-} .$$

Moreover, $A_{10}^{\pm}$ is one-dimensional over K .

Proof. From AP1 we get $u\Phi(f^+,f^-) = \langle ue^- \langle f^+ f^- e^+\rangle\rangle = \langle\langle ue^- f^+\rangle f^- e^+\rangle + \langle f^+ \langle e^- u f^-\rangle e^+\rangle$ $- \langle f^+ f^- \langle ue^- e^+\rangle\rangle$, and $\langle ue^- f^+\rangle = \langle f^+ f^- u\rangle = 0$ by the composition rules for the Peirce spaces. By 9.6.9, we have $\langle f^+ \langle e^- u f^-\rangle e^+\rangle = -\langle e^+ \langle e^- u f^-\rangle f^+\rangle = -J_{f^+}(J_{f^-}(u))$. This proves the first part of (1) and the second follows by passing to A^{op} . Now choose (f^+,f^-) such that $\Phi(f^+,f^-) = 1$ which is possible since Φ is non-degenerate. Let $H = \{h \in A_{10}^+ \mid \Phi(h,f^-) = 0\}$. Then H has codimension one in A_{10}^+ . If $h \in H$ then $J_h \circ J_{f^-} = 0$ and $J_{f^-} \circ J_{f^+} = -\mathrm{Id}$ by (1) so that $J_h \circ J_{f^-} \circ J_{f^+}$ $= -J_h = 0$ which implies $h = 0$. Indeed, otherwise we could find $h' \in A_{10}^-$ such that $\Phi(h,h') = 1$ by non-degeneracy of Φ , and then $-J_h \circ J_{h'}$ is the identity on A_{10}^+ , a contradiction. Thus $H = 0$ and A_{10}^+ is one-dimensional. Again by non-degeneracy of Φ so is A_{10}^- .

11.11. THEOREM. The simple alternative pairs A over k containing an idempotent e with $A_{00}(e) = 0$ are up to isomorphism precisely the following.

(A) Associative pairs $A(M,R,\Phi)$ (as in 6.4) where R is a simple associative k-algebra with unity, Φ is non-degenerate, and there exists $(e^+,e^-) \in M$ such that $\Phi(e^+,e^-) = 1$.

(A') The reverse A' of an associative pair A of type (A) (cf. 6.3).

(B) Alternative pairs (C, C^{op}) (as in 6.5) where C is a Cayley algebra over an extension field K of k .

(C) Alternative pairs $A(X,K,\alpha)$ (as in 6.6) where X is a vector space of dimension ≥ 4 over an extension field K of k and α is non-degenerate.

The pairs of type (A) and (A') are associative, the others are properly alternative.

Proof. Let A and e be as in the statement of the Theorem, and first assume that A is associative. Then $R = A_{11}^{+}$ is a simple associative k-algebra with unity (11.4). By 11.8, either $A_{10} = 0$ or $A_{01} = 0$. Suppose that $A_{01} = 0$. By 11.6, A_{10}^{σ} is a left-R^{σ}-module and $\Phi_{\sigma}: A_{10}^{\sigma} \times A_{10}^{-\sigma} \to R^{\sigma}$ is a hermitian form which is non-degenerate (the case $A_{10} = 0$ is not excluded). We extend Φ_{σ} to a hermitian form on $A^{\sigma} \times A^{-\sigma}$ with values in R^{σ} by setting

$$\Phi_{\sigma}(a+x,b+y) = a\bar{b} + \Phi_{\sigma}(x,y),$$

for $(a,b) \in A_{11}^{\sigma} \times A_{11}^{-\sigma}$, $(x,y) \in A_{10}^{\sigma} \times A_{10}^{-\sigma}$. One checks immediately that Φ_{σ} is non-degenerate. We have (for $c \in A_{11}^{\sigma}$, $z \in A_{10}^{\sigma}$)

$$\langle a+x,b+y,c+z\rangle = \langle abc\rangle + \langle xyc\rangle + \langle abz\rangle + \langle xyz\rangle,$$

the other products being zero by 9.3. Now $\langle abc\rangle = a\bar{b}c$, $\langle xyc\rangle = \langle xye^{\sigma}\rangle c = \Phi_{\sigma}(x,y)c$, $\langle abz\rangle = a\bar{b}z$, $\langle xyz\rangle = \Phi_{\sigma}(x,y)z$ by 9.6. Hence

$$\langle a+x,b+y,c+z\rangle = \Phi_{\sigma}(a+x,b+y)(c+z)$$

and A is indeed of type (A). Next assume that $A_{10} = 0$ and A is still associative. Then the reverse A' of A satisfies $A'_{01} = A_{10} = 0$, and hence A' is of type (A) which implies that $A'' = A$ is of type (A').

Now let A be properly alternative. By 11.8, A_{10} and A_{01} are either both zero or both non-zero. In the first case, $A = A_{11} = (R,R^{op})$ where R is a simple unital properly alternative algebra. By Kleinfeld[1], R is a Cayley algebra over its center K which is an extension field of k, and so A is of type (B). In the second case, let $R = K$ as in 11.9, and pick $(f^{+},f^{-}) \in A_{10}$ such that $\Phi(f^{+},f^{-}) = 1$. Then $A_{10}^{\sigma} = K.f^{\sigma}$ by 11.10 so that $A^{\sigma} = K.e^{\sigma} \oplus K.f^{\sigma} \oplus$

A^σ_{01} . Define $J_\sigma: A^{-\sigma} \to A^\sigma$ by

$$J_\sigma(e^{-\sigma}) = f^\sigma, \quad J_\sigma(f^\sigma) = -e^{-\sigma}, \quad J_\sigma \mid A^{-\sigma}_{01} = J_{f^{-\sigma}} ,$$

where $J_{f^{-\sigma}}$ is as in 11.10. Then we have $J_\sigma \circ J_{-\sigma} = -\mathrm{Id}$ by 11.10. Also define alternating bilinear forms $\alpha_\sigma: A^\sigma \times A^\sigma \to K$ by

$$\alpha_\sigma(e^\sigma,f^\sigma) = 1, \quad \alpha_\sigma(u,v).f^\sigma = uv \quad \text{for} \quad u,v \in A^\sigma_{01}, \quad \alpha_\sigma(K.e^\sigma \oplus K.f^\sigma, A^\sigma_{01}) = 0.$$

We claim that

$$\alpha_\sigma(J_\sigma x, J_\sigma y) = \alpha_{-\sigma}(x,y),$$

for all $x,y \in A^{-\sigma}$. If $x,y \in K.e^{-\sigma} \oplus K.f^{-\sigma}$ this is clear from the definitions. Let $u \in A^\sigma_{01}$, $v \in A^{-\sigma}_{01}$. Then by AP2,

$$\langle\langle e^\sigma vu\rangle e^{-\sigma} f\rangle + \langle\langle e^\sigma v f^\sigma\rangle e^{-\sigma} u\rangle = \langle e^\sigma, v, \langle u e^{-\sigma} f^\sigma\rangle + \langle f^\sigma e^{-\sigma} u\rangle\rangle = 0$$

since $\langle u e^{-\sigma} f^\sigma\rangle = \langle f^\sigma e^{-\sigma} u\rangle = 0$ by 9.3. This means that

$$\Psi_\sigma(u,v).f^\sigma + J_\sigma(v).u = (\Psi_\sigma(u,v) + \alpha_\sigma(J_\sigma v,u)).f^\sigma = 0,$$

and therefore we have

$$\Psi_\sigma(u,v) = \alpha_\sigma(u,J_\sigma v). \tag{1}$$

Now if $w \in A^{-\sigma}_{01}$ then $\alpha_\sigma(J_\sigma w, J_\sigma v) = \Psi_\sigma(J_\sigma w, v) = \Psi_{-\sigma}(v, J_\sigma w) = \alpha_{-\sigma}(v, J_{-\sigma} J_\sigma w)$ $= \alpha_{-\sigma}(v,-w) = \alpha_{-\sigma}(w,v)$. Using (1) and 9.6 it is now easy to verify that in each of the 13 cases listed in 1.11 $\langle xyz\rangle$ is given by

$$\langle xyz\rangle = x.\alpha_\sigma(z,J_\sigma(y)) + J_\sigma(y)\alpha_\sigma(x,z).$$

Thus if we set $X = A^+$, $\alpha = \alpha_+$ then $A = A(X,K,\alpha)$ as in 6.6. From (1) and the non-degeneracy of Ψ it follows that α is non-degenerate. Hence $\dim_K X$ is even and therefore ≥ 4 which implies that A is of type (C).

Next we show that the pairs listed are simple and contain idempotents e with $A_{00}(e) = 0$. The pairs of type (A), (A'), and (B) are simple by 6.4 and 6.5.

If A is of type (A) then (e^+,e^-) is an idempotent with $A_{00}(e) = 0$, and in case (B), $(1,1)$ is such an idempotent (where 1 is the unit element of C). Let $A = A(X,K,\alpha)$ be of type (C). Choose $a,b \in X = A^+$ such that $\alpha(a,b) = 1$. Let $(e^+,e^-) = (a,-J_-(b))$. Then $\alpha_\sigma(e^\sigma,J_\sigma e^{-\sigma}) = 1$ and $e = (e^+,e^-)$ is an idempotent of A. Let $f^\sigma = J_\sigma(e^{-\sigma})$. Then we have for $x \in A^\sigma$:

$$\langle e^\sigma e^{-\sigma} x\rangle = e^\sigma.\alpha_\sigma(x,f^\sigma) + f^\sigma.\alpha_\sigma(e^\sigma,x), \tag{2}$$

$$\langle x e^{-\sigma} e^\sigma\rangle = x + f^\sigma.\alpha_\sigma(x,e^\sigma) . \tag{3}$$

Now if $x \in A^\sigma_{00}(e)$ then x is a multiple of f^σ by (3) and since α_σ is alternating, (2) implies $\alpha_\sigma(e^\sigma,x) = 0$ and hence $x = 0$. From (2) and (3) we also see that $A^\sigma_{10} = K.f^\sigma$ and $A^\sigma_{11} = K.e^\sigma$. Now let B be a non-trivial ideal of A. Then we may choose $e = (e^+,e^-) \in B$ as above, and by definition of the Peirce spaces and the fact that B is an ideal we have $A_{ij} = B_{ij}$ for ij = 11, 10, 01. Since $A_{00} = 0$ we have $B = A$. Therefore A is simple. Since A^σ is at least 4-dimensional over K while A^σ_{11} and A^σ_{10} are one-dimensional we must have $A_{01} \neq 0$, and by 11.8, A is properly alternative. This completes the proof.

11.12. <u>Remarks</u>. (i) Let $A = A(X,K,\alpha)$ with α non-degenerate, and let e^σ and f^σ be as above. If $\dim_K X = 2$ then $A_{01} = 0$, and by 11.8, A is associative. Indeed, it is easily seen that $A = A(M,K,\Phi)$ where $M^\pm = K^2$ and $\Phi(x,y) = x_1y_1 + x_2y_2$. In any case, A contains no invertible elements since $Q(x)y = \langle xyx\rangle = x.\alpha_\sigma(x,J_\sigma y)$ and hence $Q(x)$ has at most rank one and therefore cannot be invertible. Hence no pair of type (C) is isomorphic with a pair of type (B). Also there is obviously no overlap between type (A) or (A') and (B) or (C) since the former are associative while the latter are not.

(ii) The pairs of type (B) and (C) possess involutions. Indeed, let $A = (C,C^{op})$ be of type (B), and let j be any involution of C, for instance the standard involution. If we denote as usual by $a \mapsto \bar{a}$ the canonical antiisomorphism from

$\mathcal{C}$ to $\mathcal{C}^{op}$ and conversely then one checks immediately that $\eta_\sigma(a) = \overline{j(a)}$ defines an involution η of $(\mathcal{C},\mathcal{C}^{op})$. In other words, $\mathcal{C}$ is an alternative triple system with $\langle abc\rangle = (aj(b))c$ (cf. 6.15). If $\mathcal{A}$ is of type (C) then $\eta_\sigma = J_{-\sigma}$ defines an involution of $\mathcal{A}$. In contrast, the pairs of type (A) or (A') do not in general have involutions. For example, let $R = K$ be a field, let M^+ be an infinite-dimensional vector space over K, let M^- be the dual of M^+, and set $\Phi(x,y) = y(x)$ (= the value of the linear form y on x). Then M^+ and M^- are of different dimension, and therefore $\mathcal{A}$ can have no involutions.

(iii) The centroid of a pair of type (A) or (A') is isomorphic with the center of the simple associative algebra R. For alternative pairs of type (B) or (C) the centroid is isomorphic with K. This follows from 11.2 and 11.9.

11.13. LEMMA. Let R be a semisimple unital associative ring satisfying the dcc on principal inner ideals. Then every right ideal of R is generated by an idempotent, and R is Artinian.

Proof. We show first that every non-zero right ideal $\mathfrak{r}$ of R contains a non-zero idempotent. Not every element of $\mathfrak{r}$ can be nilpotent since otherwise $\mathfrak{r}$ would be a quasi-invertible right ideal of R, contradicting the assumption that R is semisimple. Thus let $x \in \mathfrak{r}$ be not nilpotent, and consider the chain of principal inner ideals $xRx \supset x^2Rx^2 \supset \dots$. By the dcc, $x^nRx^n = x^{n+1}Rx^{n+1}$ for some n, and hence $x^{2n} \in x^nRx^n = x^{2n}Rx^{2n}$ is regular, i.e., $x^{2n} = x^{2n}yx^{2n}$ for some y. Then $x^{2n}y$ is a non-zero idempotent contained in $\mathfrak{r}$. Observe now that R satisfies the chain condition on idempotents. Indeed, if $\{e_1,e_2,\dots\}$ is an orthogonal set of idempotents then $(1-e_1)R(1-e_1) \supset (1-e_1-e_2)R(1-e_1-e_2) \supset \dots$ is a descending chain of principal inner ideals. Therefore we may choose a maximal idempotent $e \in \mathfrak{r}$. Then we have $\mathfrak{r} = eR \oplus \mathfrak{r} \cap (1-e)R$, using the fact that e is in $\mathfrak{r}$ and that $\mathfrak{r}$ is a right ideal. Let $d \in \mathfrak{r} \cap (1-e)R$ be an idempotent. Then

$ed = 0$ which implies $(d(1-e))^2 = d^2 - d^2e - ded + dede = d - de = d(1-e)$, and hence $c = d(1-e) \in \mathcal{r}$ is an idempotent with $ec = ce = 0$. By maximality of e we have $c = 0$ and therefore $d = de = d^2 = ded = 0$. By what we proved before, the right ideal $\mathcal{r} \cap (1-e)R$ is zero, and thus $\mathcal{r} = eR$. Now let $eR \subset fR$ be right ideals where e and f are idempotents. Then $eR = e^2R \subset efR \subset eR$ which shows that $eR = efR$. Also, $(1-f)e \in (1-f)fR = 0$, i.e., $fe = e$. From this it follows easily that ef and $f - ef$ are orthogonal idempotents. Thus if $e_1R \supset e_2R \supset \ldots$ is a descending chain if right ideals then we obtain (replacing e_i by e_ie_{i-1} if necessary) a set $\{e_1-e_2, e_2-e_3, \ldots\}$ of orthogonal idempotents. Since R satisfies the cci it follows that R is Artinian.

11.14. PROPOSITION. (a) <u>If</u> $A = A(M,R,\Phi)$ <u>is an associative pair of type (A) as in 11.11, satisfying the dcc on principal inner ideals, then</u> R <u>is Artinian.</u>
(b) <u>Conversely, if</u> $A = A(M,R,\Phi)$ <u>as in 6.4 with</u> R <u>simple Artinian and</u> Φ <u>non-degenerate then</u> A <u>is von Neumann regular and satisfies the dcc and acc on principal inner ideals; in particular, it is of type (A).</u>

<u>Proof</u>. (a) Since R is simple with unity it is semisimple. Also, the dcc on principal inner ideals carries over from A to $R = A^+_{11}$ by 10.2. Now (a) follows from 11.13.
(b) Since A^{op} satisfies our hypotheses as well it suffices to show that every $x \in M^+ = A^+$ is regular, and that chains of principal inner ideals contained in M^+ are of finite length. If $x \in M^+$ then $\Phi(x,M^-)$ is a right ideal of R, and by well-known facts on semisimple Artinian rings, $\Phi(x,M^-) = eR$ where e is an idempotent of R. Hence $0 = (1-e)\Phi(x,M^-) = \Phi(x-ex,M^-)$ and since Φ is non-degenerate we have $x - ex = 0$. Also, $e = \Phi(x,y)$ for some $y \in M^-$ which implies $x = ex = \Phi(x,y)x = \langle xyx \rangle$ and x is regular. Now let $z \in M^+$ and $\Phi(z,M^-) = fR$ where f is an idempotent. Assume that the principal inner ideals generated by

x and z are contained in each other, $\langle xM^-x\rangle \subset \langle zM^-z\rangle$. Then $x = faz$ for some $a \in R$ and therefore $eR = \Phi(x,M^-) = \Phi(faz,M^-) = fa\Phi(z,M^-) = fafR \subset fR$. The element $b = faf$ belongs to the Peirce space R_{11} of R with respect to f. Now if $eR = bR = fR$ then $bR_{11} = R_{11}$ which implies that b is invertible in R_{11} (here we use that R_{11} is Artinian, too). Hence there exists $c \in R_{11}$ such that $cb = f$. It follows that $cx = cfaz = cfafz = cbz = fz = z$ and therefore $\langle zM^-z\rangle = eRz = eRcx \subset eRx = \langle xM^-x\rangle$. Altogether, we see that a chain of principal inner ideals of M^+ gives rise to a chain of right ideals of R , and that one becomes stationary if and only if the other one does. Hence A satisfies the dcc and acc on principal inner ideals. By 10.7, it satisfies the cci and hence contains a maximal idempotent e which by 10.11 satisfies $A_{00}(e) = 0$. This proves that A is of type (A).

11.15. PROPOSITION. (a) Let $A = (C,C^{op})$ be a simple alternative pair of type (B) where C is a Cayley algebra over its center K . Then the inner ideals of A contained in C are C itself and those K-subspaces of C which are totally isotropic with respect to the norm of C . In particular, A satisfies the dcc and acc on all inner ideals.

(b) Let $A = A(X,K,\alpha)$ be an alternative pair of type (C). Then the principal inner ideal generated by $x \in A^\sigma$ is $K.x$, and therefore A satisfies the dcc and acc on principal inner ideals. The inner ideals of A^σ are precisely the K-subspaces of A^σ (= X), and therefore A satisfies the dcc and acc on all inner ideals if and only if X is finite-dimensional over K .

Proof. (a) The inner ideals of C depend only on the Jordan algebra C^J associated with C , and C^J is the Jordan algebra defined by the norm form of C . Now the assertion follows from McCrimmon[8], Th. 6.

(b) For $x \in A^\sigma$ we have $\langle xyx\rangle = x.\alpha_\sigma(x,J_\sigma y)$ which shows that x is regular and that the principal inner ideal generated by x is $K.x$. If m is an inner ideal and $x \in m$ and $\lambda \in K$ then $x = \langle xyx \rangle$ for some y implies $\lambda x = \langle x, \lambda y, x \rangle \in m$ and hence m is a K-subspace of A^σ . Conversely, it is obvious that every K-subspace is an inner ideal.

11.16. THEOREM. For a semisimple alternative pair over k with dcc on principal inner ideals the following conditions are equivalent.

(i) The acc on principal inner ideals;

(ii) the chain condition on idempotents;

(iii) the existence of a maximal idempotent.

The simple pairs with these properties are up to isomorphism precisely the following.

(A) $A(M,R,\Phi)$ with R a simple Artinian algebra over k and Φ non-degenerate.

(A') The reverse A' of a pair A of type (A).

(B) (C,C^{op}) with C a Cayley algebra over an extension field K of k .

(C) $A(X,K,\alpha)$ with X a vector space over an extension field K of k and α non-degenerate.

Proof. By 10.7 and 10.10 we have (i) $\to$ (ii) $\to$ (iii). By 10.14 it suffices to prove (iii) $\to$ (i) for simple pairs. By 11.11, 11.14, and 11.15 the simple semisimple alternative pairs with dcc on principal inner ideals are precisely the ones listed, and they have acc on principal inner ideals. This proves the theorem.

11.17. PROPOSITION. Let $A = A(M,R,\Phi)$ with R simple Artinian and Φ non-degenerate. Then the following conditions are equivalent.

(i) A satisfies the dcc on all inner ideals;

(ii) A satisfies the acc on all inner ideals;

(iii) M^+ is a finitely generated R-module.

If these conditions are satisfied then we have

(a) A is isomorphic with $(M_{p,q}(\mathcal{D}), M_{p,q}(\mathcal{D}^{op}))$ where $\mathcal{D}$ is a division algebra over k and the product is given by $\langle xyz\rangle = xy^*z$ (cf. 0.5 for notation). Also $R \cong M_p(\mathcal{D})$ and Φ is given by $\Phi(x,y) = xy^*$.

(b) The reverse A' of A is isomorphic with $(M_{q,p}(\mathcal{D}^{op}), M_{q,p}(\mathcal{D}))$ under the map $x \to x^*$.

(c) The inner ideals of A contained in $A^+ = M^+ = M_{p,q}(\mathcal{D})$ are of the form eM^+f where $e \in M_p(\mathcal{D})$ and $f \in M_q(\mathcal{D})$ are idempotents.

(d) The maximum length of a chain $0 \subsetneqq m_1 \subsetneqq m_2 \subsetneqq \cdots \subsetneqq m_r = M^+$ of inner ideals is $r \leqq p+q-1$.

Proof. Clearly, every R-submodule of M^+ is an inner ideal, and therefore (i) or (ii) implies (iii) (recall that every R-module is completely reducible). Conversely, assume that (iii) holds. Since Φ is non-degenerate we have injections $i_-: M^- \to (M^+)^\vee$ (the dual of M^+ as a R-module) and $i_+: M^+ \to (M^-)^\vee$ given by $i_-(y)(x) = \Phi(x,y) = i_+(x)(y)$. Hence M^- is finitely generated as well. To show that i_- is surjective it suffices to show that $(i_-)^\vee : (M^+)^{\vee\vee} \to (M^-)^\vee$ is injective. But one checks that the diagram

$$\begin{tikzcd} M^+ \arrow[rr, "\text{can.}"] \arrow[dr, "i_+"'] & & (M^+)^{\vee\vee} \arrow[dl, "(i_-)^\vee"] \\ & (M^-)^\vee & \end{tikzcd}$$

commutes, and hence the canonical map from M^+ into $(M^+)^{\vee\vee}$ is an isomorphism. Therefore, we may identify M^- with $(M^+)^\vee$ and then Φ is given by $\Phi(x,y) =$

$y(x)$. By the structure theorem for simple Artinian rings, $R = M_p(\mathcal{D})$, a full matrix algebra over a division algebra $\mathcal{D}$. Also, M^+ is isomorphic as R-module with a finite direct sum of p-dimensional right vector spaces over $\mathcal{D}$, say $\mathcal{D}^p$ written as column vectors, and R acts by matrix multiplication on the left. Thus $M^+ = M_{p,q}(\mathcal{D})$. We may identify $(M^+)^\vee$ with $M_{p,q}(\mathcal{D}^{op})$ and Φ is then given by $\Phi(x,y) = xy^*$. This proves (a) (under the assumption of (iii)). Now (b) is immediate since $(xy^*z)^* = z^*yx^*$.

Next we prove (c). Let $m \subset M^+ = M_{p,q}(\mathcal{D})$ be an inner ideal, and choose idempotents $e \in R = M_p(\mathcal{D})$ and $f \in S = M_q(\mathcal{D})$ maximal with respect to the property that $eM^+f \subset m$. We wish to show that we have equality or, equivalently, that $(1-e)m = m(1-f) = 0$. Note first that e and f are non-zero unless m is zero. Indeed, if $x \in m$ is not zero then by regularity (11.14(b)) we have $x = \langle xyx\rangle = xy^*x$ for some $y \in M^- = M_{p,q}(\mathcal{D}^{op})$ so that $e = xy^*$ and $f = y^*x$ are non-zero idempotents of R and S respectively with $eM^+f = xy^*M^+y^*x \subset \langle xM^-x\rangle \subset m$, since m is an inner ideal. Consequently, we have $M^+f(M^-)^* = R$ and $(M^-)^*eM^+ = S$ since R and S are simple rings. Now let $x \in m$ and set $z = x(1-f)$. We will show that $z \in m$. Since z is regular we have $z \in \langle zM^-z\rangle = x(1-f)(M^-)^*x(1-f) \subset x(1-f)(M^-)^*x + x(1-f)(M^-)^*xf$. Now $x(1-f)(M^-)^*x \subset x(M^-)^*x = \langle x,M^-,x\rangle \subset m$ since m is an inner ideal. Therefore it remains to show that $x(1-f)(M^-)^*xf \subset m$. We have $(M^-)^*x \subset S = (M^-)^*eM^+$, and therefore $x(1-f)(M^-)^*xf \subset x((1-f)(M^-)^*)eM^+f = \{x,M^-(1-f^*),eM^+f\} \subset m$, since $f(1-f) = 0$ and m is an inner ideal. Here $\{abc\} = \langle abc\rangle + \langle cba\rangle = ab^*c + cb^*a$. By regularity, $z = zv^*z$ for some v , and since $1 - f$ is an idempotent we may assume that $(1-f)v^* = v^*$. Then $fv^* = zf = 0$ and therefore the idempotent $g = v^*f \in S$ satisfies $fg = gf = 0$; i.e., g is orthogonal to f and therefore $g + f$ is an idempotent. Now $eM^+g = eM^+v^*f \subset eM^+f(M^-)^*z = (eM^+f)(f(M^-)^*z) = \{eM^+f,M^-f^*,z\} \subset m$ since $zf = 0$ and m is an inner ideal. Thus also $eM^+(f+g) \subset m$, and by maximality of f we have $g = 0$ which implies $z = zg = 0$, and we have shown that $m(1-f) = 0$. In order to prove $(1-e)m = 0$ it suffices to prove $m^*(1-e^*) = 0$ for the inner ideal $m^* \subset (M^-)^*$ of A' (using (b)), and the proof of this is the same as above.

Now we prove (d) and give first a geometric interpretation of (c) as follows. Let $X = \mathcal{D}^q$, $Y = \mathcal{D}^p$ (right vector spaces of column vectors over $\mathcal{D}$). Then $M^+ = M_{p,q}(\mathcal{D})$ is naturally isomorphic with $\mathrm{Hom}_{\mathcal{D}}(X,Y)$, and the inner ideals contained in M^+ are precisely the spaces $m_{U,V} = \{x \in M^+ \mid \mathrm{Ker}(x) \supset U,\ \mathrm{Im}(x) \subset V\}$, for vector subspaces $U \subset X$, $V \subset Y$. Clearly $m_{U,V} \subset m_{U',V'}$ if and only if $U' \subset U$ and $V' \supset V$. This implies (d), and also proves that (iii) implies (i) and (ii).

11.18. THEOREM. <u>A semisimple alternative pair over</u> k <u>with dcc on all inner ideals satisfies the acc on all inner ideals</u>. <u>The simple pairs of this kind are</u>:

(A) $(M_{p,q}(\mathcal{D}), M_{p,q}(\mathcal{D}^{op}))$ <u>with</u> $\mathcal{D}$ <u>a division algebra over</u> k <u>and</u> $\langle xyz\rangle = xy^*z$.

(B) (C, C^{op}) <u>with</u> C <u>a Cayley algebra over an extension field</u> K <u>of</u> k .

(C) $A(X,K,\alpha)$ <u>with</u> X <u>a finite-dimensional vector space over an extension field</u> K <u>of</u> k <u>and</u> α <u>non-degenerate</u>.

<u>Proof</u>. By 10.7, pairs with dcc on all inner ideals have maximal idempotents, and are therefore (10.14) direct products of simple ones. By 11.11 - 11.17 a simple pair is one of the above, and conversely the pairs listed have acc on all inner ideals.

§12. Classification of Jordan pairs

12.1. LEMMA. *Let V be a Jordan pair and let e be an idempotent of V such that $V_0(e) = 0$. Let $U_i \subset V_i = V_i(e)$ be pairs of submodules. Then $U = U_2 \oplus U_1$ is an ideal of V if and only if U_i is an ideal of the Jordan pair V_i and (with the notations of 8.0)*

(1) $$U_2^\sigma \circ V_1^\sigma + V_2^\sigma \circ U_1^\sigma \subset U_1^\sigma,$$

(2) $$\{V_1^\sigma, U_1^{-\sigma}, e^\sigma\} \subset U_2^\sigma .$$

In this case, U_1 is an ideal of the alternative pair $A = V_1$ (cf. 8.2).

Proof. The necessity of these conditions is obvious, and from 8.1 it follows easily that they are sufficient. By 8.2.2, U_1 is an ideal of A.

12.2. LEMMA. *With the above notations, let U_1 be an ideal of the alternative pair $A = V_1$ such that $V_2^\sigma \circ U_1^\sigma \subset U_1^\sigma$. Set $U_2^\sigma = \{V_1^\sigma, U_1^{-\sigma}, e^\sigma\} + \{U_1^\sigma, V_1^{-\sigma}, e^\sigma\}$ and $U_2 = (U_2^+, U_2^-)$. Then $U = U_2 \oplus U_1$ is an ideal of V.*

Proof. Since U_1 is an ideal of A we have by 8.2.2 that

$$U_2^\sigma \circ V_1^\sigma = \langle V_1^\sigma, B_1^{-\sigma}, V_1^\sigma \rangle + \langle U_1^\sigma, V_1^{-\sigma}, V_1^\sigma \rangle \subset U_1^\sigma .$$

Hence (1) and (2) of 12.1 are satisfied. It remains to show that U_2 is an ideal of V_2. From JP14 and 8.1 it follows that $\{V_2^\sigma, V_2^{-\sigma}, U_2^\sigma\} \subset U_2^\sigma$. Using JP12 we see that

$$Q(V_2^\sigma)\{e^{-\sigma}, U_1^\sigma, V_1^{-\sigma}\} \subset \{V_2^\sigma, V_1^{-\sigma}, U_1^\sigma \circ V_2^\sigma\} \subset \{V_2^\sigma, V_1^{-\sigma}, U_1^\sigma\} \subset U_2^\sigma ,$$

and similarly $Q(V_2^\sigma)\{e^{-\sigma}, V_1^\sigma, U_1^{-\sigma}\} \subset U_2^\sigma$. Finally it follows from JP20 that $Q(U_2^\sigma)V_2^{-\sigma} \subset U_2^\sigma$.

12.3. LEMMA. Let U_2 be an ideal of V_2 such that $\{V_1^\sigma, V_1^{-\sigma}, U_2^\sigma\} \subset U_2^\sigma$. Set $U_1^\sigma = U_2^\sigma \circ V_1^\sigma$ and $U_1 = (U_1^+, U_1^-)$. Then $U = U_2 \oplus U_1$ is an ideal of V.

Proof. By 8.1.1 we have $V_2^\sigma \circ U_1^\sigma = V_2^\sigma \circ (U_2^\sigma \circ V_1^\sigma) \subset U_2^\sigma \circ (V_2^\sigma \circ V_1^\sigma) + (V_2^\sigma \circ U_2^\sigma) \circ V_1^\sigma \subset U_2^\sigma \circ V_1^\sigma = U_1^\sigma$ and hence (1) of 12.1 is satisfied. Also

$$\{V_1^\sigma, U_1^{-\sigma}, e^\sigma\} = \{V_1^\sigma, U_2^{-\sigma} \circ V_1^{-\sigma}, e^\sigma\} \subset \{V_1^\sigma, V_1^{-\sigma}, U_2^\sigma\} \subset U_2^\sigma$$

by 8.1. Thus 12.1.2 is verified, and it remains to show that U_1 is an ideal of V_1. By 8.1.7, $\{U_1^\sigma, V_1^{-\sigma}, e^\sigma\} \subset U_2^\sigma$, and hence by 8.2.2,

$$\{U_1^\sigma, V_1^{-\sigma}, V_1^\sigma\} \subset \{V_1^\sigma, U_1^{-\sigma}, V_1^\sigma\} \subset U_1^\sigma .$$

Furthermore,

$$\{V_1^\sigma, V_1^{-\sigma}, U_1^\sigma\} \subset V_2^\sigma \circ U_1^\sigma = V_2^\sigma \circ (U_2^\sigma \circ V_1^\sigma) \subset (V_2^\sigma \circ U_2^\sigma) \circ V_1^\sigma + U_2^\sigma \circ (V_2^\sigma \circ V_1^\sigma) \subset U_2^\sigma \circ V_1^\sigma = U_1^\sigma ,$$

using 8.1.4. Thus U_1 is an ideal of the alternative pair A and therefore also of the Jordan pair $V_1 = A^J$. Now the lemma follows from 12.1.

12.4. LEMMA. Let $V = V_2 \oplus V_1$ and $A = V_1$ be as in 12.1. Assume that V is simple and that $A \neq 0$. Then the homomorphism h from V into the standard imbedding W of A (cf. 8.12) is an isomorphism, and the Jordan structure algebra J of A coincides with the inner structure algebra F (cf. 8.8).

Proof. Since h is the identity on V_1 it follows by simplicity of V that h is injective. By 12.2, $U_2 \oplus V_1$ where $U_2^\sigma = \{V_1^\sigma, V_1^{-\sigma}, e^\sigma\}$ is an ideal of V. Hence $V_2^+ = \{V_1^+, V_1^-, e^+\}$, and $e^+ \in V_2^+$ implies $1 = f(e^+) \in f(\{V_1^+, V_1^-, e^+\}) = F$ and therefore $J = F$ (cf. 8.8, 8.10 for notation). In particular, $f\colon V_2^+ \to J$ is surjective, and by definition of h this implies that h is surjective.

12.5. THEOREM. The simple Jordan pairs V containing an idempotent e such that $V_0(e) = 0$ are up to isomorphism either

(i) Jordan pairs associated with simple unital Jordan algebras, or

(ii) standard imbeddings of simple alternative pairs for which Jordan structure algebra and inner structure algebra coincide.

Proof. Let V and e be as in the statement of the theorem. If $V_1(e) = 0$ then e^+ is invertible in $V = V_2(e)$ with inverse e^-, and V is isomorphic with the Jordan pair (J,J) where $J = V^+_{e^-}$ is a simple unital Jordan algebra, by 1.6 and 1.11. Thus V is of type (i). Conversely, if J is a simple Jordan algebra with unit element 1 then (J,J) is a simple Jordan pair, and $(1,1)$ is an idempotent with $V_1(e) = V_0(e) = 0$.

Now assume that $V_1(e) \neq 0$. By 12.4, V is isomorphic with the standard imbedding of $A = V_1(e)$, the Jordan structure algebra coincides with the inner structure algebra, and hence $V_2^\sigma = \{V_1^\sigma, V_1^{-\sigma}, e^\sigma\}$. Let U_1 be an ideal of A. Then $V_2^\sigma \circ U_1^\sigma = \{V_1^\sigma, V_1^{-\sigma}, e^\sigma\} \circ U_1^\sigma = \{A^\sigma, A^{-\sigma}, U_1^\sigma\} \subset U_1^\sigma$, and by 12.2 $U_2 \oplus U_1$ is an ideal of V. This shows that A is simple and hence V is of type (ii). Conversely, let A be a simple alternative pair whose structure algebra coincides with the inner structure algebra, and let $W = W_2 \oplus W_1$ be the standard imbedding of A. Then we have $W_2^\sigma = \{W_1^\sigma, W_1^{-\sigma}, e^\sigma\}$. If $U = U_2 \oplus U_1$ is an ideal of W then by 12.1, U_1 is an ideal of A and is therefore either zero or all of A. In the first case, $U_2^\sigma \circ W_1^\sigma \subset U_1^\sigma = 0$ which implies $U_2 = 0$ by definition of the standard imbedding. In the second case, $U_2^\sigma \supset \{W_1^\sigma, W_1^{-\sigma}, e^\sigma\} = W_2^\sigma$ and therefore $U_2 = W_2$.

12.6. Remarks. (a) If $V = V_2 \oplus V_1$ is simple of type (ii) then it is unknown whether V_2 is simple (or, equivalently, whether the Jordan structure algebra of V_1 is simple). For alternative pairs, the corresponding result is true (11.4). If, however, V satisfies the dcc on principal inner ideals then V_2 is simple in view of 10.14.

(b) In general, the structure Jordan algebra and the inner structure algebra of a simple alternative pair are not the same, and the standard imbedding is not simple; cf. 8.9(c). If they are, however, then A is semisimple. Indeed, the standard imbedding of A is simple and contains a non-zero idempotent and is therefore semisimple. This implies that A is semisimple by 5.8.

(c) Let $V = V_2 \oplus V_1$ be of type (ii), and let $A = V_1$. Then the centroids of V and A are naturally isomorphic. Indeed, if $a \in Z(V)$ then a leaves V_2 and V_1 invariant, and the restriction of a to V_1 belongs to the centroid of A. The map $a \mapsto a|V_1$ is an isomorphism between $Z(V)$ and $Z(A)$. The details are left as an exercise.

12.7. LEMMA. *Let $V = V_2 \oplus V_1$ be a simple Jordan pair as in 12.5, and assume that V satisfies the dcc on principal inner ideals. Then V_2 and V_1 satisfy the dcc on principal inner ideals, and V_1 satisfies the chain condition on idempotents.*

Proof. The first statement follows from 10.2. Assume that $V_1 = A \neq 0$ and let $c = (c^+, c^-)$ be an idempotent of A. Then $\ell(c) = \ell(c^+, c^-)$ is an idempotent of the Jordan structure algebra J of A by 8.4.1, and if c and d are orthogonal idempotents of A then $\ell(c)$ and $\ell(d)$ are orthogonal idempotents of J, by 8.5.1 and 8.5.3. Since $V_2 \cong (J,J)$ it follows that J has dcc on principal inner ideals (10.7). If $\{c_1, c_2, \ldots\}$ is an orthogonal set of idempotents of A then $J_2(1-\ell(c_1)) \supset J_2(1-\ell(c_1)-\ell(c_2)) \supset \ldots$ is a descending chain of principal inner ideals of J. It follows that A satisfies the chain condition on idempotents.

12.8. In view of 12.5 and 12.7, the classification of simple Jordan pairs with dcc on principal inner ideals and containing an idempotent e with $\mathcal{V}_0(e) = 0$ amounts to the following.

(i) Classify simple unital Jordan algebras with dcc on principal inner ideals (up to isotopy, cf. 1.12).

(ii) Classify simple alternative pairs with dcc on principal inner ideals and chain condition on idempotents for which the Jordan structure algebra and the inner structure algebra coincide, and determine their standard imbeddings.

By the "Second Structure Theorem" (Jacobson[3]), the simple unital Jordan algebras over k with dcc on principal inner ideals are up to isotopy the following.

(0) Jordan division algebras over k.

(I) $\mathcal{R}^J$ where $\mathcal{R}$ is a simple Artinian algebra over k.

(II) $H_n(\mathcal{Q},K)$, $n \geqq 2$, $\mathcal{Q}$ a split quaternion algebra over an extension field K of k.

(III) $H_n(\mathcal{D},\mathcal{D}_0)$, $n \geqq 2$, $\mathcal{D}$ a division algebra with involution, and $\mathcal{D}_0$ an ample subspace of $\mathcal{D}$.

(IV) Outer ideals containing 1 in Jordan algebras of non-degenerate quadratic forms with base point over an extension field K of k.

(V) $H_3(C,K)$, C a Cayley algebra over an extension field K of k.

By McCrimmon[8] all these algebras satisfy the acc on principal inner ideals. Note that $H_n(\mathcal{Q},K)$ is isomorphic with the Jordan algebra $\mathcal{J}$ of symplectic symmetric $2n \times 2n$ matrices over K by McCrimmon[8], p. 459. The Jordan pair associated with $\mathcal{J}$ is isomorphic with $(A_{2n}(K),A_{2n}(K))$ under the map $X \mapsto SX$ where S is as in 8.16. In case (IV) let q be a non-degenerate quadratic form on a vector space $\mathcal{X}$ over K, and let $1 \in \mathcal{X}$ be such that $q(1) = 1$. Then the associated Jordan algebra J has unit element 1 and quadratic operators $U_x y = q(x,\bar{y})x - q(x)\bar{y}$ where $\bar{y} = q(y,1) - y$ (cf. Jacobson[3]). On the other hand, $V = (X,X)$ is a Jordan pair with $Q(x)y = q(x,y)x - q(x)y$, and one checks easily that $h = (h_+,h_-) : (J,J) \to V$ given by $h_+(x) = x$ and $h_-(y) = \bar{y}$ is an isomorphism of Jordan pairs. Hence the Jordan pair associated with J depends, as it should, only on the quadratic form q and not on the choice of the base point.

By the results of 11.16 we know all the simple alternative pairs with dcc on principal inner ideals and chain condition on idempotents. We proceed to determine their structure algebras and standard imbeddings.

12.9. Type (A) and (A'). Let $A = A(M,R,\Phi)$ with R simple Artinian and Φ non-degenerate. By 8.9(a), $J = F \cong R^J$, and by 8.14, the standard imbedding of A is "of the same type", $W = A(N,R,\Psi)^J$ with the same R, and Ψ non-degenerate. Since (principal) inner ideals are the same for an alternative pair and the associated Jordan pair it follows from 11.14(b) that W satisfies the dcc and acc on principal inner ideals. Next let A' be the reverse of $A(M,R,\Phi)$ so that the product is given by $\langle xyz\rangle' = \langle zyx\rangle = \Phi(z,y)x$. If $F = J$ then $1 \in F$ and hence there exist finitely many $(x_i,y_i) \in M$ such that $z = \Sigma \langle x_i y_i z\rangle' = \Sigma\, \Phi(z,y_i)x_i$ for all $z \in M^+$. This means that M^+ is finitely generated as an R-module. By 11.17, we have $A \cong (M_{p,q}(\mathcal{D}),M_{p,q}(\mathcal{D}^{op}))$ and $A' \cong (M_{q,p}(\mathcal{D}^{op}),M_{q,p}(\mathcal{D}))$. It follows that A' is again of type (A) (with $R = M_q(\mathcal{D})$) and we are reduced to the previous case.

12.10. Type (B). Let $A = (C,C^{op})$ with C a Cayley algebra over an extension field K of k. Thus C is of dimension 8 over K. By 8.9(b) we have $F = J \cong C^J$, and by 8.15, the standard imbedding is $W = (M_{1,2}(C),M_{1,2}(C^{op}))$ with $Q(x)y = x(y^*x)$. Using the canonical involution of C we may identify W with

$(M_{1,2}(\mathcal{C}),M_{1,2}(\mathcal{C}))$ where $Q(x)y = x({}^t\bar{y}x)$ and the bar now stands for the canonical involution. Let $\mathcal{E} = H_3(\mathcal{C},K)$ be the exceptional Jordan algebra of 3×3 hermitian matrices over $\mathcal{C}$ with diagonal coefficients in K, and define a map $h = (h_+,h_-)$ from $\mathcal{W}$ into the Jordan pair $(\mathcal{E},\mathcal{E})$ by

$$h_\pm(x_1,x_2) = \begin{pmatrix} 0 & x_1 & x_2 \\ \bar{x}_1 & 0 & 0 \\ \bar{x}_2 & 0 & 0 \end{pmatrix}.$$

A computation shows that h is a homomorphism; in fact, h is an isomorphism between $\mathcal{W}$ and $(\mathcal{E}_1,\mathcal{E}_1)$ where $\mathcal{E}_1$ is the Peirce-1-space in the Peirce decomposition $\mathcal{E} = \mathcal{E}_2 + \mathcal{E}_1 + \mathcal{E}_0$ with respect to the idempotent

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

of $\mathcal{E}$. Since $\mathcal{E}$ is regular (cf. McCrimmon[8]) so is $\mathcal{E}_1$. Indeed, if $x \in \mathcal{E}_1$ and $x = U_x y$ for some $y \in \mathcal{E}$ then we have $U_x y_2 \in \mathcal{E}_0$ and $U_x y_0 \in \mathcal{E}_2$ so that $x = U_x y_1$ (y_i denotes the component of y in $\mathcal{E}_i$). It follows that $\mathcal{W}$ is regular, and this implies that every inner ideal is a K-subspace. By finite-dimensionality, $\mathcal{W}$ satisfies the dcc and acc on all inner ideals.

12.11. Type (C). Let $\mathcal{A} = \mathcal{A}(X,K,\alpha)$ with X a vector space over an extension field K of k and α non-degenerate. If X is infinite-dimensional then by 8.9(c) we have $\mathcal{F} \neq \mathcal{J}$. If $\dim_K X = 2m < \infty$ we have $\mathcal{F} = \mathcal{J}$, and by 8.16 the standard imbedding $\mathcal{W}$ is $(A_n(K),A_n(K))$ where $n = 2m+1$ and $Q(x)y = x{}^tyx = -xyx$. We show that $\mathcal{W}$ is regular. An $n \times n$ matrix x is alternating if and only if $(xu,u) = 0$ for all $u \in K^n$, where $(u,v) = \Sigma\, u_iv_i$ is the usual scalar product. By linearization this implies $(xu,v) = -(u,xv)$ and hence $\mathrm{Ker}(x)$ is the orthogonal complement of $\mathrm{Im}(x)$ with respect to $(\ ,\ )$. Choose any subspace

U of K^n such that $U \oplus \mathrm{Ker}(x) = K^n$, and let V be the orthogogonal complement of U. Then $\mathrm{Im}(x) \oplus V = K^n$. Since x induces a vector space isomorphism, say x_o, between U and $\mathrm{Im}(x)$ we may define y by $y|\mathrm{Im}(x) = -x_o^{-1}$, $y|V = 0$. Then one checks that $y \in A_n(K)$ and that $Q(x)y = -xyx = x$.

The regularity of $\mathcal{W}$ implies that every inner ideal is a K-subspace, and hence $\mathcal{W}$ satisfies the dcc and acc on all inner ideals.

12.12. THEOREM. *For a semisimple Jordan pair over* k *with dcc on principal inner ideals the following conditions are equivalent.*

(i) *The acc on principal inner ideals;*

(ii) *the chain condition on idempotents;*

(iii) *the existence of a maximal idempotent.*

The simple pairs with these properties are up to isomorphism precisely the following.

(0) *Jordan division pairs over* k.

(I) $A(M,R,\Phi)^J$ *with* R *a simple Artinian* k-*algebra and* Φ *non-degenerate.*

(II) $(A_n(K), A_n(K))$ *with* K *an extension field of* k; $n \geq 4$.

(III) $(H_n(\mathcal{D},\mathcal{D}_0), H_n(\mathcal{D},\mathcal{D}_0))$ *with* $\mathcal{D}$ *a division algebra with involution over* k *and* $\mathcal{D}_0$ *an ample subspace;* $n \geq 2$.

(IV) (I,I) *where* I *is an outer ideal containing* 1 *in a Jordan algebra of a non-degenerate quadratic form on a vector space over an extension field of* k.

(V) $(M_{1,2}(C), M_{1,2}(C^{op}))$ *with* C *a Cayley algebra over an extension field* K *of* k.

(VI) $(H_3(C,K), H_3(C,K))$ *with* C *as above.*

Proof. The implications (i) $\to$ (ii) $\to$ (iii) follow from 10.7 and 10.10. By 10.14, we only have to prove (iii) $\to$ (i) for simple pairs. Now the theorem follows from 12.8 - 12.11.

12.13. Remarks. (a) Clearly the pairs of type (0), (III), (IV), and (VI) contain invertible elements. The pairs of type (II) contain invertible elements if and only if n is even, and the pairs of type (I) do if and only if $M^{\pm} = R$ (these two correspond to the Jordan algebras of type (II) and (I) in 12.8). Finally, the pairs of type (V) contain no invertible elements.

(b) The pairs of type (0), (II), (III), (V), and (VI) satisfy the acc and dcc on all inner ideals. By 11.17, a Jordan pair of type (I) has dcc and acc on all inner ideals if and only if M^{+} is a finitely generated R-module, and then it is isomorphic with $(M_{p,q}(\mathcal{D}), M_{p,q}(\mathcal{D}^{op}))$ where $\mathcal{D}$ is a division algebra. Here we may assume that $p \leq q$ since the map $x \mapsto x^{*}$ defines an isomorphism between the Jordan pairs $(M_{p,q}(\mathcal{D}), M_{p,q}(\mathcal{D}^{op}))^{J}$ and $(M_{q,p}(\mathcal{D}^{op}), M_{q,p}(\mathcal{D}))^{J}$ (the associative pairs are not isomorphic!). By McCrimmon[8], Jordan algebras (and therefore Jordan pairs) defined by quadratic forms have dcc and acc on all inner ideals if and only if there exist no totally isotropic subspaces of infinite dimension.

NOTES

The theory of inner ideals of Jordan pairs is very similar to (but simpler than) the corresponding theory for Jordan algebras. Proposition 10.4 is the Jordan pair version of McCrimmon[8], Prop. 1. The Minimal Inner Ideal Theorem (cf. Jacobson[3]) takes the form 10.5; thus the troublesome minimal inner ideals of type II of the Jordan algebra theory don't occur for Jordan pairs. It is an open problem whether the radical of a Jordan pair or a Jordan algebra with dcc on (principal) inner ideals is nilpotent or solvable. The short and elegant proof of von Neumann regularity (10.15 - 10.17) presented here is due to K. Meyberg[7]. Earlier proofs (Meyberg[5], McCrimmon[2]) were considerably more complicated, and also needed stronger hypotheses (chain condition on all inner ideals or a minimum and maximum condition for certain inner ideals).

The first classification of alternative triple systems (finite-dimensional over algebraically closed fields of characteristic $\neq 2$) was given in Loos[3]. Meyberg([6]) noticed that the same methods work for simple alternative triple systems (over rings) containing an idempotent e with $A_{00}(e) = 0$. The generalization of this to alternative pairs (11.11) is relatively straightforward. Proposition 11.17, (c) and (d), is the "rectangular matrix version" of Th. 1 and Corollary of McCrimmon[8].

By 1.13 and 6.15, the classification of involutions of alternative and Jordan pairs is equivalent with the classification of alternative and Jordan triple systems. This classification has been carried out in the finite-dimensional case over algebraically closed fields of characteristic $\neq 2$ (Loos[2-4]) but is an open problem in general.

CHAPTER IV

FINITE-DIMENSIONAL JORDAN PAIRS

§13. Universal enveloping algebras

13.1. DEFINITION. Let $V = (V^+, V^-)$ be a Jordan pair over the ring k, and let (d,q) be a representation of V into an associative algebra A with idempotents e_+ and e_- (cf. 2.3 for the definition). We say that (d,q) is <u>universal</u> if for every representation (d',q') of V into A' there exists a unique homomorphism $\phi : A \to A'$ of associative algebras such that

(1) $$\phi(e_\pm) = e'_\pm , \quad \phi\circ d_\pm = d'_\pm , \quad \phi\circ q_\pm = q'_\pm .$$

By standard arguments (cf. Cohn[1]) universal representations exist and are unique up to isomorphism. The algebra A of a universal representation (d,q) is called the <u>universal enveloping algebra</u> of V and is denoted by $U(V)$. It has idempotents e_+ and e_- such that $e_+ + e_- = 1$, and we denote the Peirce spaces with respect to $e_\pm$ by $U^{\sigma\tau}(V)$ ($\sigma,\tau = \pm$).

13.2. PROPOSITION. (a) $U(V)$ <u>is functorial in</u> V : <u>A homomorphism</u> $h\colon V \to W$ <u>of Jordan pairs induces a homomorphism</u> $U(h)\colon U(V) \to U(W)$ <u>such that</u> $U(h)(e_\sigma) = e_\sigma$, $U(h)(d_\sigma(x,y)) = d_\sigma(h_\sigma(x),h_{-\sigma}(y))$, $U(h)(q_\sigma(x)) = q_\sigma(h_\sigma(x))$, for all $x,y \in V^\pm$.

(b) *If* I *is an ideal of* V *and* $\tilde{I}$ *is the ideal of* $U(V)$ *generated by* $d_\sigma(I^\sigma, V^{-\sigma})$, $d_\sigma(V^\sigma, I^{-\sigma})$, $q_\sigma(I^\sigma)$, $q_\sigma(I^\sigma, V^\sigma)$ $(\sigma = \pm)$ *then* $U(V/I) = U(V)/\tilde{I}$.

(c) $U(V)$ *is compatible with extension of scalars: If* R *is an extension of* k *then the homomorphism* $U(V)\otimes R \to U(V\otimes R)$ *induced by* $V \to V\otimes R$ *is an isomorphism.*

(d) *There exists a unique involution* $*$ *of* $U(V)$ *such that* $e_+{}^* = e_-$, $q_+(x)^* = q_+(x)$, $q_-(y)^* = q_-(y)$, $d_+(x,y)^* = d_-(y,x)$; *i.e., the universal representation* (d,q) *is a* $*$*-representation in the sense of 2.5. If* (d',q') *is a* $*$*-representation of* V *in* A' *then the homomorphism* $\phi\colon U(V) \to A'$ *is compatible with the involutions.*

(e) $U(V) = k.e_+ \oplus k.e_- \oplus U_0(V)$ *where* $U_0(V)$ *is the subalgebra generated by* $d_\sigma(V^\sigma, V^{-\sigma})$, *and* $q_\sigma(V^\sigma)$ $(\sigma = \pm)$. $U_0(V)$ *is actually an ideal of* $U(V)$. *Also* $U^{\sigma\sigma}(V) = k.e_\sigma \oplus U_0^{\sigma\sigma}(V)$ *where* $U_0^{\sigma\sigma}(V)$ *is the subalgebra generated by* $d_\sigma(V^\sigma, V^{-\sigma})$ *and* $q_\sigma(V^\sigma)q_{-\sigma}(V^{-\sigma})$, *and* $U^{\sigma,-\sigma}(V) = U^{\sigma\sigma}(V)q_\sigma(V^\sigma)U^{-\sigma,-\sigma}(V)$. *As before,* $U_0^{\sigma\sigma}(V)$ *is an ideal of* $U^{\sigma\sigma}(V)$.

The proof of (a), (b), and (c) is straight-forward. For (d), one checks that $d'_\sigma(x,y) = d_{-\sigma}(y,x)$, $q'_\sigma(x) = q_\sigma(x)$ defines a representation of V in $U(V)^{op}$. The induced homomorphism $\phi\colon U(V) \to U(V)^{op}$ is the desired involution. Finally, the universal envelope of the trivial Jordan pair $(0,0)$ is obviously $k.e_+ \oplus k.e_-$, and $U(V) = k.e_+ \oplus k.e_- \oplus U_0(V)$ where $U_0(V)$ is the kernel of the homomorphism $U(V) \to U(0)$ induced by $V \to 0$. By (b), $U_0(V)$ is the ideal generated by $d_\sigma(V^\sigma, V^{-\sigma})$ and $q_\sigma(V^\sigma)$ $(\sigma = \pm)$. But the subalgebra generated by these elements is already an ideal. The last assertion follows from the rules for the Peirce decomposition in associative algebras.

13.3. PROPOSITION. (a) The universal envelope of V^{op} is $U(V)$ with the universal representation $d_\sigma^{op} = d_{-\sigma}$, $q_\sigma^{op} = q_{-\sigma}$, and idempotents $e_\sigma^{op} = e_{-\sigma}$.

(b) An antiautomorphism η of V induces an automorphism $U(\eta)$ of $U(V)$ such that $U(\eta)(e_\sigma) = e_{-\sigma}$, $U(\eta)(d_\sigma(x,y)) = d_{-\sigma}(\eta_\sigma(x),\eta_{-\sigma}(y))$, $U(\eta)(q_\sigma(x)) = q_{-\sigma}(\eta_\sigma(x))$. If η is an involution then $U(\eta)$ is of period two.

The proof is left as an exercise.

13.4. To simplify notation, we will again omit subscripts σ in d_σ and q_σ (cf. 2.0). Then we have the following identities, valid for any representation (d,q) of a Jordan pair V.

(1) $$[d(u,v),d(x,y)] = d(\{uvx\},y) - d(x,\{vuy\}),$$

(2) $$d(x,y)^4 = d(x,y)^2 d(Q_x y,y) + d(Q_x y,y)d(x,y)^2 + d(Q_x y,y)^2 - d(Q(Q_x y)y,y) - d(x,Q(Q_y x)x),$$

(3) $$\begin{aligned}[q(u)q(v),q(x)q(y)] &= q(\{uvx\})q(y) - q(x)q(\{vuy\}) \\ &+ q(Q_x v,Q_u v)q(y) - q(x)q(Q_y u,Q_v u) \\ &+ q(x)q(y,v)q(u)q(y,v) - q(x,u)q(v)q(x,u)q(y).\end{aligned}$$

Proof. (1) follows from JP15 by the permanence principle (2.8). By 2.3.3 we have $d(x,y)^2 - d(Q_x y,y) = 2q(x)q(y)$. If we square this and observe 2.3.5 and 2.6.1 we obtain (2). From 2.6.7 it follows that

$$\begin{aligned}&q(x)q(y)q(u)q(v) + q(x)q(v)q(u)q(y) \\ &= q(x)q(\{yuv\}) + q(x)q(Q_y u,Q_v u) - q(x)q(y,v)q(u)q(y,v)\end{aligned}$$

and

$$\begin{aligned}&q(x)q(v)q(u)q(y) + q(u)q(v)q(x)q(y) \\ &= q(\{xvu\})q(y) + q(Q_x v,Q_u v)q(y) - q(x,u)q(v)q(x,u)q(y).\end{aligned}$$

Now (3) follows by subtracting these two formulas.

13.5. LEMMA. Let $U(V)$ be the universal envelope of V. Then the subalgebra $\mathcal{D}$ of $U^{++}(V)$ generated by $d(V^+, V^-)$ is an ideal of $U^{++}(V)$. If 2 is invertible in k then $U_0^{++}(V) = \mathcal{D}$ (cf. 13.2(e)).

Proof. From 2.6.3 and 2.3.2 it follows that

$$\begin{aligned} q(x)q(y)d(u,v) &= q(x)(d(y,u)q(y,v) - q(Q_y u,v)) \\ &= -d(u,y)q(x)q(y,v) + q(x,\{xyu\})q(y,v) - q(x)q(Q_y u,v), \end{aligned}$$

and by 2.3.3 and 2.3.4 this belongs to $\mathcal{D}$. Similarly, one shows that $d(u,v)q(x)q(y) \in \mathcal{D}$, and therefore $\mathcal{D}$ is an ideal of $U^{++}(V)$. If $1/2 \in k$ then by 2.3.3 we have $q(x)q(y) = (1/2)(d(x,y)^2 - d(Q_x y,y)) \in \mathcal{D}$.

13.6. THEOREM. Let V be a Jordan pair over a ring k such that V^+ and V^- are finitely spanned as k-modules. Then $U(V)$ is finitely spanned as a k-module.

Proof. From 13.2, (d) and (e), we see that it suffices to prove that $\mathcal{U} = U^{++}(V)$ is finitely spanned. We show first that $\mathcal{D}$ (as in 13.5) is finitely spanned. Let $x_1,\dots,x_m$ be a spanning set of V^+ and $y_1,\dots,y_n$ a spanning set of V^-. Set $d_{ij} = d(x_i,y_j)$, and let X_r be the k-submodule of $\mathcal{U}$ spanned by all monomials $d_{i_1j_1}\cdots d_{i_sj_s}$ where $1 \leqq s \leqq r$. Clearly, the X_r form an increasing sequence of submodules whose union is $\mathcal{D}$, and we have $X_r.X_s \subset X_{r+s}$. By 13.4.1 we have

(1) $$d_{ij}d_{k\ell} \equiv d_{k\ell}d_{ij} \mod X_1 .$$

In a monomial $x = d_{i_1j_1}\cdots d_{i_rj_r}$ where $r > 3mn$, at least one of the d_{ij} occurs at least four times. By (1), we have $x \equiv d_{ij}^4 y \mod X_{r-1}$ where $y \in X_{r-4}$. By 13.4.2, $d_{ij}^4 \in X_3$ and hence $x \in X_3X_{r-4} \subset X_{r-1}$. This shows that $X_r = X_{r-1}$, and hence $\mathcal{D} = X_{3mn}$ is finitely spanned.

Next we show that $\mathcal{U}/\mathcal{D}$ is finitely spanned. Let $q_{ij} = q(x_i)q(y_j) + \mathcal{D} \in \mathcal{U}/\mathcal{D}$. Then $\mathcal{U}/\mathcal{D}$ is generated by $1 = e_+ + \mathcal{D}$ and the q_{ij} . Let $\mathcal{Y}_r$ be the k-submodule spanned by the monomials $q_{i_1j_1} \cdots q_{i_sj_s}$, $0 \leq s \leq r$, where $\mathcal{Y}_0 = k.1$. Then $\mathcal{U}/\mathcal{D}$ is the union of the increasing sequence of the $\mathcal{Y}_r$ and we have $\mathcal{Y}_r.\mathcal{Y}_s \subset \mathcal{Y}_{r+s}$. From 13.4.3, 2.3.3, and 2.3.4 it follows that

(2) $$q_{ij}q_{k\ell} \equiv q_{k\ell}q_{ij} \quad \mathrm{mod}\ \mathcal{Y}_1 ,$$

and from 2.3.5 we get

(3) $$q_{ij}q_{i\ell} = q(Q(x_i)y_j)\cdot q(y_\ell) + \mathcal{D} \in \mathcal{Y}_1 .$$

A monomial $q_{i_1j_1} \cdots q_{i_rj_r}$ with $r > m$ must be of the form $\ldots q_{ij} \ldots q_{i\ell} \ldots$ for some index i . Now it follows from (2) and (3) similarly as before that $\mathcal{Y}_r = \mathcal{Y}_{r-1}$ and hence $\mathcal{U}/\mathcal{D} = \mathcal{Y}_m$ is finitely spanned.

13.7. Representations of Jordan algebras and Jordan triple systems. Let A be a unital associative algebra and let J be a unital Jordan algebra over k . A representation of J in A is a quadratic map $\mu : J \to A$ satisfying

(1) $$\mu(1) = 1,$$

(2) $$\nu(x,y)\mu(x) = \mu(x)\nu(y,x) = \mu(x,U_xy),$$

(3) $$\mu(U_xy) = \mu(x)\mu(y)\mu(x),$$

in all scalar extensions (cf. McCrimmon[7]). Here

(4) $$\nu(x,y) = \mu(x,1)\mu(y,1) - \mu(x,y).$$

A representation of a Jordan triple system T in A consists of a bilinear map $\ell : T \times T \to A$ and a quadratic map $p : T \to A$ satisfying in all scalar extensions

(5) $$\ell(x,y)p(x) = p(x)\ell(y,x) = p(x,P(x)y),$$

(6) $$p(x)\ell(y,z) + \ell(z,y)p(x) = p(x,\{xyz\}),$$

(7) $$\ell(x,y)\ell(x,z) = \ell(P(x)y,z) + p(x)p(y,z),$$

(8) $$\ell(z,x)\ell(y,x) = \ell(z,P(x)y) + p(y,z)p(x),$$

(9) $$p(P(x)y) = p(x)p(y)p(x).$$

(Cf. Loos[5]).

13.8. PROPOSITION. *Let* (d,q) *be a representation of the Jordan pair* V *in* A. *Let* $v \in V^-$, *and let* $J = k.1 \oplus V_v^+$ *be the unital Jordan algebra obtained from* V_v^+ *by adjoining a unit element. Then*

$$\mu(\alpha 1+x) = \alpha^2.1 + \alpha d_+(x,v) + q_+(x)q_-(v)$$

$(\alpha \in k\ ,\ x \in V^+)$ *defines a representation of* J *in* A.

Proof. Let $\bar{\mu}(\alpha 1+x) = U_{\alpha 1+x}|V_v^+ = \alpha^2 \mathrm{Id} + \alpha D(x,v) + Q(x)Q(v)$. Since V_v^+ is an ideal of J this defines a representation of J in $\mathrm{End}(V^+)$, and the identities 13.7.2 and 13.7.3 for $\bar{\mu}$ are equivalent with certain identities in $D(x,y)$'s and $Q(z)$'s. By the permanence principle, the same identities are valid with D,Q replaced by d,q, and hence μ satisfies 13.7.2 and 13.7.3. Since μ is obviously quadratic and $\mu(1) = 1$ the proposition follows.

13.9. PROPOSITION. *Let* J *be a unital Jordan algebra, and let* T *be the associated Jordan triple system obtained by "forgetting" the unit element. If* μ *is a representation of* J *and* ν *is defined as in 13.7.4 then* $(\ell,p) = (\nu,\mu)$ *is a representation of* T. *Conversely, if* (ℓ,p) *is a representation of* T *such that* $p(1) = 1$ *then* $\mu = p$ *is a representation of* J.

Proof. Since $U_x y = P(x)y$ it is clear that (5) and (9) of 13.7 hold for $(\ell,p) = (\nu,\mu)$, and the validity of (6) - (8) follows from well-known formulas for representations of Jordan algebras (see McCrimmon[7]). Conversely, if (ℓ,p) is a representation of T such that $p(1) = 1$ then it suffices to show that

$$\ell(x,y) = p(x,1)p(y,1) - p(x,y).$$

Now $\ell(1,x) = \ell(1,x)p(1) = p(1,x)$ by 13.7.5, and by 13.7.7 we have $\ell(1,y)\ell(1,z) = \ell(y,z) + p(1)p(y,z)$ which proves our assertion.

13.10. PROPOSITION. *Let T be a Jordan triple system, and let (T,T) be the associated Jordan pair (cf. 1.13).*

(a) *Every representation (ℓ,p) of T in A induces a representation (d,q) of (T,T) in the algebra $M_2(A)$ of 2×2 matrices over A by*

$$d_+(x,y) = \begin{pmatrix} \ell(x,y) & 0 \\ 0 & 0 \end{pmatrix}, \quad d_-(y,x) = \begin{pmatrix} 0 & 0 \\ 0 & \ell(y,x) \end{pmatrix},$$

$$p_+(x) = \begin{pmatrix} 0 & p(x) \\ 0 & 0 \end{pmatrix}, \quad p_-(y) = \begin{pmatrix} 0 & 0 \\ p(y) & 0 \end{pmatrix}.$$

(b) *Every representation (d,q) of (T,T) in A induces a representation (ℓ,p) of T in A by* $\ell(x,y) = d_+(x,y) + d_-(x,y)$, $p(x) = q_+(x) + q_-(x)$.

The proof is left to the reader.

13.11. Let T be a Jordan triple system. We define the concepts of universal representation and universal enveloping algebra in the same way as for Jordan pairs. Let $U(T)$ denote the universal enveloping algebra of T with universal representation (ℓ,p) . With T we associate the Jordan pair (T,T) with the canonical involution κ given by the exchange of factors (cf. 1.13). By 13.3, κ induces an automorphism $\theta = U(\kappa)$ of period 2 of $U(T,T)$ such that $\theta(e_\sigma) = e_{-\sigma}$,

$\theta(d_\sigma(x,y)) = d_{-\sigma}(x,y)$ and $\theta(q_\sigma(x)) = q_{-\sigma}(x)$. The relation between $U(T)$ and $U(T,T)$ is now as follows.

13.12. PROPOSITION. (a) <u>Let</u> U^θ <u>be the fixed point set of</u> θ <u>in</u> $U(T,T)$, <u>and define</u> $\ell : T \times T \to U^\theta$, $p : T \to U^\theta$ <u>by</u>

$$\ell(x,y) = d_+(x,y) + d_-(x,y),$$

$$p(x) = q_+(x) + q_-(x).$$

<u>Then</u> (ℓ,p) <u>is a universal representation of</u> T <u>and hence</u> $U(T) \cong U^\theta$.

(b) <u>Identify</u> $U(T)$ <u>and</u> U^θ <u>by (a), and let</u> $U^+(T) = U(T) \cap (U^{++} \oplus U^{--})$, $U^-(T) = U(T) \cap (U^{+-} \oplus U^{-+})$ <u>where</u> $U^{\sigma\tau} = U^{\sigma\tau}(T,T)$ <u>(cf. 13.1)</u>. <u>Then</u> $U(T) = U^+(T) \oplus U^-(T)$ <u>is a</u> $\mathbb{Z}/2\mathbb{Z}$<u>-graded algebra</u>. <u>We have</u> $U^+(T) = k.1 \oplus U_0^+(T)$ <u>where</u> $U_0^+(T)$ <u>is generated by</u> $\ell(T,T)$ <u>and</u> $p(T)p(T)$, <u>and</u> $U^-(T) = U^+(T)p(T)U^+(T)$.

(c) <u>The map</u> $F : U(T,T) \longrightarrow \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \; a,d \in U^+(T), \; b,c \in U^-(T)\right\}$ <u>given by</u> $F(u) = \begin{pmatrix} u_{++} + \theta u_{++}, & u_{+-} + \theta u_{+-} \\ u_{-+} + \theta u_{-+}, & u_{--} + \theta u_{--} \end{pmatrix}$ <u>where</u> $u = \Sigma\, u_{\sigma\tau}$ <u>and</u> $u_{\sigma\tau} \in U^{\sigma\tau}(T,T)$ <u>is an isomorphism</u>.

<u>Proof</u>. By 13.10(a), (ℓ,p) is a representation of T in U^θ . Let (ℓ',p') be a representation of T in some algebra A' . By 13.10(b), this induces a representation (d',q') of (T,T) in $M_2(A')$ and hence a homomorphism ψ from $U = U(T,T)$ into $M_2(A')$. One checks easily (on the generators of U) that if $\psi(u) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $\psi(\theta(u)) = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$ for all $u \in U$; i.e., θ corresponds to conjugation with the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in $M_2(A')$. Hence $\psi(U^\theta) \subset B = \left\{\begin{pmatrix} a & b \\ b & a \end{pmatrix} \middle| \; a,b \in A'\right\}$. Now the map $\chi : B \to A'$, sending $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ into $a + b$, is a homomor-

phism, and hence $\phi = \chi \circ \psi : \mathcal{U}^\theta \to A'$ is a homomorphism such that $\phi(\ell(x,y)) =$

$\psi\begin{pmatrix} \ell'(x,y) & 0 \\ 0 & \ell'(x,y) \end{pmatrix} = \ell'(x,y)$ and similarly $\phi(p(x)) = p'(x)$. It remains to be shown that ϕ is uniquely determined. For this it suffices to show that $\mathcal{U}^\theta$ is generated by 1, $\ell(\mathcal{T},\mathcal{T})$, and $p(\mathcal{T})$. Before doing so we prove (b). Since θ interchanges e_+ and e_- it is clear that $\theta(\mathcal{U}^{\sigma\tau}) = \mathcal{U}^{-\sigma,-\tau}$, and therefore

$$\mathcal{U}^\theta = \{a + b + \theta a + \theta b \mid a \in \mathcal{U}^{++},\ b \in \mathcal{U}^{+-}\}.$$

Hence $U^+(\mathcal{T}) = \mathcal{U}^\theta \cap (\mathcal{U}^{++} \oplus \mathcal{U}^{--}) = \{u + \theta u \mid u \in \mathcal{U}^{++}\} \cong \mathcal{U}^{++}$. By 13.2(e) it follows that $U^+(\mathcal{T})$ has the asserted structure, taking into account that $p(x)p(y) = (q_+(x) + q_-(x))(q_+(y) + q_-(y)) = q_+(x)q_-(y) + q_-(x)q_+(y) = q_+(x)q_-(y) + \theta(q_+(x)\cdot$ $\cdot q_-(y))$ by the Peirce rules. Also, for $u \in \mathcal{U}^{++}$, $v \in \mathcal{U}^{+-}$, $w \in \mathcal{U}^{--}$ we have $uvw + \theta(uvw) = (u+\theta u)(v+\theta v)(w+\theta w)$ by the Peirce rules. This proves (b) in view of 13.2(e), and also shows that $\mathcal{U}^\theta$ is generated by 1, $\ell(\mathcal{T},\mathcal{T})$, and $p(\mathcal{T})$. Finally, (c) follows by a straightforward computation.

13.13. PROPOSITION. Let $\mathcal{J}$ be a unital Jordan algebra, and let $U(\mathcal{J})$ be the universal enveloping algebra of $\mathcal{J}$. ($U(\mathcal{J})$ is defined analogously to the universal enveloping algebras of Jordan pairs and Jordan triple systems; it is also called the universal quadratic envelope, cf. McCrimmon[7]). Let $\mathcal{T}$ be the Jordan triple system obtained from $\mathcal{J}$ by forgetting the unit element $1_\mathcal{J}$ of $\mathcal{J}$. Then $U(\mathcal{J})$ is isomorphic with $U(\mathcal{T})/\mathcal{B}$ where $\mathcal{B}$ is the ideal of $U(\mathcal{T})$ generated by $1 - p(1_\mathcal{J})$.

Proof. By 13.9, the map $x \mapsto p(x) + \mathcal{B}$ is a representation of $\mathcal{J}$ in $U(\mathcal{T})/\mathcal{B}$ and hence induces a homomorphism $\phi : U(\mathcal{J}) \to U(\mathcal{T})/\mathcal{B}$. Also by 13.9, the universal representation $\mu : \mathcal{J} \to U(\mathcal{J})$ defines a representation (ν,μ) of $\mathcal{T}$ in $U(\mathcal{J})$ which induces a homomorphism $\eta : U(\mathcal{T}) \to U(\mathcal{J})$. Clearly, η factors through a homomorphism $\psi : U(\mathcal{T})/\mathcal{B} \to U(\mathcal{J})$, and ϕ and ψ are inverses of each other.

§14. Solvability and nilpotence

14.1. Solvability. Let V be a Jordan pair over the ring k. We define the derived series of V by

$$V^{(0)} = V, \quad V^{(1)} = (Q(V^{+})V^{-}, Q(V^{-})V^{+}), \quad V^{(n+1)} = (V^{(n)})^{(1)},$$

and say that V is solvable if $V^{(n)} = 0$ for some n. Clearly $V^{(n+1)}$ is an ideal of $V^{(n)}$ but need not be an ideal of V. Let I be an ideal of V. A standard argument shows that V is solvable if and only if I and V/I are solvable. Thus if V satisfies the acc on ideals then it contains a maximal solvable ideal, and the quotient of V by this ideal is free of non-zero solvable ideals. Note also that solvability is preserved under extension of scalars.

14.2. LEMMA. Let A and B be ideals of V, and let

$$C^{\sigma} = \{A^{\sigma}, B^{-\sigma}, V^{\sigma}\} + \{B^{\sigma}, A^{-\sigma}, V^{\sigma}\} + \{A^{\sigma}, V^{-\sigma}, B^{\sigma}\} + Q(A^{\sigma})B^{-\sigma}$$
$$+ Q(B^{\sigma})A^{-\sigma} + Q(V^{\sigma})Q(A^{-\sigma})B^{\sigma} + Q(V^{\sigma})Q(B^{-\sigma})A^{\sigma} .$$

Then $(C^{+}, C^{-}) = A\#B$ is an ideal of V.

The proof consists in a tedious but straightforward verification, using the identities JP1 - JP21.

14.3. V-solvability. Let I be an ideal of V. The V-derived series of I in V is $I = I^{<0>} \supset I^{<1>} \supset \ldots$ where $I^{<n+1>} = I^{<n>}\#I^{<n>}$, and I is called V-solvable if $I^{<n>} = 0$ for some n. By 14.2, the V-derived series of I consists of ideals of V. Note that V-solvability of I depends on the imbedding of I into V and is not an intrinsic property of I. Clearly, if I is V-

solvable then it is $\mathcal{I}$-solvable, and if it is $\mathcal{I}$-solvable then it is solvable in the sense of 14.1. Let

$$0 \to V' \xrightarrow{i} V \xrightarrow{p} V'' \to 0$$

be an exact sequence of Jordan pairs. An ideal $\mathcal{I} \supset V'$ of V is V-solvable if and only if $p(\mathcal{I})$ is V''-solvable and V' is V-solvable. From this it follows as usual that the sum of two V-solvable ideals is again V-solvable. Hence if V satisfies the acc on ideals there exists a maximal V-solvable ideal, and the quotient of V by this ideal is free of such ideals.

14.4. LEMMA. V contains a non-zero V-solvable ideal if and only if it contains a non-zero ideal $\mathcal{I}$ such that

$$Q(\mathcal{I}^\sigma) = D(\mathcal{I}^\sigma, \mathcal{I}^{-\sigma}) = 0 \ . \tag{1}$$

Proof. An ideal $\mathcal{I}$ satisfying (1) is V-solvable since $\mathcal{I}^{<1>} = 0$. Conversely, let $\mathcal{B}$ be a non-zero V-solvable ideal, and let n be the largest positive integer such that $C = \mathcal{B}^{<n>} \neq 0$. Then $C \# C = 0$ and hence

$$\{C^\sigma, C^{-\sigma}, V^\sigma\} = \{C^\sigma, V^{-\sigma}, C^\sigma\} = Q(C^\sigma)C^{-\sigma} = 0 \ .$$

Let $\mathcal{I}^\sigma = \{x \in C^\sigma \mid Q(x) = 0\}$, and set $\mathcal{I} = (\mathcal{I}^+, \mathcal{I}^-)$. In order to show that $\mathcal{I} \neq 0$ we may assume that $\mathcal{I}^+ \neq C^+$. Then there exists $a \in C^+$ and $b \in V^-$ such that $x = Q(a)b \neq 0$. But $x \in C^+$ since C is an ideal, and $Q(x)V^- = Q_aQ_bQ_aV^- \subset Q_aC^- = 0$ shows that $x \in \mathcal{I}^+$. Now we show that $\mathcal{I}$ is an ideal. Clearly, $\mathcal{I}^\pm$ is closed under scalar multplication. From $\{C^\sigma, V^{-\sigma}, C^\sigma\} = 0$ it follows that $\mathcal{I}^\pm$ is closed under addition, and therefore $\mathcal{I}$ is a submodule of V . From JP3 and JP21 it follows that $\mathcal{I}$ is an outer ideal, and since it consists of trivial elements by definition, it is an ideal of V .

14.5. PROPOSITION. A V-solvable ideal of V is contained in rad V and a solvable ideal is contained in Nil V.

Proof. The first inclusion follows from 14.4, and the second from the formula
$Q(x^{(n,y)}).y^{(n,x)} = Q(x^{(n,y)})Q(y).x^{(n-1,y)} = U_{x^n}.x^{n-1} = x^{2n+n-1} = x^{(3n-1,y)}$
(cf. 3.8.1; $x^n = x^{(n,y)}$ and $U_x = Q(x)Q(y)$ in the Jordan algebra $V_y^{\pm}$).

14.6. Nilpotence. Let $V_{(n)}$ denote the k-linear span of all products in V of degree $\geq n$ where $Q(x)y$ and $\{xyz\}$ are considered to be of degree 3. Thus

$$V = V_{(1)} \supset V_{(3)} \supset V_{(5)} \supset \cdots ,$$

and $V_{(n)}$ is an ideal of V, in fact, we have

$$Q(V^{\sigma}_{(m)})V^{-\sigma}_{(n)} \subset V^{\sigma}_{(2m+n)}, \quad \{V^{\sigma}_{(m)}, V^{-\sigma}_{(n)}, V^{\sigma}_{(p)}\} \subset V^{\sigma}_{(m+n+p)} .$$

We say V is nilpotent if $V_{(n)} = 0$ for some n. If V is nilpotent then it is solvable and even V-solvable since

$$V^{(n)} \subset V^{<n>} \subset V_{(3^n)} .$$

Let $M(V)$ be the multiplication algebra of V (cf. 2.4), and let $M_0 = M_0(V)$ be the subalgebra of $M(V)$ generated by $d_{\pm}(x,y)$ and $q_{\pm}(x)$ as in 2.4; in other words, M_0 is the image of $U_0(V)$ under the homomorphism $U(V) \to \mathrm{End}(V^+ \times V^-)$ induced by the regular representation (cf. 13.2). We have $M_0 = \bigoplus M_0^{\sigma\tau}$ where $M_0^{\sigma\tau} = M_0 \cap M^{\sigma\tau}(V)$ and $M^{\sigma\tau}(V)$ is the image of $U^{\sigma\tau}(V)$. We set

$$M_0.V = (M_0^{++}.V^+ + M_0^{+-}.V^-,\ M_0^{-+}.V^+ + M_0^{--}.V^-) .$$

14.7. PROPOSITION. $V_{(3^n)} \subset (M_0)^n.V \subset V_{(2n+1)}$.

Proof. The second inclusion is obvious from the definitions. For the first inclusion we note that if $a \in (M_0^n.V)^+$ and $b \in V^-$ then $Q(a) \in (M_0^{n+1}.V)^+$. Indeed, for $n = 0$ this is obvious, and for the induction step one uses JP3 and JP20. Now clearly $V_{(3)} = M_0.V$. Assume that $V_{(3^n)} \subset M_0^n.V$ and let $x \in V^+_{(3^{n+1})}$. Then x is a sum of elements of the form $Q(a)b$ and $\{abc\}$ where a or b or c is in $V_{(3^n)} \subset M_0^n.V$. If $a \in V^+_{(3^n)}$ then $Q(a)b \in (M_0^{n+1}.V)^+$ by the remark above, and $\{abc\} = D(c,b)a \in M_0^{++}.a \subset (M_0^{n+1}.V)^+$. If b or c are in $V_{(3^n)}$ we have similarly $Q(a)b \in M_0^{+-}.b \subset (M_0^{n+1}.V)^+$ and $\{abc\} = Q(a,c)b = D(c,b)a \in (M_0^{n+1}.V)^+$. Since all this holds with $+$ and $-$ interchanged as well the proposition follows.

14.8. COROLLARY. V _is nilpotent if and only if_ $M_0(V)$ _is nilpotent_.

The following is the main lemma for the proof of the equivalence of solvability and nilpotence for finite-dimensional Jordan pairs. We say a Jordan pair V is _finite-dimensional_ if k is a field and both V^+ and V^- are finite-dimensional over k.

14.9. LEMMA. _Let_ V _be a finite-dimensional Jordan pair over a field_ k, _and let_ I _be an ideal of_ V _such that the subalgebra of_ $U^{++}(V)$ _generated by_ $d(I^+,I^-)$ _and_ $q(I^+)q(I^-)$ _is nilpotent. Then the ideal_ $\tilde{I}$ _of_ $U(V)$ _generated by_ $d(I^\sigma,V^{-\sigma})$, $d(V^\sigma,I^{-\sigma})$, $q(I^\sigma)$, $q(I^\sigma,V^\sigma)$ $(\sigma = \pm)$ _is nilpotent_. (Cf. 13.2 for notation).

Proof. By 13.6, the universal enveloping algebra $U(V)$ is finite-dimensional. Clearly, $\tilde{I}$ is the sum of its intersections with the Peirce spaces $U^{\sigma\tau}(V)$. Since a finite-dimensional algebra is nilpotent if it is spanned by nilpotent elements and since $U^{\sigma,-\sigma}(V)^2 = 0$ by the Peirce rules it suffices to prove that $\tilde{I} \cap U^{\sigma\sigma}(V)$ is nilpotent. Now I is invariant under the involution $*$ of $U(V)$ which interchanges $U^{++}(V)$ and $U^{--}(V)$ (cf. 13.2(d)) and hence we only have to show that $\tilde{I} \cap U^{++}(V)$ is nilpotent. This will be done in several steps. To simplify notation, let $\mathcal{U} = U^{++}(V)$ and denote by A the nilpotent subalgebra of $\mathcal{U}$ generated by $d(I^+, I^-)$.

1°. The subalgebra $\mathcal{B}$ of $\mathcal{U}$ generated by $S = d(V^+, I^-) \cup d(I^+, V^-)$ is nilpotent.

Let $\mathcal{D}$ be the subalgebra of $\mathcal{U}$ generated by $d(V^+, V^-)$ (which is actually an ideal by 13.5). From 13.4.1 and the fact that I is an ideal of V it follows that $\mathcal{D}A \subset A\mathcal{D} + A$. This implies $A' = A\mathcal{D} + A$ is a nilpotent ideal of $\mathcal{D}$, and it will therefore be sufficient to show that $\mathcal{B}/\mathcal{B} \cap A'$ is nilpotent. Let $u \mapsto \bar{u}$ denote the canonical map $\mathcal{D} \to \mathcal{D}/A'$. By 13.4.1 we have

$$[\overline{d(u,v)}, \overline{d(x,y)}] = \overline{d(\{uvx\},y)}$$

for $u,x \in I^+$, $v,y \in V^-$, or $x \in I^+$, $v \in I^-$, $u \in V^+$, $y \in V^-$, and also

$$[\overline{d(u,v)}, \overline{d(x,y)}] = \overline{d(-x,\{vuy\})}$$

for $u,x \in V^+$, $v,y \in I^-$. This proves $[\bar{S},\bar{S}] \subset \bar{S}$. Furthermore, it follows from 13.4.2 and 2.6.1 that $\overline{d(x,y)}^4 = 0$ if $x \in I^+$ or $y \in I^-$. By Engel's theorem (cf. Jacobson[1]) the subalgebra of $\mathcal{D}/A'$ generated by $\bar{S}$, i.e., $\bar{\mathcal{B}} = \mathcal{B}/\mathcal{B} \cap A'$, is nilpotent. (The proof of Engel's theorem given in Jacobson[1] is for algebras of linear transformations; the general case follows by applying this to the left regular representation).

2°. The ideal $\mathcal{C}$ of $\mathcal{U}$ generated by $\mathcal{B}$ (resp. S) is nilpotent.

By 13.4.1 we have $\mathcal{D}\mathcal{B} \subset \mathcal{B}\mathcal{D} + \mathcal{B}$. From 2.3.2 - 2.3.4 one derives the identity

$$[q(u)q(v), d(x,y)] = q(u)q(v,\{vxy\}) - q(u,\{uyx\})q(v)$$

$$= d(u,v)d(u,\{vxy\}) - d(Q_u v,\{vxy\}) - d(\{uyx\},v)d(u,v) + d(\{uyx\},Q_v u)\ .$$

This implies $[q(u)q(v),d(x,y)] \in \mathcal{BD} + \mathcal{B}$ whenever $x \in I^+$ or $y \in I^-$. Since $\mathcal{U}$ is generated by $1 = e_+$, $\mathcal{D}$, and $q(V^+)q(V^-)$ it follows that $\mathcal{UB} \subset \mathcal{BU}$, and $\mathcal{C} = \mathcal{BU}$ is nilpotent.

Now let $\mathcal{W} = \mathcal{U}/\mathcal{C}$, and for $(x,y) \in V$ set $f(x,y) = q(x)q(y) + \mathcal{C}$. Also denote by $\mathcal{E}$ the subalgebra of $\mathcal{W}$ generated by $f(I^+,I^-)$ which is nilpotent by hypothesis.

3^o. <u>The ideal</u> $\mathcal{G}$ <u>of</u> $\mathcal{W}$ <u>generated by</u> $T = f(I^+,V^-) \cup f(V^+,I^-)$ <u>is nilpotent</u>.

From 13.4.3 and 2.3.3 and 2.3.4 it follows that

$$(1) \qquad [f(u,v),f(x,y)] = f(\{uvx\},y) - f(x,\{vuy\})$$

whenever one of u,v,x,y is in I. Hence we have $\mathcal{WE} \subset \mathcal{EW}$, and therefore $\mathcal{E}' = \mathcal{EW}$ is a nilpotent ideal of $\mathcal{W}$. Let $\mathcal{F}$ be the subalgebra of $\mathcal{W}$ generated by T. We will show that $\mathcal{F}/\mathcal{F}\cap\mathcal{E}'$ is nilpotent. Denoting by $u \mapsto \bar{u}$ the canonical map $\mathcal{W} \to \mathcal{W}/\mathcal{E}'$ we have by (1), similarly as in the proof of 1^o,

$$[\overline{f(u,v)},\overline{f(x,y)}] = \overline{f(\{uvx\},y)}$$

if $u,x \in I^+$, $v,y \in V^-$, or $x \in I^+$, $v \in I^-$, $u \in V^+$, $y \in V^-$, and

$$[\overline{f(u,v)},\overline{f(x,y)}] = \overline{f(-x,\{vuy\})}$$

for $u,x \in V^+$, $v,y \in I^-$ which proves $[\bar{T},\bar{T}] \subset \bar{T}$. From 2.3.5 it follows that $\overline{f(x,y)}^3 = \overline{f(Q_x y,Q_y x)} = 0$ if $x \in I^+$ or $y \in I^-$. Now Engel's theorem implies that the subalgebra $\bar{\mathcal{F}} = \mathcal{F}/\mathcal{F}\cap\mathcal{E}'$ generated by $\bar{T}$ is nilpotent, and therefore $\mathcal{F}$ is nilpotent. Again by (1) we see that $\mathcal{WF} \subset \mathcal{FW}$ and hence $\mathcal{G} = \mathcal{FW}$ is nilpotent.

Note now that $\tilde{I} \cap U^{++}(V)$ is the ideal of $\mathcal{U} = U^{++}(V)$ generated by $d(I^+,V^-) \cup d(V^+,I^-) \cup q(I^+)q(V^-) \cup q(V^+)q(I^-)$ since by 2.3.3 and 2.3.4 we may omit $q(V^+)q(I^-,V^-)$ and $q(I^+,V^+)q(V^-)$ from the generators. Hence we have that

$C \subset I \cap U^{++}(V)$, and $\left(I \cap U^{++}(V)\right)/C = G$. Therefore $I \cap U^{++}(V)$ is nilpotent, and the lemma is proved.

14.10. THEOREM. Let V be a finite-dimensional Jordan pair over a field k and let I be an ideal of V . Then the following conditions are equivalent.

(i) I is solvable;

(ii) I is I-solvable;

(iii) I is V-solvable;

(iv) I is nilpotent;

(v) $U_0(I)$ is nilpotent;

(vi) the ideal $\tilde{I} = \mathrm{Ker}(U(V) \to U(V/I))$ of $U(V)$, generated by $d(I^\sigma, V^{-\sigma})$, $d(V^\sigma, I^{-\sigma})$, $q(I^\sigma)$, $q(I^\sigma, V^\sigma)$ ($\sigma = \pm$) is nilpotent.

Proof. The implications (iv) $\to$ (ii) $\to$ (i) and (iii) $\to$ (ii) are obvious. Since $M_0(I)$ is a homomorphic image of $U_0(I)$ and also of a subalgebra of $\tilde{I}$, we have (v) $\to$ (iv) and (vi) $\to$ (iv) by 14.8. The implication (v) $\to$ (vi) follows from 14.9. We prove (i) $\to$ (v) by induction on the length of the derived series of I . First assume that I is trivial (i.e., $I^{(1)} = 0$) and let $\mathcal{D}(I) \subset U_0^{++}(I)$ be the ideal generated by $d(I^+, I^-)$ (cf. 13.5). By 13.4.1 and 13.4.2 we see that $\mathcal{D}(I)$ is commutative, and generated by nilpotent elements. Hence $\mathcal{D}(I)$ is nilpotent. Now $U_0^{++}(I)/\mathcal{D}(I)$ is generated by $q(x)q(y) + \mathcal{D}(I)$ where $x \in I^+$, $y \in I^-$. From 13.4.3 and 2.3.3 and 2.3.4 we see that $U_0^{++}(I)/\mathcal{D}(I)$ is commutative, and since $0 = (q(x)q(y))^2 = q(Q_x y)q(y)$, it is nilpotent. Thus $U_0^{++}(I)$ is nilpotent, and by 14.9, $U_0(I)$ is nilpotent. In the general case, we have $U_0(I^{(1)})$ nilpotent by induction, and hence $\widetilde{I^{(1)}}$ nilpotent by 14.9. Also $I/I^{(1)}$ is trivial and hence $U_0(I/I^{(1)}) = U_0(I)/\widetilde{I^{(1)}}$ is nilpotent. Therefore $U_0(I)$ is nilpotent.

Finally we show that (vi) $\to$ (iii). Let $I = I^{\langle 0\rangle} \supset I^{\langle 1\rangle} \supset \ldots$ be the V-derived series of I , and assume that $I^{\langle n\rangle} = I^{\langle n+1\rangle} = (I^{\langle n\rangle})^{\langle 1\rangle}$. Then the ideal $\widetilde{I^{<n>}}$ of $U(V)$ is contained in $\tilde{I}$ and is therefore nilpotent. Replacing $I^{<n>}$ by I we have to show that $I = 0$. Let $f : U(V) \to \mathrm{End}(V^+ \times V^-)$ denote the homomorphism induced by the regular representation. From the definitions it follows easily that $I^{<1>} \subset f(\tilde{I}).I$, and if $I \neq 0$ then $f(\tilde{I}).I \subsetneqq I$ by nilpotence of $f(\tilde{I})$. Hence we have $I = 0$.

14.11. THEOREM. The radical of a finite-dimensional Jordan pair is nilpotent.

Proof. Since V satisfies all chain conditions we have $\mathrm{rad}\, V = \mathrm{Nil}\, V = \mathrm{Rad}\, V$ by 10.8. By 14.10, it suffices to show that $\mathrm{rad}\, V$ is solvable. Let S be the maximal solvable ideal of V which by 14.5 is contained in $\mathrm{rad}\, V$. After dividing by S, we may assume that V has no non-zero solvable ideals and have to show that $\mathrm{rad}\, V = 0$. If this is not so then V contains non-zero trivial elements and hence $T(V)$, the linear span of all trivial elements, is a non-zero ideal of V (cf. 4.6). Replacing V by $T(V)$ it suffices to show: If V is spanned by trivial elements then it is solvable. Let B be a subpair of V, maximal among the subpairs which are both solvable and are spanned by trivial elements. Clearly, 0 is such a subpair. Assume that $B \neq V$. Then there exists a trivial element w , say $w \in V^+$, which is not in B^+ . By 14.10, $U_0(B)$ is nilpotent, and hence its image N in $M_0(V)$ under the homomorphism induced by the regular representation is nilpotent. Therefore there exists an integer $n \geq 0$ such that $N^n.w \not\subset B$ but $N^{n+1}.w \subset B$ (by $N.w$ we mean of course the pair $(N^{++}.w, N^{-+}.w)$ where $N^{\sigma\tau}$ is the image of $U_0^{\sigma\tau}(B)$). Since B is spanned by trivial elements N is generated by $D(a,b)$ and $Q(a,c)$ where a,b,c are trivial elements contained in $B^{\pm}$. Hence there exists an element $P = P_1 \dots P_m \in N^n$, each P_i of the form $D(a,b)$ or $Q(a,b)$, such that $z = P.w$ is not in B , say $z \notin B^+$. From JP20 it follows that $\{uvw\}$ is trivial if u,v,w are trivial. Hence z is a trivial element, and we have $N.z \subset B$, in particular, $\{B^+, B^-, z\}$

$\subset B^+$ and $Q(B^-)z \subset B^-$. Let $C = (k.z + B^+, B^-)$. Then C is spanned by trivial elements, and $Q(C^+)C^- = Q(B^+)B^- + \{B^+, B^-, z\} \subset B^+$, $Q(C^-)C^+ = Q(B^-)B^+ + Q(B^-)z \subset B^-$. Thus $C^{(1)} \subset B$ which implies $C^{(n+1)} \subset B^{(n)}$ and C is solvable, contradicting the maximality of B . This completes the proof.

14.12. COROLLARY. A finite-dimensional simple Jordan pair is semisimple.

14.13. Solvability and nilpotence for alternative pairs. Let $A = (A^+, A^-)$ be an alternative pair over a ring k . The derived series of A is defined by

$$A^{(1)} = (\langle A^+A^-A^+\rangle, \langle A^-A^+A^-\rangle), \quad A^{(n+1)} = (A^{(n)})^{(1)} .$$

We say A is solvable if $A^{(n)} = 0$ for some n . Obviously $A^{(n)} \supset (A^J)^{(n)}$ where A^J is the associated Jordan pair. Let $A_{(n)}$ be the k-linear span of all products of degree $\geqq n$ where $\langle xyz\rangle$ is considered to be of degree 3 . Then $A = A_{(1)} \supset A_{(3)} \supset A_{(5)} \supset \cdots$ is a sequence of ideals of A , and A is called nilpotent if $A_{(n)} = 0$ for some n . Again we have $A_{(n)} \supset (A^J)_{(n)}$. Also, it is clear that $A^{(n)} \subset A_{(3^n)}$ and hence nilpotence implies solvability.

We define the multiplication algebra $M_0(A)$ as follows. Write the elements of $A^+ \times A^-$ as column vectors, and define endomorphisms of $A^+ \times A^-$ by

$$\lambda_+(x,y) = \begin{pmatrix} L_+(x,y) & 0 \\ 0 & 0 \end{pmatrix}, \quad \lambda_-(y,x) = \begin{pmatrix} 0 & 0 \\ 0 & L_-(y,x) \end{pmatrix},$$

$$\rho_+(x,y) = \begin{pmatrix} R_+(x,y) & 0 \\ 0 & 0 \end{pmatrix}, \quad \rho_-(y,x) = \begin{pmatrix} 0 & 0 \\ 0 & R_-(y,x) \end{pmatrix},$$

$$\mu_+(x,y) = \begin{pmatrix} 0 & M_+(x,y) \\ 0 & 0 \end{pmatrix}, \quad \mu_-(x,y) = \begin{pmatrix} 0 & 0 \\ M_-(x,y) & 0 \end{pmatrix}.$$

Then $M_0(A)$ is the subalgebra of $\mathrm{End}(A^+ \times A^-)$ generated by all $\lambda_\sigma(A^\sigma,A^{-\sigma})$, $\rho_\sigma(A^\sigma,A^{-\sigma})$, $\mu_\sigma(A^\sigma,A^\sigma)$, and $M(A) = k.c_+ + k.c_- + M_0(A)$ where

$$c_+ = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & 0 \end{pmatrix}, \quad c_- = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{Id} \end{pmatrix}.$$

A proof similar to (but simpler than) that of 14.7 shows that

$$A_{(3^n)} \subset M_0(A)^n.A \subset A_{(2n+1)} ,$$

and hence A is nilpotent if and only if $M_0(A)$ is nilpotent.

14.14. THEOREM. <u>For an ideal</u> B <u>of a finite-dimensional alternative pair</u> A , <u>the following conditions are equivalent</u>.

(i) B <u>is nilpotent</u>;

(ii) B <u>is solvable</u>;

(iii) B <u>is radical</u>;

(iv) <u>the ideal</u> $\mathcal{B}$ <u>of</u> $M(A)$ <u>generated by</u> $\lambda_\sigma(A^\sigma,B^{-\sigma})$, $\lambda_\sigma(B^\sigma,A^{-\sigma})$, $\rho_\sigma(A^\sigma,B^{-\sigma})$, $\rho_\sigma(B^\sigma,A^{-\sigma})$, $\mu_\sigma(A^\sigma,B^\sigma)$, $\mu_\sigma(B^\sigma,A^\sigma)$ ($\sigma = \pm$) <u>is nilpotent</u>.

<u>Proof</u>. The implications (iv) $\to$ (i) $\to$ (ii) are obvious.

(ii) $\to$ (iii): B is a solvable ideal of the Jordan pair A^J and hence contained in $\mathrm{Rad}\,A^J = \mathrm{Rad}\,A$ by 14.5.

(iii) $\to$ (iv): Let $W = W_2 \oplus W_1$ be the standard imbedding of A (8.12). By construction, it is clear that W is finite-dimensional. We have $\mathrm{Rad}\,W = \mathrm{Rad}\,W_2 + \mathrm{Rad}\,W_1$ (cf. 5.8) and hence $B \subset \mathrm{Rad}\,A = \mathrm{Rad}\,W_1 \subset \mathrm{Rad}\,W$. From the formula $\langle xyz\rangle = \{\{xye^\sigma\}e^{-\sigma}z\}$ (cf. 8.12) it follows that

$$\lambda_\sigma(x,y) = d_\sigma(\{xye^\sigma\},e^{-\sigma})|W_1 , \tag{1}$$

$$\rho_\sigma(z,y) = d_\sigma(z,e^{-\sigma})d_\sigma(e^\sigma,y)|W_1 , \tag{2}$$

(3) $$\mu_\sigma(x,z) = d_\sigma(z,e^{-\sigma})q_\sigma(x,e^\sigma)|\mathcal{W}_1 ,$$

where (d,q) is the regular representation of $\mathcal{W}$ (cf. 2.4). Let $\mathcal{I} = \operatorname{Rad}\mathcal{W}$ which is solvable by 14.11. By 14.10, the ideal of $M(\mathcal{W})$ generated by $d_\sigma(\mathcal{I}^\sigma,\mathcal{W}^{-\sigma})$, $d_\sigma(\mathcal{W}^\sigma,\mathcal{I}^{-\sigma})$, $q_\sigma(\mathcal{I}^\sigma,\mathcal{W}^\sigma)$ is nilpotent, since it is a homomorphic image of $\tilde{\mathcal{I}}$. Hence it follows from (1) - (3) that $\tilde{\mathcal{B}}$ is nilpotent.

§15. Cartan subpairs

15.0. In this section, k always denotes a field and $\bar{k}$ its algebraic closure. Jordan pairs over k are finite-dimensional. If $\mathcal{V} = (\mathcal{V}^+,\mathcal{V}^-)$ is a Jordan pair over k then we define the dimension of $\mathcal{V}$ as

$$\dim \mathcal{V} = \max(\dim \mathcal{V}^+,\dim \mathcal{V}^-).$$

We often write $\bar{\mathcal{V}}$ instead of $\mathcal{V}_{\bar{k}} = \mathcal{V}\otimes\bar{k}$. By finite-dimensionality, $\mathcal{V}$ satisfies all chain conditions. Hence the various radicals coincide (10.8), and $\operatorname{Rad}\mathcal{V}$ is the maximal solvable (= nilpotent) ideal of $\mathcal{V}$ (14.11). Recall also that a frame of $\mathcal{V}$ is a maximal orthogonal system of local idempotents (10.12).

15.1. Galois descent. Let X be a finite-dimensional vector space over k, let K be an extension field of k, and let Y be a subspace of $X_K = X\otimes K$. We say that Y is defined over k if $X\cap Y$ spans Y over K. Here we identify X with $X\otimes 1$ in X_K. If K is Galois over k (not necessarily of finite degree) and $\Gamma = \mathrm{Gal}(K/k)$ is the (topological) Galois group, acting on X_K by semilinear transformations, then Y is defined over k if and only if it is stable under Γ. For a proof, see Borel[1], p. 52ff.

15.2. Separability. Let V be a Jordan pair over k, and let K be an extension field of k. By 4.12 and 10.8 we have

(1) $$(\mathrm{Rad}\, V)_K \subset \mathrm{Rad}(V_K),$$

but in general, we will not have equality. If, however, K is a separable extension of k then equality holds in (1). Indeed, let $\bar{K}$ be the algebraic closure of K, and let k_s be the separable closure of k in $\bar{K}$ so that $K \subset k_s$. It suffices to prove the assertion for k_s instead of K so we assume $K = k_s$. Let $_{\underline{\underline{Z}}}(V_K)$ be the Jordan pair over $\underline{\underline{Z}}$ obtained from V_K by restricting the scalars to $\underline{\underline{Z}}$ (cf. 0.3). Then $\Gamma = \mathrm{Gal}(K/k)$ acts on $_{\underline{\underline{Z}}}(V_K)$ by automorphisms, and hence $\mathrm{Rad}(_{\underline{\underline{Z}}}(V_K)) = \mathrm{Rad}\, V_K$ (cf. 4.1) is stable under Γ. By 15.1, it is defined over k. Since obviously $\mathrm{Rad}(V_K) \cap V = \mathrm{Rad}\, V$ the assertion follows. As a consequence, we see that $\mathrm{Rad}\, V_{\bar{k}}$ is defined over $k^{p^{-\infty}}$ where $p = \mathrm{char}\, k$.

A Jordan pair V is called separable if V_K is semisimple, for every extension field K of k. Obviously, separability is preserved under base field extension. By the above, V is separable if and only if $V \otimes k^{p^{-\infty}}$ is semisimple. In particular, a semisimple Jordan pair over a perfect field is separable.

15.3. Associators. Let V be a Jordan pair over a ring R and let $u,x,z \in V^{\sigma}$ and $v,y \in V^{-\sigma}$. The associator of (u,v,x,y,z) is defined as

$$[uvxyz] = \{\{uvx\}yz\} - \{uv\{xyz\}\} .$$

It is clearly linear in each variable. For $x = (x^+,x^-)$, $y = (y^+,y^-) \in V$ we define $A_\sigma(x,y) \in \mathrm{End}(V^\sigma)$ by

$$A_\sigma(x,y).z = [x^\sigma, x^{-\sigma}, y^\sigma, y^{-\sigma}, z].$$

We say V is <u>associative</u> if all associators vanish. In this case, V is an associative pair with $\langle xyz\rangle = \{xyz\}$. Indeed, by JP14, $[uvxyz] = \{x\{vuy\}z\} - \{xy\{uvz\}\} = 0$. A finite-dimensional Jordan pair over a field is called a <u>torus</u> if it is associative and separable.

As an example of an associative Jordan pair, let U be the subpair of a Jordan pair V generated by a pair $(x,y) \in V$. It is not hard to show that U^+ (resp. U^-) is spanned by all powers $x^{(n,y)}$ (resp. $y^{(n,x)}$). From the formula

$$\{x^{(m,y)}, y^{(n,x)}, x^{(p,y)}\} = 2x^{(m+n+p-1,y)}$$

(proved by induction) it follows that U is associative.

Let J be a Jordan algebra. The associator of $x,y,z \in J$ is defined by

$$[xyz] = (x\circ y)\circ z - x\circ(y\circ z),$$

and J is called associative if all associators vanish. If $J = V_v^+$ then $x\circ y = \{xvy\}$ and hence $[xyz] = [xvyvz]$. In particular, if V is an associative Jordan pair then all the Jordan algebras V_v^+ are associative.

15.4. LEMMA. <u>Let V be a Jordan pair over a ring R. Let $(e_1,\dots,e_r)$ be an orthogonal system of idempotents, let $x^\sigma = \lambda_1^\sigma e_1^\sigma + \dots + \lambda_r^\sigma e_r^\sigma$, $y^\sigma = \mu_1^\sigma e_1^\sigma + \dots + \mu_r^\sigma e_r^\sigma$ where $\lambda_i^\sigma, \mu_i^\sigma \in R$. Set $\lambda_i = \lambda_i^+\lambda_i^-$, $\mu_i = \mu_i^+\mu_i^-$, $\lambda_0 = \mu_0 = 0$, and denote by E_{ij}^σ the projection onto the Peirce space V_{ij}^σ (cf. 5.14). Then</u>

$$A_\sigma(x,y) = \sum_{0\le i\le j\le r} (\lambda_i - \lambda_j)(\mu_i - \mu_j)E_{ij}^\sigma .$$

<u>In particular, if $e = (e^+, e^-)$ is an idempotent then $A_\sigma(e,e)$ is the projection onto the Peirce space $V_1^\sigma(e)$.</u>

Proof. Let $z \in V^{\sigma}_{ij}$. Then it follows from 5.14 that $\{x^{\sigma},x^{-\sigma},z\} = (\lambda_i+\lambda_j)z$. Hence $\{x^{\sigma},x^{-\sigma},y^{\sigma}\} = 2\,\Sigma\,\lambda_{\ell}\mu_{\ell}e^{\sigma}_{\ell}$, and

$$\begin{aligned} A_{\sigma}(x,y).z &= 2\{\Sigma\lambda_{\ell}\mu_{\ell}e^{\sigma}_{\ell},y^{-\sigma},z\} - \{x^{\sigma},x^{-\sigma},(\mu_i+\mu_j)z\} \\ &= (2(\lambda_i\mu_i+\lambda_j\mu_j) - (\lambda_i+\lambda_j)(\mu_i+\mu_j))z = (\lambda_i-\lambda_j)(\mu_i-\mu_j)z . \end{aligned}$$

15.5. LEMMA. (a) Let J be a finite-dimensional Jordan division algebra over k, and let K be the subalgebra generated by (1 and) $a \in J$. Then K "is a field"; i.e., there exists a bilinear multiplication xy on K such that K is an extension field of k and $U_x y = x^2 y$ for all $x,y \in K$.

(b) A finite-dimensional Jordan division algebra (resp. Jordan division pair) over an algebraically closed field k is isomorphic with k^J (resp. (k^J,k^J)).

Proof. Let $f\colon k[T] \to J$ be the homomorphism of unital Jordan algebras sending T into a . It suffices to show that the kernel of f is an ideal of the associative algebra $k[T]$. This follows from Jacobson[3], p. 1.62, since J is non-degenerate. Now (b) follows immediately from (a).

15.6. PROPOSITION. A Jordan pair V over k is a torus if and only if V is a finite direct product of Jordan pairs of the form (K^J,K^J) where K is a finite separable extension field of k .

Proof. Clearly a Jordan pair of the indicated form is a torus. Conversely, let V be a torus, and let $E = (e_1,..,e_r)$ be a frame of V . Since V is semisimple the e_i are division idempotents (cf. 5.12). By 15.4 and associativity of V we have $V_{ij} = 0$ for $i \neq j$ in the Peirce decomposition with respect to E , and hence $V = V_{11} \oplus \dots \oplus V_{rr}$. By the composition rules for the Peirce spaces the V_{ii} are ideals. Therefore we may assume that V is a division pair. Let $v \in V^-$

be non-zero, and let J be the Jordan division algebra V_v^+ with unit element v^{-1}. Since $V \cong (J,J)$ it suffices to show that J "is" a separable extension of k (in the sense of 15.5). If char $k \neq 2$ we simply set $xy = (1/2)(x\circ y)$. Assume that k has characteristic 2 and let $\bar{J} = J \otimes \bar{k}$. Since $\bar{V} = (\bar{J},\bar{J})$ is a torus over $\bar{k}$ we may choose a frame $(c_1,\dots,c_n)$ and then have as before that $\bar{V} = \bar{V}_{11} \oplus \dots \oplus \bar{V}_{nn}$ where the $\bar{V}_{ii}$ are Jordan division pairs over $\bar{k}$. By 15.5, $\bar{V}_{ii}^{\pm} = \bar{k}.c_i^{\pm}$ and therefore $v = \nu_1 c_1^- + \dots + \nu_n c_n^-$ where $\nu_i \neq 0$. Replacing c_i^- by $\nu_i c_i^-$ and c_i^+ by $\nu_i^{-1} c_i^+$ we may assume that $1 = v^{-1} = \Sigma\, c_i^+$ is the unit element of J, and hence $\bar{J} = \bar{k}.c_1^+ \oplus \dots \oplus \bar{k}.c_n^+$ where the c_i^+ are orthogonal idempotents of $\bar{J}$. Now we distinguish two cases.

(a) k is infinite. Let $b_1,\dots,b_n$ be a basis of J over k, let $b = b_1 \wedge \dots \wedge b_n$ (which is a basis for the one-dimensional n-th exterior power of J) and define a polynomial function f on J by $f(x).b = 1 \wedge x \wedge x^2 \wedge \dots \wedge x^{n-1}$. Then f has a natural extension to $\bar{J}$, and if $x = \xi_1 c_1^+ + \dots + \xi_n c_n^+ \in \bar{J}$ then $f(x)$ is up to a non-zero scalar factor the Vandermonde determinant of $\xi_1,\dots,\xi_n$. Hence $f \neq 0$, and since k is infinite there exists $x \in J$ such that $f(x) \neq 0$. This means that J is generated by x, and by 15.5 it is a field extension of k which is separable since $\bar{J} = \bar{k}^n$.

(b) k is finite. Since the characteristic is 2 we have $x\circ y = 2\Sigma\xi_i\eta_i c_i^+ = 0$ for all $x = \Sigma\xi_i c_i^+$, $y = \Sigma\eta_i c_i^+$ in $\bar{J}$. The map $x \mapsto x^2$ from J into itself is injective since J is a division algebra, and therefore surjective since J is finite. Also $(x+y)^2 = x^2 + x\circ y + y^2 = x^2 + y^2$. Thus if $x \to \sqrt{x}$ denotes its inverse then we have $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$, and $\sqrt{\lambda x} = \sqrt{\lambda}\sqrt{x}$ for all $\lambda \in k$. We set $xy = U_{\sqrt{x}}y$. Then this is a k-bilinear multiplication on J making J into an extension field of k and such that $U_x y = x^2 y$ (the fact that xy is associative and commutative needs to be checked only in $\bar{J}$ where it is trivial).

15.7. DEFINITION. Let V be a Jordan pair over k. If $X = (X^+, X^-) \subset (V^+, V^-)$ is a pair of subsets we define the _centralizer_ of X, $\mathrm{Cent}(X)$, to be the pair (C^+, C^-) where

$$C^\sigma = \{z \in V^\sigma \mid A_\sigma(x,y)z = 0, \ \text{ for all } x,y \in X\} .$$

The _normalizer_ $\mathrm{Norm}(X)$ of X is the pair (N^+, N^-) where

$$N^\sigma = \{z \in V^\sigma \mid A_\sigma(x,y)z \in X , \ \text{ for all } \ x,y \in X\}.$$

Clearly, $\mathrm{Cent}(X)$ is a pair of vector subspaces, and so is $\mathrm{Norm}(X)$ provided X is a pair of vector subspaces.

Generalizing associativity, a Jordan pair is called _associator nilpotent_ if there exists an integer n such that any product of more than n of the transformations $A_\sigma(x,y)$ vanishes. Since this is a multilinear condition, associator nilpotence is preserved under base field extension. A nilpotent Jordan pair is obviously associator nilpotent. Finally, a subpair C of V is called a _Cartan subpair_ if it is associator nilpotent and equal to its own normalizer.

15.8. THEOREM.(a) _Consider the following conditions on a Jordan pair_ C _over_ k.

(i) C _is associator nilpotent;_

(ii) $A_\pm(x,x)$ _is nilpotent for every_ $x \in C$;

(iii) $C_1(e) = 0$ _for every idempotent_ e _of_ C ;

(iv) $C = C_0 \oplus C_1 \oplus \dots \oplus C_r$ _(direct sum of ideals) where_ C_0 _is nilpotent and_ C_i _is local, for_ $i = 1,\dots,r$.

Then we have the implications (i) → (ii) → (iii) → (iv). _If_ k _is algebraically closed then also_ (iv) → (i).

(b) _Let_ k _be algebraically closed, and let_ C _be an associator nilpotent Jordan pair, contained in some Jordan pair_ V _as a subpair. Denote by_ $A^\sigma \subset \mathrm{End}(V^\sigma)$

the subalgebra generated by all $A_\sigma(x,y)$, $x,y \in C$. Let $(e_1,..,e_r)$ be a frame of C, and let E^σ_{ij} be the projection of V^σ onto the Peirce space V^σ_{ij}. Then E^σ_{ij} is a central idempotent of A^σ for $i \neq j$, and

$$(1) \qquad A^\sigma = \sum_{i \neq j} k.E^\sigma_{ij} \oplus R^\sigma$$

where R^σ is a nilpotent ideal of A^σ.

Proof. Obviously we have (i) → (ii), and (ii) → (iii) follows from 15.4. For (iii) → (iv), pick a frame $(e_1,\ldots,e_r)$ of C. Then $C_{ij} = 0$ for $i \neq j$ and hence $C = C_0 \oplus C_1 \oplus \ldots \oplus C_r$ where we set $C_i = C_{ii}$. By 10.11 and 14.11, $C_0 \subset \mathrm{Rad}\, C$ is nilpotent. For (iv) → (i) it suffices to prove (b) under the assumption that $C = C_{00} \oplus C_{11} \oplus \ldots \oplus C_{rr}$ where $C = \Sigma\, C_{ij}$ is the Peirce decomposition with respect to a frame $(e_1,\ldots,e_r)$ of C. Since C_{ii} is local for $i > 0$ the quotient $C_{ii}/\mathrm{Rad}\, C_{ii}$ is a division pair by 4.4, and therefore isomorphic with (k^J,k^J) by 15.5. Hence $C^\pm_{ii} = k.e^\pm_i \oplus \mathrm{Rad}\, C^\pm_{ii}$ and

$$(2) \qquad C^\pm = k.e^\pm_1 \oplus \ldots \oplus k.e^\pm_r \oplus \mathrm{Rad}\, C^\pm .$$

From the Peirce decomposition rules it follows that every transformation in A^σ leaves the Peirce spaces V^σ_{ij} invariant. By 15.4 it follows that E^σ_{ij} is a central idempotent of A^σ for $i \neq j$. Now (1) follows from (2), 14.10, 14.11, and 15.4.

15.9. COROLLARY. (a) Cartan subpairs are maximal among associator nilpotent subpairs.

(b) Let k be algebraically closed. Then C is a Cartan subpair of V if and only if it is of the form $C = V_{00} \oplus V_{11} \oplus \ldots \oplus V_{rr} = \mathrm{Cent}(T)$ where $V = \Sigma\, V_{ij}$ is the Peirce decomposition of V with respect to a frame $(e_1,\ldots,e_r)$ of V and $T = (\Sigma\, k.e^+_i, \Sigma\, k.e^-_i)$ is the torus spanned by $(e_1,\ldots,e_r)$.

Proof. For the proof of (a) we may assume k to be algebraically closed. Let $\mathcal{C}$ be a Cartan subpair of $\mathcal{V}$, contained in some associator nilpotent subpair $\mathcal{D}$ of $\mathcal{V}$. Pick a frame $(e_1,\ldots,e_r)$ of $\mathcal{C}$. Then by 15.8(a) we have $\mathcal{C} = \Sigma \mathcal{C}_{ii} \subset \mathcal{D} = \Sigma \mathcal{D}_{ii} \subset \Sigma \mathcal{V}_{ii}$. Assume that there exists a z in $\Sigma \mathcal{V}^\sigma_{ii}$ but not in $\mathcal{C}^\sigma$. Since $A^\sigma.z = \mathcal{R}^\sigma.z$ and $\mathcal{R}^\sigma$ is nilpotent there exists $P \in \mathcal{R}^\sigma$ such that $w = P.z \notin \mathcal{C}^\sigma$ but $A_\sigma(x,y)w \in \mathcal{C}^\sigma$ for all $x,y \in \mathcal{C}$, contradicting the fact that $\mathcal{C}$ is its own normalizer. Hence $\mathcal{C} = \mathcal{D} = \Sigma \mathcal{V}_{ii}$; in particular, $(e_1,\ldots,e_r)$ is a frame of $\mathcal{V}$. This proves (a) and one half of (b). Now let $(e_1,\ldots,e_r)$ be a frame of $\mathcal{V}$, and set $\mathcal{C} = \Sigma \mathcal{V}_{ii}$. Then by 15.8(b), $\mathcal{C}$ is associator nilpotent, and if z belongs to the normalizer of $\mathcal{C}$ then $A^\sigma.z \subset \mathcal{C}^\sigma$ which by 15.8.1 implies $z \in \mathcal{C}^\sigma$. This shows that $\mathcal{C}$ is a Cartan subpair of $\mathcal{V}$. Finally, it is clear by 15.4 that the centralizer of $\mathcal{T}$ is $\mathcal{C}$.

15.10. PROPOSITION. Let $\mathcal{C}$ be a Cartan subpair of $\mathcal{V}$. Then there exists a torus $\mathcal{T} \subset \mathcal{C}$ with the following properties.

(i) $\mathcal{C}_K = \mathcal{T}_K \oplus \operatorname{Rad} \mathcal{C}_K$ for any perfect extension field K of k.

(ii) $\mathcal{C} = \operatorname{Cent}(\mathcal{T})$.

Proof. Pick a maximal idempotent $e \in \mathcal{C}$ so that $\mathcal{C}_0(e) \subset \operatorname{Rad} \mathcal{C}$. This is possible by 10.11. Let $\overline{\mathcal{V}} = \mathcal{V}_{\bar{k}}$ and $\overline{\mathcal{C}} = \mathcal{C}_{\bar{k}}$. Then $\overline{\mathcal{C}}$ is a Cartan subpair of $\overline{\mathcal{V}}$, and by 15.9 we have $\overline{\mathcal{C}} = \Sigma \overline{\mathcal{V}}_{ii}$ with respect to a frame $(e_1,\ldots,e_r)$ of $\overline{\mathcal{V}}$. Since e is still a maximal idempotent of $\overline{\mathcal{C}}$ we have $e^\sigma = \Sigma c_i^\sigma$ where $c_i = (c_i^+, c_i^-)$ is a non-zero idempotent contained in $\overline{\mathcal{V}}_{ii}$. Replacing e_i by c_i we may assume that $e = \Sigma e_i$. Let $\mathcal{U} = \Sigma \bar{k}.e_i \subset \overline{\mathcal{C}}$ be the torus spanned by the e_i. Then $\overline{\mathcal{C}} = \operatorname{Cent}(\mathcal{U})$, and since $\overline{\mathcal{V}}_{ii} = \bar{k}.e_i \oplus \operatorname{Rad} \overline{\mathcal{V}}_{ii}$ we have $\overline{\mathcal{C}} = \mathcal{U} \oplus \operatorname{Rad} \overline{\mathcal{C}}$. Thus if we can show that $\mathcal{U}$ is defined over k then $\mathcal{T} = \mathcal{U} \cap \mathcal{V}$ is the desired torus. We distinguish two cases.

(a) char $k = 0$. Then k is perfect, and by 15.1 we have to show that $\mathcal{U}$ is invariant under the Galois group $\Gamma = \mathrm{Gal}(\bar{k}/k)$. Since $e \in \mathcal{C}$ it is fixed under Γ . Hence Γ permutes the e_i and leaves $\mathcal{U}$ invariant.

(b) char $k = p > 0$. Let $\mathcal{J}$ be the Jordan algebra $\mathcal{C}^+_{e^-}$, and let $f_r: \bar{\mathcal{J}} \to \bar{\mathcal{J}}$ be the map $f_r(x) = x^{p^r}$, $r \geq 0$. Then f_r is defined over k (see Borel[1], pp. 43). We have $\bar{\mathcal{J}} = \sum_{i=0}^{r} \mathcal{J}_i$ where $\mathcal{J}_i = \bar{k}.e_i^+ \oplus \mathrm{Rad}\, \mathcal{J}_i$ is local for i positive, and $\mathcal{J}_0$ is nilpotent. Thus if $x \in \bar{\mathcal{J}}$ we have $x = \sum \lambda_i e_i^+ + n_i$ where $\lambda_i \in \bar{k}$, the n_i are nilpotent, and we set $\lambda_0 = e_0^+ = 0$. It follows that $x^{p^r} = \sum \lambda_i^{p^r} e_i^+ + n_i^{p^r}$, and for sufficiently large r we have $x^{p^r} = \sum \lambda_i^{p^r} e_i^+ \in \mathcal{U}^+$. Since we can extract p^r-th roots in $\bar{k}$ we see that $\mathcal{U}^+ = f_r(\bar{\mathcal{C}}^+)$, and is therefore defined over k (see Borel[1], p.57, Cor. 14.5). Similarly, one shows that $\mathcal{U}^-$ is defined over k .

15.11. PROPOSITION. Let $x = (x^+, x^-) \in \mathcal{V}$ and let $\mathcal{N}(x) = (\mathcal{N}^+(x), \mathcal{N}^-(x)) \subset (\mathcal{V}^+, \mathcal{V}^-)$ be defined by

$$\mathcal{N}^\sigma(x) = \{z \in \mathcal{V}^\sigma \mid A_\sigma(x,x)^n.z = 0, \quad \text{for some integer } n > 0\}.$$

Then $\mathcal{N}(x)$ is a subpair of $\mathcal{V}$ which is equal to its own normalizer.

Proof. From the definition it is obvious that $\mathcal{N}(x)$ is its own normalizer. To show that it is a subpair we may assume that k is algebraically closed. Let $\mathcal{C}$ be the subpair of $\mathcal{V}$ generated by x which is associative by 15.3. Choosing a frame $(e_1,\dots,e_r)$ of $\mathcal{C}$ we have

$$x^\sigma = \sum_{i=1}^{r} \lambda_i^\sigma e_i^\sigma + n^\sigma \tag{1}$$

where $n^\sigma \in \mathrm{Rad}\, \mathcal{C}^\sigma$. Since x generates $\mathcal{C}$ it follows that all $\lambda_i = \lambda_i^+ \lambda_i^-$ are different from zero, and $\lambda_i \neq \lambda_j$ for $i \neq j$. Let $s^\sigma = \sum \lambda_i^\sigma e_i^\sigma$ and let $s = (s^+, s^-)$. Then it follows from 15.8(b) that $A_\sigma(s,s)$ is the semisimple part in the Jordan decomposition of $A_\sigma(x,x)$. By 15.4, $N(x) = (\mathrm{Ker}\, A_+(s,s), \mathrm{Ker}\, A_-(s,s)) = \Sigma\, V_{ii}$ is a subpair of V.

Remark. Formula (1) may be regarded as a "Jordan decomposition" of x. In contrast to the familiar Jordan decomposition of matrices, however, nilpotent and semisimple part are not uniquely determined whereas the $\lambda_i = \lambda_i^+\lambda_i^-$ are uniquely determined by x (although the $\lambda_i^\pm$ are not) and play the role of eigenvalues of x.

15.12. Associator regularity. Let V be a Jordan pair over k. For $x = (x^+,x^-) \in V_R$ and $R \in k\text{-}\underline{\underline{\mathrm{alg}}}$ let

$$P(x,T) = T^n + c_1(x)T^{n-1} + \dots + c_n(x)$$

be the product of the characteristic polynomials of $A_+(x,x)$ and $A_-(x,x)$ (so that $n = \dim V^+ + \dim V^-$). Then it is clear that by "varying R" we have defined polynomial functions c_i on $V^+ \times V^-$ (cf. 18.1), and hence $P(T) = T^n + c_1T^{n-1} + \dots + c_n$ is a polynomial in T whose coefficients are polynomial functions on $V^+ \times V^-$. Let m be the largest integer such that $P(T)$ is divisible by T^m, so that $P(T) = T^n + c_1T^{n-1} + \dots + c_{n-m}T^m$ where $c_{n-m} \neq 0$ but $c_{n-i} = 0$ for all $i < m$. We remark that $m \geq 2$ since $A_\pm(x,x).x^\pm = 0$. An element $x \in \overline{V}$ is called associator regular if $c_{n-m}(x) \neq 0$. It is clear that the associator regular elements form a Zariski open and dense subset of $\overline{V}$ which is invariant under the automorphism group $\mathrm{Aut}(\overline{V})$. If k is infinite then V itself contains associator regular elements but if k is finite this need not be so.

15.13. PROPOSITION. Let m be defined as above, and let $x = (x^+,x^-) \in V$. Then $\dim V^+(x) + \dim V^-(x) \geq m$, and equality holds if and only if x is associator regular. In this case, $N(x)$ is a Cartan subpair of V, and it is the only one

<u>which contains</u> x .

<u>Proof</u>. The first assertion holds since $\dim N^+(x) + \dim N^-(x)$ is the multiplicity of zero as a root of $P(x,T)$. Now let x be associator regular. By 15.11 we only have to show that $C = N(x)$ is associator nilpotent. Since we may assume k to be algebraically closed it suffices (by 15.8(a)) to show that the restriction of $A_\sigma(y,y)$ to C^σ is nilpotent, for all $y \in C$. Since $A_\sigma(x,x)$ induces an injective endomorphism on V^σ/C^σ by definition of C , there exists an open dense subset W of C such that this is still the case for all $y \in W$; i.e., $N(y) \subset C$ for all $y \in W$. On the other hand, $\dim N^+(y) + \dim N^-(y) \geq m = \dim C^+ + \dim C^-$ and hence $N(y) = C$ for all $y \in W$. This means that $A_\sigma(y,y)$ restricted to C^σ is nilpotent, for all $y \in W$, and since W is dense in C , this is true for all $y \in C$. Assume now that x is contained in some other Cartan subpair C' . Then $C' \subset N(x)$ since C' is associator nilpotent, and hence $C' = N(x)$ by 15.9(a).

15.14. Let V be a Jordan pair over k , and let $\overline{V} = V_{\overline{k}}$. Let $X \subset \overline{V}^+ \times \overline{V}^-$ be the set of quasi-invertible pairs. Then X is the open and dense subset defined by $\det B(x,y) \neq 0$ and is therefore an irreducible smooth algebraic variety defined over k . The map $\beta\colon X \to GL(\overline{V}^+) \times GL(\overline{V}^-)$ (cf. 3.9) is a morphism of k-varieties. By Borel[1], p. 106, the inner automorphism group $G = \mathrm{Inn}(\overline{V})$ which is generated by $\beta(X)$ is a connected algebraic group which is defined over k . From 3.11 it follows that the Lie algebra $\mathfrak{g}$ of G contains all inner derivations $\delta(x,y)$, $(x,y) \in \overline{V}$. We recall that a morphism ϕ between smooth irreducible algebraic varieties is dominant and separable if and only if the differential of ϕ at some point (and hence at all points of a dense open subset) is surjective (see Borel[1], p. 75, Th. 17.3).

15.15. THEOREM. Let V be a Jordan pair over an algebraically closed field k, and let B be a subpair of V. Consider the following conditions.

(i) B contains a Cartan subpair of V;

(ii) B contains the centralizer of some torus of V;

(iii) the orbit map $\phi: G \times B \to V$ is dominant and separable;

(iv) the orbit of B under G contains an open dense subset of V;

(v) B contains associator regular elements.

Then we have the implications (i) $\leftrightarrow$ (ii) $\to$ (iii) $\to$ (iv) $\to$ (v). If V is semi-simple then all five conditions are equivalent.

Proof. (i) $\to$ (ii): This follows from 15.10.

(ii) $\to$ (iii): Let T be a torus of V such that $C = \mathrm{Cent}(T) \subset B$. By 15.6 we have $T = (k^J, k^J)^r$ and hence there exists an orthogonal system $(e_1,\dots,e_r)$ of idempotents of V such that T is the torus spanned by the e_i. Then $C = \Sigma V_{ii}$ by 15.4. Let $x^\sigma = \lambda_1^\sigma e_1^\sigma + \dots + \lambda_r^\sigma e_r^\sigma$ with $\lambda_i^\sigma \in k$ be an element of T^σ and set $\lambda_i = \lambda_i^+\lambda_i^-$ for $i = 1,\dots,r$, and $\lambda_0^\sigma = \lambda_0 = 0$. Since k is infinite we may choose the λ's in such a way that $\lambda_i \neq \lambda_j$ for $i \neq j$, $i,j = 0,\dots,r$. It will be sufficient to show that the differential $(d\phi)_a$ of ϕ at the point $a = (\mathrm{Id}, (x^+,x^-)) \in G \times B$ is surjective. The tangent space of $G \times B$ at a may be identified with $g \times B$, and then $(d\phi)_a$ is given by

(1) $$(d\phi)_a(\Delta,y) = \Delta(x) + y ,$$

where $\Delta = (\Delta_+,\Delta_-) \in g$ and $\Delta(x) = (\Delta_+(x^+),\Delta_-(x^-))$. Since $C \subset B$ it suffices to show that every $z = (z^+,z^-) \in V_{ij}$ for $i \neq j$ is of the form $\Delta(x)$ for some $\Delta \in g$. Let $w^\sigma = \Sigma\xi_\ell^\sigma e_\ell^\sigma$, and let $\Delta' = \delta(z^+,w^-) = (D(z^+,w^-), -D(w^-,z^+)) \in g$. Then we have by the rules for the Peirce decomposition

$$\Delta'_+(x^+) = \{z^+,w^-,x^+\} = (\xi_i^-\lambda_i^+ + \xi_j^-\lambda_j^+)z^+$$

and

$$\Delta'_-(x^-) = -\{w^-,z^+,x^-\} = -(\xi_i^-\lambda_j^- + \xi_j^-\lambda_i^-)\{e_i^-,z^+,e_j^-\} .$$

Choose now the ξ's in such a way that $\xi_\ell^- = 0$ for $\ell \neq i,j$ and

$$\xi_i^-\lambda_i^+ + \xi_j^-\lambda_j^+ = 1,$$

$$\xi_i^-\lambda_j^- + \xi_j^-\lambda_i^- = 0.$$

This is possible since the determinant of this system of linear equations is $\begin{vmatrix} \lambda_i^+ & \lambda_j^+ \\ \lambda_j^- & \lambda_i^- \end{vmatrix} = \lambda_i^+\lambda_i^- - \lambda_j^+\lambda_j^- = \lambda_i - \lambda_j \neq 0$. Then we have $\Delta'(x) = (z^+,0)$. In a similar way, one finds w^+ such that for $\Delta'' = \delta(w^+,z^-)$ we have $\Delta''(x) = (0,z^-)$. Then $\Delta = \Delta' + \Delta''$ is the desired element of $\mathcal{g}$.

(iii) $\rightarrow$ (iv): This follows from Borel[1], p. 38f.

(iv) $\rightarrow$ (v): The set V_{reg} of associator regular elements of V is dense and therefore intersects the dense orbit $G.\mathcal{B}$. Since V_{reg} is invariant under G it must intersect $\mathcal{B}$. Therefore $\mathcal{B}$ contains associator regular elements.

(ii) $\rightarrow$ (i): Let $x = (x^+,x^-)$ be as in the proof of (ii) $\rightarrow$ (iii), and let $z = \Sigma\, z_{ij} \in \Sigma V^\sigma_{ij}$. By 15.4 we have $A_\sigma(x,x).z = \Sigma\, (\lambda_i - \lambda_j)^2 z_{ij}$. Since $\mathcal{B}^\sigma \supset \Sigma\, V^\sigma_{ii}$ we see that $A_\sigma(x,x).z \in \mathcal{B}^\sigma$ implies $z \in \mathcal{B}^\sigma$, i.e., $A_\sigma(x,x)$ induces an injective map on $V^\sigma/\mathcal{B}^\sigma$. It follows that the set W of all elements in $\mathcal{B}$ with this property is open and dense in $\mathcal{B}$. By (ii) $\rightarrow$ (v) so is the set of associator regular elements of $\mathcal{B}$. Hence there exists $y \in W$ which is associator regular. Then $N(y)$ is a Cartan subpair contained in $\mathcal{B}$ (15.13).

Finally, we prove (v) $\rightarrow$ (i) in case V is semisimple. Let $x \in \mathcal{B}$ be associator regular, and let $C = N(x)$ be the Cartan subpair determined by x . Then $C = \Sigma\, V_{ii}$ with respect to a frame $(e_1,\ldots,e_r)$, and since V is semisimple we have $V_{00} = 0$ and $V^\sigma_{ii} = k.e^\sigma_i$. Also if $x^\sigma = \Sigma\, \lambda^\sigma_i e^\sigma_i$ then we must have $\lambda_i - \lambda_j \neq 0$ for $0 \leq i \leq j \leq r$ (where again $\lambda_i = \lambda_i^+\lambda_i^-$ and $\lambda_0 = 0$) since otherwise $N(x)$ would be bigger than C by 15.4. Replacing e_i^+ by $(\lambda_i^-)^{-1}e_i^+$ and e_i^- by $\lambda_i^- e_i^-$ we may assume that $x^- = \Sigma\, e_i^- \in B^-$ and $x^+ = \Sigma\, \lambda_i e_i^+$. Since all powers $(x^+)^{(n,x^-)}$ belong to B^+ a Vandermonde determinant argument shows that $e_i^+ \in B^+$, and hence $e_i^- = Q(e^-)e_i^+ \in B^-$. This implies $C \subset B$.

15.16. COROLLARY. <u>If</u> k <u>is infinite then every Cartan subpair</u> C <u>of</u> V <u>contains associator regular elements</u>.

<u>Proof</u>. By 15.15, $C_{\bar{k}}$ contains associator regular elements and hence the function c_{n-m} (cf. 15.12) does not vanish on $C_{\bar{k}}$. Since k is infinite it does not vanish on C.

15.17. THEOREM. <u>Let</u> k <u>be algebraically closed</u>. <u>Then any two Cartan subpairs of a Jordan pair</u> V <u>over</u> k <u>are conjugate by an inner automorphism of</u> V.

<u>Proof</u>. Let C and C' be Cartan subpairs, and let V_{reg} be the set of associator regular elements of V. Then $G.C \cap G.C' \cap V_{\mathrm{reg}}$ is not empty by 15.15, and since V_{reg} is invariant under G there exists an associator regular element x in C and $g \in G$ such that $g(x) \in C'$. By 15.13 we have $C = N(x)$ and $g(C) = N(g(x)) = C'$.

15.18. DEFINITION. Let k be algebraically closed, and let $(e_1,\dots,e_r)$ be a frame of V. Then $C = \Sigma\, V_{ii}$ is a Cartan subpair, and we have $r = \dim(C/\mathrm{Rad}\, C)$. Since any two Cartan subpairs are conjugate we see that any two frames of V must have the same number r of elements. The non-negative integer $r = \mathrm{rank}\, V$ is called the <u>rank</u> of V. If k is not algebraically closed we define $\mathrm{rank}\, V = \mathrm{rank}\, V_{\bar{k}}$. Clearly the rank of V is invariant under extension of the base field.

If I is an ideal contained in the radical of V then

$$\operatorname{rank} V = \operatorname{rank}(V/I) .$$

By 10.12 we have $\operatorname{rank} V = 0$ if and only if V is radical. From the definition it is immediate that $\operatorname{rank} V^{op} = \operatorname{rank} V$ and that the rank of a direct product of Jordan pairs is given by $\operatorname{rank}(V \times W) = \operatorname{rank} V + \operatorname{rank} W$.

15.19. LEMMA. <u>For any torus</u> T <u>of</u> V <u>we have</u> $\dim T \leq \operatorname{rank} V$. <u>If equality holds then</u> $\operatorname{Cent}(T)$ <u>is a Cartan subpair of</u> V .

<u>Proof</u>. We may assume that k is algebraically closed and, by passing to $V/\operatorname{Rad} V$, that V is semisimple. Then $T \cong (k^J, k^J)^s$ by 15.6; i.e., $T^\sigma = \Sigma\, k.e_i^\sigma$ where the e_i are orthogonal idempotents of V . Picking in each V_{ii} a division idempotent d_i we obtain an orthogonal system $(d_1,\ldots,d_s)$ of division idempotents, and hence $s = \dim T \leq r = \operatorname{rank} V$. If $s = r$ then the e_i are division idempotents, i.e., $V_{ii} = k.e_i$. Indeed, if (say) e_1 is not a division idempotent then V_{11} is at least of dimension 2 . Picking a frame $(c_1,\ldots,c_t)$ in V_{11} we have $t \geq 2$ since otherwise V_{11} would be one-dimensional. But then $(c_1,\ldots,c_t,d_2,\ldots,d_r)$ is an orthogonal system of more than r division idempotents which is impossible. Hence $\operatorname{Cent}(T)$ is a Cartan subpair by 15.9.

<u>Remark</u>. A maximal torus (i.e., a torus not properly contained in any other torus) need not be a torus of maximum dimension. For example, let V be the Jordan pair of 2×2 symmetric matrices over a field of characteristic 2 , and let T be the torus spanned by $e^{\pm} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then T is maximal but not of maximal dimension among the tori of V since $\operatorname{rank} V = 2$. This phenomenon is of course related to the fact that a primitive idempotent need not be a local idempotent (cf. 5.12), and occurs only in characteristic 2 .

15.20. THEOREM. Let V be a Jordan pair over an arbitrary field k. Then V contains Cartan subpairs. Also, V contains tori of dimension rank V, and the Cartan subpairs of V are precisely the centralizers of such tori.

Proof. If C is a Cartan subpair of V let $T \subset C$ be a torus as in 15.10. Then C is the centralizer of T, and $\dim T = \dim(C_{\bar{k}}/\mathrm{Rad}(C_{\bar{k}})) = r = \mathrm{rank}\, V$. Conversely, if T is a torus of dimension r then by 15.19, $\mathrm{Cent}(T)$ is a Cartan subpair. Thus we only have to prove the existence of Cartan subpairs. If k is infinite then V contains associator regular elements and hence Cartan subpairs (15.13). Assume therefore that k is finite, and let q be the number of elements of k. Then the Galois group Γ of $\bar{k}$ over k is generated (topologically) by the Frobenius map $f: \lambda \mapsto \lambda^q$. If X is an algebraic variety defined over k then Γ acts on X and we denote by $x \mapsto x^{(q)}$ the action of f on X. (If $X \subset \bar{k}^n$ then $x^{(q)}$ is simply given by raising each coordinate of x to the q-th power). In particular, consider the inner automorphism group $G = \mathrm{Inn}(\bar{V})$. By 15.14, G is connected and defined over k. Hence the map $g \mapsto g^{-1}g^{(q)}$ from G into itself is surjective (Borel[1], p. 369, 16.4). Let C be a Cartan subpair of $\bar{V}$. Since f acting on $\bar{V}$ is an automorphism of $\underline{\underline{Z}}(\bar{V})$ which is semilinear (i.e., $(\lambda x)^{(q)} = \lambda^q x^{(q)}$ for $\lambda \in \bar{k}$) it is clear that $C^{(q)}$ is a Cartan subpair of $\bar{V}$. By 15.17 there exists $g \in G$ such that $g(C^{(q)}) = C$, and $g = h^{-1}h^{(q)}$ for some $h \in G$. This implies $h(C) = h^{(q)}(C^{(q)}) = (h(C))^{(q)}$ and hence $h(C)$ is a Cartan subpair of $\bar{V}$ which is stable under Γ. By 15.1, $h(C)$ is defined over k, and therefore $h(C) \cap V$ is a Cartan subpair of V.

§16. The generic minimum polynomial

16.0. In this section, k is a field with algebraic closure $\bar{k}$, and k-alg denotes the category of commutative unital k-algebras. All Jordan pairs over k are finite-dimensional. The terminology of § 18 will be used freely. If $V = (V^+, V^-)$ is a Jordan pair over k then we denote by

$$A = \mathcal{O}(V) = \mathcal{O}(V^+ \times V^-)$$

the algebra of polynomial functions on $V^+ \times V^-$, and by

$$F = \mathcal{R}(V) = \mathcal{R}(V^+ \times V^-)$$

its quotient field, the field of rational functions on $V^+ \times V^-$. The generic point of V^+ (resp. V^-) is denoted by X (resp. Y), and the generic point of k is identified with the indeterminate T (cf. 18.4). Also, $\overline{V} = V_{\bar{k}}$ is the set of geometric points of V (18.10).

16.1. Let V be a Jordan pair over k . For every $R \in k$-alg and $(t,(x,y)) \in R \times V_R$ we set

$$\chi(t,x,y) = \det(t^2 \mathrm{Id}_{V^+} - tD(x,y) + Q(x)Q(y)) . \tag{1}$$

This defines a polynomial function $\chi = \chi(T,X,Y) \in \mathcal{O}(k \times V) = \mathcal{O}(V) \otimes k[T] = A[T]$, called the characteristic polynomial of V . If t is invertible in R then $t^2\mathrm{Id} - tD(x,y) + Q(x)Q(y) = t^2 B(t^{-1}x,y)$ and therefore

$$\chi(t,x,y) = t^{2n}.\det B(t^{-1}x,y) \tag{2}$$

where $n = \dim V^+$. Note that $\chi(T,X,Y)$ is homogeneous of degree $2n$ in (T,X) and (T,Y), and also monic in T .

16.2. DEFINITION. Consider the field $\mathcal{R}(k \times V)$ of rational functions on $k \times V$ which may be identified with $F(T)$. The element $(T^{-1}X,Y) \in V_{F(T)}$ is quasi-invertible since $\det B(T^{-1}X,Y) = T^{-2n}\chi(T,X,Y)$ is a non-zero element of $F(T)$. Hence $q(T,X,Y) = (T^{-1}X)^Y$ is an element of $V^+_{F(T)}$, i.e., a rational map from $k \times V$ into V^+ (cf. 18.7). Note that χ is a denominator of q since

$$(T^{-1}X)^Y = \chi(T,X,Y)^{-1}.\mathrm{adj}(T^2\mathrm{Id} - D(X,Y) + Q(X)Q(Y)).(TX - Q(X)Y)$$

where adj denotes the adjoint linear transformation. Now let

$$q(T,X,Y) = p(T,X,Y)/m(T,X,Y)$$

be a reduced expression. Since the exact denominator m divides χ it is homogeneous in (T,X) and (T,Y) and hence we can write

$$m(T,X,Y) = \sum_{i=0}^{h} (-1)^i m_i(X,Y)T^{h-i}$$

where $m_i = m_i(X,Y) \in A$ is homogeneous of bidegree (i,i) and $m_0 = 1$. Normalized in this way, $m = m(T,X,Y)$ is called the <u>generic minimum polynomial</u> of V. The bilinear form $m_1\colon V^+ \times V^- \to k$ is called the <u>generic trace</u> of V. The number h is called the <u>height</u> of V. The largest integer s such that $m_s \neq 0$ is called the <u>degree</u> of V. Clearly we have $0 \leq s \leq h$. The difference $h - s$ is called the <u>excess</u> of V. Thus V has excess 0 if and only if $m(0,X,Y) \neq 0$. By 18.9, m is invariant under base field extension, and therefore s and h are invariant under base field extension.

16.3. <u>Examples</u>. (i) Let V be nilpotent. Then

$$(1) \qquad (T^{-1}X)^Y = \sum_{i=1}^{h} T^{-i}X^{(i,Y)} = T^{-h}\sum_{i=1}^{h} T^{h-i}X^{(i,Y)}$$

where h is the largest integer such that $X^{(h,Y)} = 0$. Hence $m(T,X,Y) = T^h$ and V has degree zero and height h. Conversely, if V has degree zero then $m(T,X,Y) = T^h$, and hence $X^Y = p(1,X,Y)$ is a polynomial in X and Y . It follows that all $(x,y) \in V$ are quasi-invertible, and hence $V = \mathrm{Rad}\, V$ is nilpotent by 14.11.

(ii) Let $V = (J,J)$ where J is a unital finite-dimensional Jordan algebra over k . Let $N(X)$ be the generic norm of J , i.e., the exact denominator of the rational map X^{-1} , normalized by $N(1) = 1$ (see also Jacobson[4] and McCrimmon[9]). Since X^{-1} is homogeneous so is $N(X)$, and by the degree of J we mean the degree, say s , of $N(X)$. By 3.13, we have $(T^{-1}X)^Y = (TX^{-1} - Y)^{-1}$, and by 18.13, $m(T,X,Y)$ is the exact denominator of the rational map $f(T,Y) = (TX^{-1} - Y)^{-1}$ from $K \times J_K$ into J_K where $K = \mathcal{L}(J)$. On the other hand, it is easily seen that $N(TX^{-1} - Y)$ is an exact denominator of $f(T,Y)$. Hence the two differ only by a factor in K which is seen to be $N(X)$ by specializing $Y \to 1$. Thus we have

$$m(T,X,Y) = N(X)N(TX^{-1} - Y) . \tag{2}$$

It follows that the degrees of V and J coincide. Also, V has height s and therefore excess zero.

16.4. Let V be a Jordan pair over k , and let $(x,y) \in V$. Then $q(T,x,y) = (T^{-1}x)^y$ is a rational map from k into V^+ whose exact denominator $\mu_{x,y}(T) \in k[T]$, normalized such that the highest power of T has coefficient one, is called the minimum polynomial of (x,y) . Clearly $\mu_{x,y}(T)$ divides $m(T,x,y)$, and by 18.13 there exists a dense open subset W of $\overline{V}$ such that $\mu_{x,y}(T) = m(T,x,y)$ for all $(x,y) \in W$. The minimum polynomial is invariant under extension of the base field. Also, it follows from 18.13 that the minimum polynomial of the generic point $(X,Y) \in V_F$ is just the generic minimum polynomial of V , i.e., $\mu_{X,Y}(T) = m(T,X,Y)$.

16.5. PROPOSITION. (a) <u>The minimum polynomial</u> $\mu_{x,y}(T)$ <u>of</u> $(x,y) \in V$ <u>is uniquely determined (up to a scalar factor) as the polynomial of smallest degree among all polynomials</u> $f(T) = \sum_{i=0}^{c} \lambda_i T^{c-i} \in k[T]$ <u>such that</u>

$$(1) \qquad \sum_{i=0}^{c} \lambda_i x^{(n-i,y)} = 0 \text{ for all } n > c \,.$$

(b) <u>If</u> $\mu_{x,y}(T) = \sum_{i=0}^{d} \mu_i T^{d-i}$ <u>then</u> $(T^{-1}x)^y = \nu_{x,y}(T)/\mu_{x,y}(T)$ <u>where</u>

$$(2) \qquad \nu_{x,y}(T) = \sum_{i=0}^{d-1} \left(\sum_{j=0}^{i} \mu_j x^{(i+1-j,y)} \right) T^{d-i-1} \,.$$

<u>Proof</u>. Write $\mu(T)$ instead of $\mu_{x,y}(T)$, and let

$$\nu(T) = \sum_{i=0}^{b} a_i T^{b-i}$$

be the exact numerator of $(T^{-1}x)^y$, where $a_i \in V^+$ and $a_0 \neq 0$. In $V \otimes k(T)$ we have $(Tx)^y = \nu(T^{-1})/\mu(T^{-1}) = T^{d-b}\nu^*(T)/\mu^*(T)$ where

$$\mu^*(T) = \sum_{i=0}^{d} \mu_i T^i \,, \qquad \nu^*(T) = \sum_{i=0}^{b} a_i T^i \,.$$

Applying $B(Tx,y)$ to this we get

$$\mu^*(T)(Tx - T^2 Q(x)y) = T^{d-b}.B(Tx,y).\nu^*(T)$$

and comparing coefficients at powers of T we see that $d - b = 1$ and $a_0 = x$. Hence it follows that

$$(3) \qquad \mu^*(T)(Tx)^y = T.\nu^*(T) \,.$$

Now let $R = k(\varepsilon)$ with $\varepsilon^r = 0$. Then $(\varepsilon x,y)$ is quasi-invertible in V_R with quasi-inverse $(\varepsilon x)^y = \sum_{i\geq 1} \varepsilon^i x^{(i,y)}$ (cf. 3.8) and it follows from (3) that $\mu^*(\varepsilon)(\varepsilon x)^y = \varepsilon\nu^*(\varepsilon)$, i.e., we have

$$\left(\sum_{i=0}^{d}\mu_i\varepsilon^i\right)\left(\sum_{i=1}^{r}\varepsilon^i x^{(i,y)}\right) = \left(\sum_{i=0}^{d-1}a_i\varepsilon^{i+1}\right).$$

We compare coefficients at powers of ε and obtain, since r can be chosen to be arbitrarily large, that

$$(4)\qquad a_i = \sum_{j=0}^{i}\mu_j x^{(i+1-j,y)} \quad \text{for } i = 0,1,\dots,d-1,$$

$$(5)\qquad 0 = \sum_{j=0}^{d}\mu_j x^{(n-j,y)} \quad \text{for all } n > d.$$

Now (2) follows from (4), and by (5), $\mu(T)$ satisfies (1). Conversely, let $f(T)$ be a polynomial satisfying (1), let S be another indeterminate, and let $h(T,S) \in k[T,S]$ be the unique polynomial such that

$$f(T) - f(S) = (T-S)h(T,S).$$

Then we have

$$(6)\qquad (TS-S^2)(f(T)-f(S)) = (T-S)^2.S.h(T,S).$$

Consider the Jordan homomorphism from $S.k[T,S]$ into $V_y^+\otimes k[T]$ such that $S^i \mapsto x^{(i,y)}$ for $i \geqq 1$. Applying this homomorphism to (6) and using (1) we get

$$(Tx - x^{(2,y)})f(T) = (T^2\mathrm{Id} - TD(x,y) + Q(x)Q(y))g(T)$$

where $g(T)$ is the image of $S.h(T,S)$. This means that

$$(T^{-1}x)^y = (T^2\mathrm{Id} - TD(x,y) + Q_xQ_y)^{-1}(Tx-Q_xy) = g(T)/f(T)$$

in $V\otimes k(T)$ and shows that $f(T)$ is a denominator of $(T^{-1}x)^y$.

16.6. COROLLARY. (a) <u>The generic minimum polynomial</u> $m(T,X,Y)$ <u>of</u> V <u>is uniquely determined as the polynomial of smallest degree among all monic polynomials</u> $f(T)$

$= \sum_{i=0}^{c} f_i T^{c-i} \in F[T]$ <u>such that</u> $\sum_{i=0}^{c} f_i X^{(n-i,Y)} = 0$ <u>for all</u> $n > c$.

(b) <u>The exact numerator of</u> $(T^{-1}X)^Y$ is

$$p(T,X,Y) = \sum_{i=0}^{h-1} \left(\sum_{j=0}^{i} (-1)^j m_j(X,Y) X^{(i+1-j,Y)} \right) T^{h-i-1} .$$

This follows from 16.5 applied to the element $(X,Y) \in V_F$ since $m(T,X,Y) = \mu_{X,Y}(T)$ (cf. 16.4).

16.7. PROPOSITION. <u>The generic minimum polynomial is invariant under the automorphism group of</u> V ; <u>more precisely</u>: <u>If</u> $R \in$ k-<u>alg</u> <u>and</u> $g = (g_+, g_-)$ <u>is an automorphism of</u> V_R <u>then</u> $m(T, g_+X, g_-Y) = m(T,X,Y)$.

<u>Proof</u>. Consider the equations

(1) $$\sum_{i=0}^{h} (-1)^i m_i(X,Y) X^{(n-i,Y)} = 0 \quad \text{for} \quad n > h .$$

Since $g \in \mathrm{Aut}(V_R)$ extends to an automorphism of $V_R \otimes A = V \otimes A \otimes R$ we can apply g to (1) and obtain

(2) $$\sum_{i=0}^{h} (-1)^i m_i(X,Y) (g_+X)^{(n-i,g_-Y)} = 0 \quad \text{for} \quad n > h .$$

Replace (X,Y) by (g_+X, g_-Y) in (1), subtract (2) from (1) and then replace (X,Y) by $(g_+^{-1}X, g_-^{-1}Y)$. This yields

(3) $$\sum_{i=0}^{h-1} (-1)^{i+1} b_{i+1} X^{(n-i,Y)} = 0 , \quad \text{for} \quad n > h-1$$

where $b_i = m_i(X,Y) - m_i(g_+X, g_-Y) \in A \otimes R$. Here we used the fact that $m_0 = 1$. Now let α be a linear form on R with values in k . Then α induces A-linear maps $A \otimes R \to A$ and $V \otimes A \otimes R \to V \otimes A$ in the obvious way. Applying α to (3) we obtain $\sum_{i=0}^{h-1}(-1)^i\alpha(b_{i+1})X^{(n-i,Y)} = 0$, for $n > h-1$. This implies $\alpha(b_i) = 0$ by 16.6(a). Since R is free over k there exist sufficiently many linear forms on R and it follows that $b_i = 0$.

16.8. COROLLARY. The coefficients m_i of the generic minimum polynomial are Lie invariant under $\mathrm{Der}(V)$; i.e., if $\Delta = (\Delta_+,\Delta_-)$ is a derivation of V then

(1) $$dm_i(X,Y)(\Delta_+X,\Delta_-Y) = 0 .$$

In particular, the generic trace m_1 satisfies

(2) $$m_1(\{uvx\},y) = m_1(x,\{vuy\}) .$$

Proof. We have $\mathrm{Id} + \varepsilon\Delta \in \mathrm{Aut}(V_{k(\varepsilon)})$ where $k(\varepsilon)$ is the algebra of dual numbers over k . By 16.7, we get $m_i(X,Y) = m_i(X+\varepsilon\Delta_+X, Y+\varepsilon\Delta_-Y) = m_i(X,Y) + \varepsilon dm_i(X,Y)(\Delta_+X, \Delta_-Y)$ (cf. the definition of the derivative in 18.6). This proves (1). Now (2) is the special case where $\Delta = \delta(u,v) = (D(u,v),-D(v,u))$ is an inner derivation.

16.9. DEFINITION. The generic norm of a Jordan pair V over k is defined by

(1) $$N(X,Y) = m(1,X,Y) = \sum_{i=0}^{s}(-1)^i m_i(X,Y) .$$

Thus $N = N(X,Y) \in A$, and the degree of V is the degree of N in X (or Y). If we set $P(X,Y) = p(1,X,Y)$ then by 16.2,

(2) $$X^Y = P(X,Y)/N(X,Y) .$$

We claim that (2) is a reduced expression. Indeed, if P and N had a non-constant factor in common then so would p and m since

(3) $$m(T,X,Y) = T^hN(T^{-1}X,Y) \quad\text{and}\quad p(T,X,Y) = T^hP(T^{-1}X,Y) .$$

By 16.7, the generic norm is invariant under the automorphism group of V. Let $R \in k$-alg and $(x,y) \in V_R$. Then (x,y) is quasi-invertible if and only if $N(x,y)$ is invertible in R, and then

(4) $$x^y = P(x,y)N(x,y)^{-1} .$$

Indeed, if (x,y) is quasi-invertible then $\det B(x,y) = \chi(1,x,y)$ is invertible and since m divides χ it follows that $N(x,y) = m(1,x,y)$ is invertible. If conversely $N(x,y)$ is invertible in R then there exists a unique homomorphism $A[N^{-1}] \to R$ extending the homomorphism $\widehat{(x,y)}: A \to R$ (cf. 18.1). The induced homomorphism $V \otimes A[N^{-1}] \to V \otimes R$ of Jordan pairs, applied to (2), yields (4).

16.10. PROPOSITION. *Let N be the generic norm of V, and let $\deg V$ denote the degree of V.*

(a) *The generic norm of V^{op} is given by*

(1) $$N^{op}(Y,X) = N(X,Y) ,$$

and hence we have $\deg V = \deg V^{op}$.

(b) *If $V = V_1 \times V_2$ is a direct product of Jordan pairs V_i with generic norms $N_i(X_i,Y_i)$ then the generic norm of V is given by*

(2) $$N(X,Y) = N_1(X_1,Y_1).N_2(X_2,Y_2) ,$$

and hence $\deg(V_1 \times V_2) = \deg V_1 + \deg V_2$.

(c) *If V and V^{op} (resp. V_1 and V_2) have excess zero then (1) (resp. (2)) holds for the generic minimum polynomial instead of the generic norm as well.*

Proof. By 3.3 we have $X^Y = X + Q(X)Y^X$ which shows that N^{op} is a denominator of X^Y. If we interchange V and V^{op} we see that N is a denominator of Y^X. This implies (a) since N is the exact denominator of X^Y by 16.9. Part (b) is obvious, and (c) follows from $m(T,X,Y) = T^{\deg V}.N(T^{-1}X,Y)$ (cf. 16.9.3) in case V has excess zero.

Remark. If V doesn't have excess zero then (c) becomes false (e.g., let $V = (k,0)$ with trivial multiplication). Also the height of V and of V^{op} will in general not be the same, although it can be shown that they differ at most by 1 .

16.11. THEOREM. Let $R \in k$-alg , and let $(x,y) \in V_R$ be quasi-invertible. Then

(1) $$N(x,y)N(x^y,z) = N(x,y+z),$$

(2) $$N(x,y)N(w,y^x) = N(x+w,y),$$

for all $(w,z) \in V_R$. (Compare with JP33 and JP34).

Proof. It suffices to prove (1) since (2) will follow by passing to V^{op} and using 16.10(a). By 18.3 there exists an open dense subset U of $\overline{V}^+$ such that $P(x,Y)/N(x,Y)$ is a reduced expression, for all $x \in U$. Let $W \subset \overline{V}$ be the set of quasi-invertible elements. Then W is open and dense, and the map $W \to \overline{V}^+$ given by the quasi-inverse is surjective since $x^0 = x$. Hence the inverse image

$$W' = \{(x,y) \in W \mid x^y \in U\}$$

is open and dense in $\overline{V}$, and for all $(x,y) \in W'$, the exact denominator of the rational map $(x^y)^Y$ is $N(x^y,Y)$. On the other hand, by 3.7,

$$(x^y)^Y = x^{y+Y} = P(x,y+Y)/N(x,y+Y)$$

and therefore $N(x,y+Y)$ is also a denominator. It follows that $N(x,y+z) = N(x^y,z).f(x,y)$ for all $z \in \overline{V^-}$ where $f(x,y) \in \bar{k}$, and for $z = 0$ we see that $f(x,y) = N(x,y)$. Thus (1) folds for all (x,y,z) in the open and dense subset $W'' = W' \times \overline{V^-}$ of $\overline{V} \times \overline{V^-}$. Every function $\phi \in \mathcal{O}(V \times V^-)[N^{-1}]$ defines a function $\bar{\phi}$ on W'' , and $\bar{\phi} = 0$ implies $\phi = 0$ since W'' is open and dense. In particular, $\phi = N(X,Y).N(X^Y,Z) - N(X,Y+Z)$ vanishes on W'' and hence

$$N(X,Y).N(X^Y,Z) = N(X,Y+Z)\ . \tag{3}$$

(Here (X,Y,Z) is the generic point of $V^+ \times V^- \times V^-$). Now if $R \in$ k-alg and $(x,y,z) \in V_R \times V_R^-$ is such that $N(x,y)$ is invertible in R then we have a homomorphism $\mathcal{O}(V \times V^-)[N(X,Y)^{-1}] \to R$, induced by $(X,Y,Z) \mapsto (x,y,z)$. Applying this homomorphism to (3) we get (1).

16.12. COROLLARY. *The generic trace* m_1 *satisfies the identities*

$$m_1(Q(x)y,z) = m_1(Q(x)z,y), \tag{1}$$

$$m_1(x,Q(y)w) = m_1(w,Q(y)x), \tag{2}$$

for all $(x,y),\ (w,z) \in V$.

Proof. Let $R = k(\varepsilon,\delta)$ be the commutative k-algebra with generators ε and δ and relations $\varepsilon^2 = \delta^2 = 0$. Then $(x,\varepsilon y)$ is quasi-invertible in V_R and we have $x^{\varepsilon y} = x + \varepsilon x^{(2,y)} = x + \varepsilon Q(x)y$. In the same way, $x^{\delta z} = x + \delta Q(x)z$. Now $N(x,\varepsilon y)N(x^{\varepsilon y},\delta z) = N(x,\varepsilon y+\delta z) = N(x,\delta z)N(x^{\delta z},\varepsilon y)$, and if we expand this we get

$$(1-\varepsilon m_1(x,y))(1-\delta m_1(x+\varepsilon Q_x y,z)) = (1-\delta m_1(x,z))(1-\varepsilon m_1(x+\delta Q_x z,y))$$

Comparing the terms at $\varepsilon\delta$ gives (1), and (2) follows by passing to V^{op}.

16.13. COROLLARY. (a) *An element* $y \in V^-$ *belongs to* Rad V^- *if and only if* $N(x,y) = 1$ *for all* $x \in V^+$. *Analogously,* $x \in \text{Rad } V^+$ *if and only if* $N(x,y) = 1$ *for all* $y \in V^-$.

(b) *Let* k *be infinite, and let* $y,z \in V^-$. *Then* $z-y \in \text{Rad } V^-$ *if and only if* $N(x,y) = N(x,z)$ *for all* $x \in V^+$. *In the same way, if* x *and* w *are in* V^+

<u>then</u> $x-w$ <u>is in</u> Rad V^+ <u>if and only if</u> $N(x,y) = N(w,y)$ <u>for all</u> $y \in V^-$.

<u>Proof</u>. (a) If $N(x,y) = 1$ for all x then (x,y) is quasi-invertible for all x by 16.9, and hence $y \in \text{Rad } V^-$. Conversely, if $y \in \text{Rad } V^-$ then also $y \in \text{Rad } \overline{V^-}$ by 15.2. Hence $N(x,y)$ is invertible for all $x \in \overline{V^+}$ and this implies that $N(x,y)$ must be a constant, independent of x . For $x = 0$ the constant turns out to be one. The second statement follows by passing to V^{op} .

(b) Let $U = \{x \in \overline{V^+} \mid (x,y) \text{ is quasi-invertible}\}$ and $U' = \{x^y \mid x \in U\}$. Then both U and U' are open and dense in $\overline{V^+}$ (this follows from the fact that $(u^{-y})^y = u$ and therefore $U' = \{u \in \overline{V^+} \mid (u,-y) \text{ quasi-invertible}\}$). If $z-y \in \text{Rad } V^-$ then by (a) and 16.11 we have $N(x,y) = N(x,y)N(x^y,z-y) = N(x,y+z-y) = N(x,z)$ for all $x \in U$, and by density this holds for all x . (Note that so far we haven't used that k is infinite). For the converse, assume that $N(x,y) = N(x,z)$ for all $x \in V^+$. Since k is infinite this still holds for all $x \in \overline{V^+}$. This implies $N(x^y,z-y) = N(x,z)N(x,y)^{-1} = 1$ for all $x \in U$, i.e., $N(u,z-y) = 1$ for all $u \in U'$. By density of U' we have this for all x , and therefore $z-y \in \text{Rad } V^-$ by what we proved under (a). Again, the second statement follows by passing to V^{op} .

16.14. PROPOSITION. <u>Let</u> k <u>be infinite and let</u> V <u>be a semisimple Jordan pair over</u> k . <u>Then</u> $g = (g_+,g_-) \in GL(V^+) \times GL(V^-)$ <u>is an automorphism of</u> V <u>if and only if</u>

(1) $$N(g_+x,g_-y) = N(x,y)$$

for all $(x,y) \in V$.

<u>Proof</u>. That (1) is necessary follows from 16.7. Assume that (1) holds, and let $(x,y) \in V$ be quasi-invertible. By 16.9 and (1) so is (g_+x,g_-y) . Moreover, by 16.11, $N(g_+x,g_-y)N(g_+(x^y),g_-z) = N(x,y)N(x^y,z) = N(x,y+z) = N(g_+x,g_-y + g_-z) =$

$N(g_+x,g_-y)N(g_+(x)^{g_-(y)},g_-(z))$ and therefore $N(g_+(x^y),g_-(z)) = N(g_+(x)^{g_-(y)},g_-z)$ for all $z \in V^-$. By 16.13(b) this implies $g_+(x^y) = g_+(x)^{g_-(y)}$ since V is semisimple, and by 16.9.3 we get $g_+P(x,y) = P(g_+x,g_-y)$. If we compare homogeneous components in (1) (which we are allowed to do since k is infinite) we get $m_1(g_+x,g_-y) = m_1(x,y)$. The homogeneous component of bidegree (2,1) of $P(x,y)$ is $x^{(2,y)} - x.m_1(x,y) = Q(x)y - x.m_1(x,y)$. Hence we have

$$g_+Q(x)y = Q(g_+x)g_-y$$

for all quasi-invertible (x,y). By density and since k is infinite this holds for all (x,y). Similarly one proves that $g_-Q(y)x = Q(g_-y)g_+x$, and therefore g is an automorphism.

16.15. THEOREM. (a) *Let C be a Cartan subpair of V. Then the generic minimum polynomial m_C of C is the restriction of the generic minimum polynomial m of V to C; i.e., for all $(x,y) \in C_R$, $R \in k$-alg, we have*

(1) $$m_C(T,x,y) = m(T,x,y) .$$

(b) *Let k be algebraically closed, and let $C = \sum_{i=0}^{r} V_{ii}$ with respect to a frame $(e_1,\ldots,e_r)$ of V as in 15.9(b). For $i = 1,\ldots,r$ let s_i be the degree of V_{ii}, let $s = s_1 + \ldots + s_r$, and let $e = \max(0,h_0-s)$ where h_0 is the height of V_{00}. Define linear forms $\lambda_i^\sigma\colon C^\sigma \to k$ by $z \equiv \lambda_i^\sigma(z).e_i^\sigma \bmod \operatorname{Rad} V_{ii}^\sigma$. Then s is the degree and e is the excess of V, and for all $(x,y) \in C$ we have the formulas*

(2) $$m(T,x,y) = T^e.\prod_{i=1}^{r}(T - \lambda_i^+(x)\lambda_i^-(y))^{s_i},$$

(3) $$N(x,y) = \prod_{i=1}^{r}(1 - \lambda_i^+(x)\lambda_i^-(y))^{s_i} .$$

Proof. (a) We may assume k to be algebraically closed, and then it suffices to prove (1) for $R = k$. Let G be the inner automorphism group of V, and let $W \subset V$ be an open and dense subset such that $m(T,x,y) = \mu_{x,y}(T)$ for all $(x,y) \in W$ (cf. 16.4). Replacing W by $G.W$ if necessary we may assume that W is stable under G. By 15.15, $G.C$ contains an open dense subset of V. Hence $G.C \cap W = G.(C \cap W)$ is not empty, and $C \cap W$ is an open and dense subset of C. For $(x,y) \in C \cap W$ we have $m_C(T,x,y) = \mu_{x,y}(T) = m(T,x,y)$, and since $C \cap W$ is dense in C, (1) holds for all $(x,y) \in C$.

(b) For $i = 1,\dots,r$ let J be the unital Jordan algebra $(V^+_{ii})_{e^-_i}$ (whose unit element is $1 = e^+_i$) so that $V_{ii} = (J,J)$ (cf. 1.10). Since e_i is a local idempotent we have $J = k.1 \oplus N$ where $N = \operatorname{Rad} V^+_{ii}$ is a nilpotent ideal. Let τ be the unique linear form on J vanishing on N and such that $\tau(1) = 1$ (in fact, $\tau = \lambda^+_i$). Also let s_i be the index of nilpotency of J, i.e., the smallest integer such that $n^{s_i+t} = 0$ for all $t \in \underline{\underline{N}}$ and all $n \in N$. Then it is easily seen that the generic norm of J is given by $N(x) = \tau(x)^{s_i}$. Also we have $\tau(x^{-1}) = \tau(x)^{-1}$ and hence it follows from 16.3(ii) that the generic minimum polynomial m_i of $(J,J) \cong V_{ii}$ is given by $m_i(T,x,y) = (T-\tau(x)\tau(y))^{s_i}$. Under the isomorphism $(J,J) \cong V_{ii}$ given by $(x,y) \mapsto (x,Q(e^-_i)y)$ (cf. 1.11) this implies that V_{ii} has generic minimum polynomial $m_i(T,x,y) = (T-\lambda^+_i(x)\lambda^-_i(y))^{s_i}$. By 16.3 (ii), V_{ii} has excess zero, and hence 16.10 implies that the generic minimum polynomial of $C' = V_{11} \oplus \dots \oplus V_{rr}$ is given by

$$m'(T,x,y) = \prod_{i=1}^{r} (T - \lambda^+_i(x)\lambda^-_i(y))^{s_i},$$

and its degree is $s = s_1 + \dots + s_r$. Next consider V_{00} which is contained in $\operatorname{Rad} V$ and is therefore nilpotent. By 16.3(i), it has degree zero, and its generic minimum polynomial is T^{h_0}. Since $m'(T,x,y) = T^s$ if $(x,y) \in V_{00}$ we get (2), and for $T \to 1$ we have (3).

16.16. COROLLARY. Let V be separable and not zero.

(a) The rank, degree, and height of V all coincide, and the excess of V is zero.

(b) The discriminant of $m(T,X,Y)$ (considered as a polynomial in T with coefficients in A) is non-zero (V is "generically unramified").

(c) The generic trace of V is not zero. If char $k \neq 2$ then the generic trace is a non-degenerate bilinear form on $V^+ \times V^-$.

Proof. We may assume k to be algebraically closed. Then we have (with the notations of 16.15) $V^\sigma_{ii} = k.e^\sigma_i$ for $i = 1,\dots,r$, and $V_{00} = 0$. Hence $h_0 = 0$ and $s_i = 1$ which proves $r = \operatorname{rank} V = s = \deg V$, and $e = \max(0,-r) = 0$. Formula (2) of 16.15 now reads

$$m(T,x,y) = \prod_{i=1}^{r} (T - \lambda_i^+(x)\lambda_i^-(y)) \ . \tag{1}$$

Let $d = d(X,Y) \in A$ be the discriminant of $m(T,X,Y) \in A[T]$. If we choose $(x,y) \in C$ such that $\lambda_i^+(x)\lambda_i^-(y) \neq \lambda_j^+(x)\lambda_j^-(y)$ for $i \neq j$ then it follows that the discriminant of $m(T,x,y)$, which is $d(x,y)$, is non-zero. Hence d is non-zero as an element of A . From (1) it follows that the generic trace is given by

$$m_1(x,y) = \sum_{i=1}^{r} \lambda_i^+(x)\lambda_i^-(y) \tag{2}$$

for $(x,y) \in C$ which shows that $m_1 \neq 0$. To prove the last statement, decompose V into simple factors (10.14). By 16.10, the generic trace of V is just the direct sum of the generic traces of the simple factors. Therefore we may assume V to be simple. Let $K^+ = \{x \in V^+ \mid m_1(x,V^-) = 0\}$ and $K^- = \{y \in V^- \mid m_1(V^+,y) = 0\}$. Then it follows from 16.8.2 and 16.12 that $K = (K^+,K^-)$ is an outer ideal of V . If char $k \neq 2$ then K is an ideal of V by 1.3, and since $m_1 \neq 0$ we have $K = 0$ and m_1 is non-degenerate.

16.17. <u>Remark.</u> If char $k = 2$ then m_1 may well be degenerate; for example, if V is the Jordan pair of symmetric matrices over k. On the other hand, the generic norm is always non-degenerate in the sense of 16.13. Consequently, if V has rank one then m_1 is non-degenerate regardless of characteristic, since then $N(x,y) = 1 - m_1(x,y)$. Also, if V is outer simple in the sense that it has no proper outer ideals then m_1 is non-degenerate.

16.18. <u>Alternative pairs.</u> Let $A = (A^+,A^-)$ be a finite-dimensional alternative pair over k. The generic minimum polynomial, norm, and trace of A are by definition the generic minimum plynomial, norm ,and trace of the associated Jordan pair A^J. Let $(u,v) \in A$. Recall that $(L(u,v),-L(v,u))$ and $(R(u,v),-R(v,u))$ are derivations of A^J (cf. 7.5). By 16.8 it follows that

(1) $$m_1(\langle uvx\rangle,y) = m_1(x,\langle vuy\rangle),$$

(2) $$m_1(\langle xvu\rangle,y) = m_1(x,\langle yuv\rangle),$$

for all $(x,y) \in A$. This has the following consequence.

16.19. PROPOSITION. <u>A finite-dimensional alternative pair</u> A <u>is separable if and only if the generic trace of</u> A <u>is non-degenerate.</u>

<u>Proof.</u> We may assume that k is algebraically closed. Let m_1 be non-degenerate. If $y \in \operatorname{Rad} A^-$ then $N(x,y) = 1$ for all $x \in A^+$, by 16.13. If we replace x by tx where $t \in k$ and compare coefficients at powers of t (which we can do since k is infinite) we get $m_1(x,y) = 0$ for all $x \in A^+$ and hence $y = 0$. Similarly one proves $\operatorname{Rad} A^+ = 0$. For the converse, it suffices (by 10.14) to show that m_1 is non-degenerate for A simple. Let $K^+ = \{x \in A^+ \mid m_1(x,A^-) = 0\}$ and $K^- = \{y \in A^- \mid m_1(A^+,y) = 0\}$. Then it follows from (1) and (2) of 16.18 that

K is an ideal of A which is proper by 16.16(c). Hence $K = 0$ and m_1 is non-degenerate.

16.20. COROLLARY. For a separable alternative pair A (resp. Jordan pair V) we have $\dim A^+ = \dim A^-$ (resp. $\dim V^+ = \dim V^-$).

Proof. For alternative pairs this is clear by 16.19. In the Jordan case, choose a maximal idempotent e so that $V = V_2(e) \oplus V_1(e)$. Then $\dim V_2^+(e) = \dim V_2^-(e)$ since $Q(e^+): V_2^-(e) \to V_2^+(e)$ is in particular a vector space isomorphism. Also, $V_1(e)$ is a separable alternative pair and hence $\dim V_1^+(e) = \dim V_1^-(e)$.

§17. Simple Jordan pairs

17.0. In this section, k is an algebraically closed field, and V denotes a simple finite-dimensional Jordan pair over k.

17.1. THEOREM. Let $(c_1,\dots,c_r)$ and $(e_1,\dots,e_r)$ be frames of V. Then there exists an inner automorphism g of V such that $g(c_i) = e_i$, $i = 1,\dots,r$.

Proof. Let S and T be the tori spanned by $(c_1,\dots,c_r)$ and $(e_1,\dots,e_r)$. Since V is semisimple, S and T are Cartan subpairs of V. By 15.17, there exists an inner automorphism h such that $h(S) = T$. The $(k.e_i^+, k.e_i^-)$ are the simple ideals of T, and decomposition into simple ideals is unique up to order.

Hence $h(c_i^\sigma) = t_i^\sigma e_{\pi(i)}^\sigma$ where π is a permutation of $\{1,\dots,r\}$ and the t_i^σ are non-zero elements of k. Since h maps idempotents into idempotents we have $t_i^- = (t_i^+)^{-1}$. It follows that $\theta(t_1,\dots,t_r)h(c_i) = e_{\pi(i)}$ where $\theta(t_1,\dots t_r)$ is the inner automorphism defined as in 5.13. Therefore it suffices to show that there exists an inner automorphism f such that $f(e_i) = e_{\pi(i)}$, $i = 1,\dots,r$. Let $e = e_1 + \dots + e_r$. Then $V = V_2(e) \oplus V_1(e)$ and $V_2 = V_2(e)$ is a simple Jordan pair (10.14). Let J be the unital Jordan algebra $(V_2^+)_{e^-}$, with unit element $e^+ = 1$. Then $V_2 = (J,J)$ and J is simple (1.6). Also, $(e_1^+,\dots,e_r^+)$ is an orthogonal system of idempotents of J, and the Peirce spaces of J are $J_{ij} = V_{ij}^+$ $(1 \leq i \leq j \leq r)$. Hence $J_{ij} \neq 0$ for $i \neq j$, and there exists $y \in J_{ij}$ such that $y^2 = e_i + e_j$ (cf. Jacobson[3], p. 3.25 and 3.61). (In other words, the e_i^+ are strongly connected idempotents of J). Let $x = e_i^+ + e_j^+ - y$. Then $x^2 = (e_i^+ + e_j^+)^2 - (e_i^+ + e_j^+)\circ y + y^2 = e_i^+ + e_j^+ - 2y + e_i^+ + e_j^+ = 2x$ by the rules for the Peirce decomposition. This means

$$Q(x)e^- = 2x, \tag{1}$$

by definition of the squaring in J. Now it follows from JP25 that $B(x,e^-)^2 = \mathrm{Id}$ and $B(e^-,x)^2 = \mathrm{Id}$, and hence $\beta(x,e^-) = (B(x,e^-),B(e^-,x)^{-1})$ is an inner automorphism of period two of V. Furthermore, we have $U_y.e_i^+ = Q(y)Q(e^-).e_i^+ = Q(y)e_i^-$ $\in V_{jj}^+ = k.e_j^+$ and similarly $U_y.e_j^+ \in k.e_i^+$. Since $e_i^+ + e_j^+ = y^2 = U_y.e^+ = U_y(e_i^+ + e_j^+) = Q(y)(e_i^- + e_j^-)$ we get $Q(y)e_i^- = e_j^+$ and $Q(y)e_j^- = e_i^+$. An easy computation shows that $\beta(x,e^-)$ interchanges e_i and e_j and leaves all the other idempotents fixed. Since the symmetric group is generated by transpositions the assertion follows.

17.2. <u>Numerical invariants</u>. Let $(e_1,\dots,e_r)$ be a frame of V. By 17.1, the Peirce spaces V_{ij}^+ $(1 \leq i < j \leq r)$ all have the same dimension, and we set

(1) $$a = \dim V_{ij}^{+} \quad (1 \leqq i < j \leqq r).$$

If $r = 1$ we set $a = 0$. The same holds for the Peirce spaces V_{i0}^{+} $(1 \leqq i \leqq r)$, and their common dimension will be denoted by

(2) $$b = \dim V_{i0}^{+} \quad (1 \leqq i \leqq r).$$

By 17.1, the numbers a and b depend only on V and not on the choice of the frame $(e_1,\ldots,e_r)$. Also, a and b are the same for V and V^{op} . Indeed, $Q(e^+): V_{ij}^- \to V_{ij}^+$ is a linear isomorphism (where we set $e = e_1 + \ldots + e_r$) for $1 \leqq i \leqq j \leqq r$, and $br = \dim V_1^+(e) = \dim V_1^-(e)$ by 16.20, since $V_1(e)$ is a semisimple (even simple) alternative pair.

The following relations follow easily from the definitions.

(3) $$d_2 = \dim V_2(e) = r + \frac{r(r-1)}{2} a ,$$

(4) $$d_1 = \dim V_1(e) = r.b ,$$

(5) $$d = \dim V = d_2 + d_1 .$$

We define the <u>genus</u> g of V by

(6) $$g = 2 + a(r-1) + b .$$

The following relations are easily verified.

(7) $$gr = 2d_2 + d_1 ,$$

(8) $$d_2 = gr - d ,$$

(9) $$d_1 = 2d - gr .$$

Hence V contains invertible elements (i.e., $V_1(e) = 0$) if and only if

(10) $$2d = gr .$$

17.3. THEOREM. The generic norm N is irreducible, and we have

(1) $$\det B(X,Y) = N(X,Y)^g,$$

(2) $$\chi(T,X,Y) = T^{d_1}.m(T,X,Y)^g,$$

where m (resp. χ) is the generic minimum polynomial (resp. the characteristic polynomial) (cf. 16.1, 16.2), and d_1 and g are as in 17.2.

Proof. Let $f_1(X,Y),\dots,f_n(X,Y)$ be the different irreducible factors of $N(X,Y)$. Since N is invariant under the automorphism group of V we have

$$f_i(h_+(X),h_-(Y)) = \rho_i(h)f_{\pi(h)(i)}(X,Y)$$

for all h in the inner automorphism group G of V, where π is a homomorphism from G into the symmetric group of n letters, and ρ_i is a character of G. The kernel of π is a closed normal subgroup of finite index in G, and since G is a connected algebraic group (cf. 15.14) it is all of G. Thus each f_i is a semi-invariant of G. Now let $(e_1,\dots,e_r)$ be a frame of V, and let T be the torus (= Cartan subpair since V is semisimple) spanned by $(e_1,\dots,e_r)$. Then the restriction of f_1 to T is a factor of the restriction of N to T, and must therefore be invariant, up to the scalar $\rho_1(h)$, under all $h \in G$ which leave T invariant. For $(x,y) \in T$ we have $N(x,y)$ given by 16.15.3 with $s_i = 1$, i.e.,

(3) $$N(x,y) = \prod_{i=1}^{r} (1 - \lambda_i^+(x)\lambda_i^-(y)).$$

Hence the irreducible factors of the restriction of N to T are the $(1-\lambda_i^+\lambda_i^-)$. By 17.1, every permutation of $(e_1,\dots,e_r)$ is induced by some $h \in G$. Since the restriction of f_1 to T is a product of some of the $(1-\lambda_i^+\lambda_i^-)$ it follows that N and f_1 agree (up to a scalar) on T. By 15.15, the orbit of T under G is dense. It follows that N and f_1 agree on this dense orbit, and therefore everywhere. This shows that $N = f_1$ is irreducible.

Now we prove (1). Since k is infinite it suffices to prove that $\det B(x,y)$ $N(x,y)^g$ for all $(x,y) \in V$, and since both sides are invariant under G and the orbit of T under G is dense we may assume that $(x,y) \in T$. Let

$$x = \sum_{i=1}^{r} \lambda_i^+(x) e_i^+ , \qquad y = \sum_{i=1}^{r} \lambda_i^-(y) e_i^- ,$$

(cf. 16.15), and set $\lambda_i = \lambda_i^+(x)\lambda_i^-(y)$ and $\lambda_0 = 0$. Then a computation shows that

$$D(x,y)z_{ij} = (\lambda_i+\lambda_j)z_{ij} \quad \text{and} \quad Q(x)Q(y)z_{ij} = \lambda_i\lambda_j z_{ij} , \tag{4}$$

for $z_{ij} \in V_{ij}^+$. This implies

$$B(x,y)z_{ij} = (1-\lambda_i)(1-\lambda_j)z_{ij} . \tag{5}$$

Now

$$\begin{aligned} \det B(x,y) &= \prod_{i=1}^{r}(1-\lambda_i)^2 . \prod_{1\leq i<j\leq r}((1-\lambda_i)(1-\lambda_j))^a . \prod_{i=1}^{r}(1-\lambda_i)^b \\ &= \prod_{i=1}^{r}(1-\lambda_i)^{2+a(r-1)+b} = \prod_{i=1}^{r}(1-\lambda_i)^g = N(x,y)^g . \end{aligned}$$

By 16.1.2, 16.9.3, and 17.2 we have

$$\begin{aligned} \chi(T,X,Y) &= T^{2\dim V^+} . \det B(T^{-1}X,Y) \\ &= T^{2(d_2+d_1)} . N(T^{-1}X,Y)^g = T^{gr+d_1} . N(T^{-1}X,Y)^g \\ &= T^{d_1} . (T^r . N(T^{-1}X,Y))^g = T^{d_1} . m(T,X,Y)^g . \end{aligned}$$

This completes the proof.

17.4. We now wish to classify the simple finite-dimensional Jordan pairs over the algebraically closed field k. By 15.5, the only finite-dimensional division pair over k is $(k,k)^J$. Also, there is (up to isomorphism) only one Cayley algebra over k . Specializing 12.12 to this situation we see that every simple finite-dimensional Jordan pair over k is contained in the following list.

$I_{p,q}$. $(M_{p,q}(k), M_{p,q}(k))$, $p \times q$ matrices over k .

II_n. $(A_n(k), A_n(k))$, alternating $n \times n$ matrices over k .

III_n. $(H_n(k), H_n(k))$, symmetric $n \times n$ matrices over k .

In these three cases, the Jordan pair structure is given by $Q(x)y = x.{}^t y.x$.

IV_n. (k^n, k^n), with $Q(x)y = q(x,y)x - q(x)y$. Here q is the standard quadratic form on k^n , given by

$$q(x_1,\dots,x_{2m}) = \sum_{i=1}^{m} x_i x_{m+i} \quad \text{if } n = 2m ,$$

$$q(x_0,\dots,x_{2m}) = x_0^2 + \sum_{i=1}^{m} x_i x_{m+i} \quad \text{if } n = 2m+1 .$$

(There are no outer ideals containing 1 in Jordan algebras of quadratic forms except the whole algebra since k is algebraically closed; cf. Jacobson[3]).

V. $(M_{1,2}(C), M_{1,2}(C^{op}))$, 1×2 matrices over the Cayley algebra C over k with $Q(x)y = x(y^*x)$.

VI. $(H_3(C,k), H_3(C,k))$, the Jordan pair associated with the exceptional Jordan algebra of 3×3 hermitian matrices over C .

Now we discuss each type in more detail.

17.5. <u>Type</u> $I_{p,q}$. The transpose $x \mapsto {}^t x$ defines an isomorphism $I_{p,q} \cong I_{q,p}$, so we may assume that $p \leq q$. A frame $(e_1,\dots,e_r)$ is given by the matrices

$$e_i^\sigma = \left(\begin{array}{cc|c} \begin{matrix}0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & 1 & \\ & & & & 0 \\ & & & & & \ddots \\ & & & & & & 0\end{matrix} & & \bigcirc \end{array}\right) \text{i-th} \qquad i = 1,\dots,p .$$
$$\underbrace{\qquad\qquad}_{p} \quad \underbrace{\qquad}_{q-p}$$

Hence the rank is p. Also we have $a = 2$ if $p \geqq 2$, $b = q - p$, $d_2 = p^2$, $d_1 = p(q - p)$, $d = pq$, $g = p + q$. The generic norm is given by

$$N(x,y) = \det(1_p - x.{}^t y),$$

where 1_p denotes the $p \times p$ unit matrix. The generic trace is $m_1(x,y) = \operatorname{trace}(x.{}^t y)$ and is non-degenerate.

17.6. <u>Type</u> II_n. The rank is $r = [n/2]$ since the matrices

$$\begin{pmatrix} \boxed{\begin{matrix}0 & 0\\ 0 & 0\end{matrix}} & & & \bigcirc \\ & \ddots & & \\ & & \boxed{\begin{matrix}0 & 1\\ -1 & 0\end{matrix}} & \\ \bigcirc & & & \ddots \end{pmatrix}$$

form a frame. We have $II_2 \cong (k,k)^J$, $II_3 \cong I_{1,3}$, under the map

$$(x_1,x_2,x_3) \longmapsto \begin{pmatrix} 0 & x_1 & x_2 \\ -x_1 & 0 & x_3 \\ -x_2 & -x_3 & 0 \end{pmatrix}$$

and hence we assume $n \geqq 4$. Then $a = 4$, $b = 2(n-2r)$ (i.e., $b = 0$ if n is even, and $b = 2$ if n is odd), $d_2 = 2r^2 - r$, $d_1 = 2r(n - 2r)$, $d = n(n-1)/2$, $g = 2n - 2$. The generic norm is the "square root" of $\det(1_n - x.{}^t y)$,

$$N(x,y)^2 = \det(1_n - x.{}^t y) = \det(1_n + xy).$$

If $n = 2r$ is even then $N(x,y) = \mathrm{Pf}(x)\mathrm{Pf}(x^{-1}-y)$ for x invertible where Pf denotes the Pfaffian (cf. Artin[1], p. 141). Thus $N(x,y)$ is a kind of Pfaffian which still makes sense for alternating matrices of odd order. The generic trace is non-degenerate, and is given by

$$m_1(x,y) = \sum_{i<j} x_{ij}y_{ij} \; .$$

17.7. <u>Type</u> III_n. Obviously $\mathrm{III}_1 = (k,k)^J$ so we assume $n \geqq 2$. The rank is n since a frame is given by the matrices

$$\begin{pmatrix} 0 & & & & & & \\ & \ddots & & & & \bigcirc & \\ & & 0 & & & & \\ & & & 1 & & & \\ & & & & 0 & & \\ & \bigcirc & & & & \ddots & \\ & & & & & & 0 \end{pmatrix}$$

We have $a = 1$, $b = 0$, $d = d_2 = n(n+1)/2$, $d_1 = 0$, $g = n + 1$. The generic n norm is

$$N(x,y) = \det(1_n - xy).$$

The generic trace is

$$m_1(xy) = \mathrm{trace}(xy) = \sum x_{ii}y_{ii} + 2 \sum_{i<j} x_{ij}y_{ij}$$

and is therefore degenerate in characteristic 2.

17.8. <u>Type</u> IV_n. We have $\mathrm{IV}_1 = (k,k)^J$ and $\mathrm{IV}_2 \cong \mathrm{IV}_1 \times \mathrm{IV}_1$ is not simple. Thus assume that $n \geqq 3$. Then IV_n is of rank 2, and we have $a = n - 2$, $b = 0$, $d = d_2 = n$, $d_1 = 0$, $g = n$. The generic norm is

$$N(x,y) = 1 - q(x,y) + q(x)q(y)$$

and the generic trace is $m_1(x,y) = q(x,y)$ which is degenerate if char $k = 2$ and n is odd.

17.9. Type V. Let c_1 and c_2 be two primitive orthogonal idempotents of the Cayley algebra C, e.g., $c_1 = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$, $c_2 = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$ if C is realized in vector matrix form. Then $e_i^{\sigma} = (c_i,0)$ $(i = 1,2)$ is a frame, and hence the rank is 2. We have $a = 6$, $b = 4$, $d_2 = d_1 = 8$, $g = 12$, $d = 16$. The generic norm and trace are given by

$$N(x,y) = n(1 - xy^*) \ , \quad m_1(x,y) = t(xy^*)$$

where n (resp. t) denotes the norm (resp. trace) of C. The generic trace is non-degenerate.

17.10. Type VI. From well-known facts on the exceptional Jordan algebra $H_3(C,k)$ it follows that the rank is 3, and we have $a = 8$, $b = 0$, $d = d_2 = 27$, $d_1 = 0$, $g = 18$. The generic norm and trace are given by $m_1(x,y) = T(x,y)$ and

$$N(x,y) = 1 - T(x,y) + T(x^{\#},y^{\#}) - N(x)N(y)$$

where $N(x)$ is the generic norm of $H_3(C,k)$ and T and $x^{\#}$ are defined as in McCrimmon[3]. The generic trace is non-degenerate.

17.11. Isomorphisms in low dimensions. A glance at the three invariants dimension, rank, and genus in the cases listed above shows that at most the following isomorphisms are possible.

(1) $$I_{1,1} \cong II_2 \cong III_1 \cong IV_1 \cong (k,k)^J,$$

(2) $$IV_2 \cong IV_1 \times IV_1,$$

(3) $$I_{1,3} \cong II_3,$$

(4) $$III_2 \cong IV_3,$$

(5) $$II_4 \cong IV_6.$$

Of these, (1) is obvious, and (2) and (3) were noted before. We show that (4) and (5) do occur. Indeed, $q(x) = \det x$ is a non-degenerate quadratic form on $H_2(k) \cong k^3$, and we have $xyx = q(x,\bar{y})x - q(x)\bar{y}$ where

$$\bar{y} = \begin{pmatrix} y_2 & -y_0 \\ -y_0 & y_1 \end{pmatrix} \quad \text{for} \quad y = \begin{pmatrix} y_1 & y_0 \\ y_0 & y_2 \end{pmatrix}.$$

Hence the map $(x,y) \to (x,\bar{y})$ from $(H_2(k),H_2(k))$ into (k^3,k^3) (with the Jordan structure defined by q) is an isomorphism (cf. also 12.19, (ii)).

Now we prove (5). Let

$$x = \begin{pmatrix} 0 & x_1 & x_2 & x_3 \\ & 0 & x_6 & -x_5 \\ & * & 0 & x_4 \\ & & & 0 \end{pmatrix} \in A_4(k) \cong k^6.$$

Then the Pfaffian $\mathrm{Pf}(x) = q(x) = x_1x_4 + x_2x_5 + x_3x_6$ is a non-degenerate quadratic form on $A_4(k) = k^6$, and one checks that $x^tyx = -xyx = q(x,\bar{y})x - q(x)\bar{y}$ where $\bar{x}$ is defined by

$$\bar{x} = \begin{pmatrix} 0 & x_4 & x_5 & x_6 \\ & 0 & x_3 & -x_2 \\ & * & 0 & x_1 \\ & & & 0 \end{pmatrix}$$

for x as above. Hence the map $(x,y) \mapsto (x,\bar{y})$ from $(A_4(k),A_4(k))$ into (k^6,k^6) (with the Jordan pair structure defined by q) is an isomorphism.

We collect our results in

17.12. THEOREM. A simple finite-dimensional Jordan pair over an algebraically closed field is up to isomorphism uniquely determined by the three invariants dimension, rank, and genus. The following list is complete and contains no repetitions.

V	dimension	rank	genus	invertible elements?
$I_{p,q}$, $1 \leq p \leq q$	pq	p	$p+q$	if $p=q$
II_n, $n \geq 5$	$\frac{n.(n-1)}{2}$	$[\frac{n}{2}]$	$2n-2$	if n even
III_n, $n \geq 2$	$\frac{n\cdot(n+1)}{2}$	n	$n+1$	yes
IV_n, $n \geq 4$	n	2	n	yes
V	16	2	12	no
VI	27	3	18	yes

§18. Appendix: Polynomial and rational functions

18.1. Let k be a commutative unital ring, and $k\text{-}\underline{\underline{alg}}$ the category of unital commutative (associative) k-algebras. For $R,S \in k\text{-}\underline{\underline{alg}}$ we denote by $\mathrm{Hom}_{k\text{-}\underline{\underline{alg}}}(R,S)$ the set of k-algebra homomorphisms from R to S. Also. let $\underline{\underline{sets}}$ denote the category of sets. By a k-functor we mean a covariant functor from $k\text{-}\underline{\underline{alg}}$ to $\underline{\underline{sets}}$. The forgetful functor, assigning to each $R \in k\text{-}\underline{\underline{alg}}$ the underlying set, is a k-functor, denoted by $\underline{\underline{\mathcal{O}}}_k$ and sometimes called the affine line over k.

Consider a k-module V, and let $\underline{\underline{V}}$ be the k-functor defined by

$$\underline{\underline{V}}(R) = V \otimes_k R = V_R ,$$

for all $R \in k\text{-}\underline{\underline{alg}}$. A polynomial function on V is a morphism (= natural transformation) $f: \underline{\underline{V}} \to \underline{\underline{\mathcal{O}}}_k$ of k-functors. In other words, giving a polynomial function on V simply means that we have maps $f_R: V_R \to R$ for each $R \in k\text{-}\underline{\underline{alg}}$, "varying functorially with R"; i.e., if $\phi: R \to S$ is a homomorphism of k-algebras then the diagram

$$(1) \qquad \begin{array}{ccc} V_R & \xrightarrow{f_R} & R \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ V_S & \xrightarrow[f_S]{} & S \end{array}$$

is commutative. Thus a polynomial function on V is not a function in the usual sense, although it does define a function $f_k: V \to k$. In general, f is not uniquely determined by f_k. This is so, however, if k is an infinite field, and

V is finite-dimensional over k (see 18.7).

Let $\mathcal{O}(V)$ be the set of all polynomial functions on V. Then it is easily seen that with the obvious definitions

$$(f+g)_R = f_R + g_R \ , \quad (fg)_R = f_R g_R \ , \quad (\alpha f)_R = \alpha f_R$$

for all $R \in k\text{-}\underline{\underline{alg}}$, $\mathcal{O}(V)$ is itself a k-algebra. Also, $\mathcal{O}(V)$ depends functorially on V : if $\Phi\colon V \to W$ is a homomorphism of k-modules then we obtain a homomorphism $\Phi^*\colon \mathcal{O}(W) \to \mathcal{O}(V)$ of k-algebras by $(\Phi^* f)_R = f_R \circ \Phi_R$ (where $\Phi_R\colon V_R \to W_R$ denotes the R-linear extension of Φ). From this one deduces easily that there is a natural homomorphism

(2) $$\mathcal{O}(V \times W) \to \mathcal{O}(V) \otimes_k \mathcal{O}(W)$$

for any two k-modules V and W.

For every $R \in k\text{-}\underline{\underline{alg}}$ we have a canonical map

(3) $$V_R \to \mathrm{Hom}_{k\text{-}\underline{\underline{alg}}}(\mathcal{O}(V), R)$$

defined by $x \mapsto \hat{x}$ where $\hat{x}\colon \mathcal{O}(V) \to R$ is evaluation at x ; $\hat{x}(f) = f_R(x)$. To simplify notation, we will write $f(x)$ instead of $f_R(x)$ so that we have

$$\hat{x}(f) = f(x)$$

for all $x \in V_R$, $R \in k\text{-}\underline{\underline{alg}}$. The maps (3) vary functorially with R ; i.e., for $\phi\colon R \to S$ the diagram

(4) $$\begin{array}{ccc} V_R & \longrightarrow & \mathrm{Hom}_{k\text{-}\underline{\underline{alg}}}(\mathcal{O}(V), R) \\ \downarrow & & \downarrow \\ V_S & \longrightarrow & \mathrm{Hom}_{k\text{-}\underline{\underline{alg}}}(\mathcal{O}(V), S) \end{array}$$

is commutative, where the vertical arrows are the obvious ones.

18.2. With every $f \in \mathcal{O}(V)$ we associate a family $(f^{(n)})_{n \in \underline{\underline{N}}}$ of elements of $\mathcal{O}(V)$, called the <u>homogeneous components</u> of f, by the equation

$$f(Tx) = \sum_{n=0}^{\infty} f^{(n)}(x) T^n \,, \tag{1}$$

for all $x \in V_R$, $R \in k$-$\underline{\underline{alg}}$. Here T is an indeterminate and $Tx \in V_{R[T]} \cong \bigoplus T^n V_R$. Hence (1) defines $f^{(n)}(x)$ uniquely, and it is easily checked that everything is functorial. We say that f is <u>homogeneous of degree</u> n if $f^{(n)} = f$. This is equivalent with

$$f(\alpha x) = \alpha^n f(x)$$

for all $\alpha \in R$, $x \in V_R$, $R \in k$-$\underline{\underline{alg}}$. (Warning: unless V is finitely spanned f may have infinitely many non-zero homogeneous components; however, for each $x \in V_R$ only finitely many of the $f^{(n)}(x)$ are different from zero).

Let V_i $(i = 1,2)$ be k-modules, and let T_i be indeterminates. For $f \in \mathcal{O}(V_1 \times V_2)$ the <u>homogeneous components of bidegree</u> (m,n) <u>of</u> f are defined by the equation

$$f(x_1,x_2) = \sum_{m,n=0}^{\infty} f^{(m,n)}(x_1,x_2) T_1^m T_2^n \,,$$

for all $(x_1,x_2) \in (V_1)_R \times (V_2)_R$, $R \in k$-$\underline{\underline{alg}}$. Similarly as before, f is called <u>homogeneous of bidegree</u> (m,n) if $f = f^{(m,n)}$; equivalently, if

$$f(\alpha_1 x_1, \alpha_2 x_2) = \alpha_1^m \alpha_2^n f(x_1,x_2)$$

for all $\alpha_i \in R$, $x_i \in (V_i)_R$, $R \in k$-$\underline{\underline{alg}}$. The <u>total degree</u> of $f^{(m,n)}$ is $m + n$, and we have

$$f^{(n)} = \sum_{i+j=n} f^{(i,j)} \,.$$

All this generalizes in the obvious way to the case of more than two modules.

The polynomial functions of degree zero on V may obviously be identified with elements of k, and a linear form f on V defines a polynomial function of degree one on V. Conversely, let $f \in \mathcal{O}(V)$ be homogeneous of degree one, and define $g \in \mathcal{O}(V \times V)$ by $g(x,y) = f(x+y)$. Then g is homogeneous of total degree one, and hence we have, expanding into bihomogeneous components, that $g = g^{(1,0)} + g^{(0,1)}$. But then $f(\alpha x+\beta y) = g(\alpha x,\beta y) = \alpha g^{(1,0)}(x,y) + \beta g^{(0,1)}(x,y)$ implies that $f = g^{(1,0)} = g^{(0,1)}$ is a linear form on V. Similarly, it can be shown that the homogeneous polynomial functions of degree two may be identified with the quadratic forms on V (see Roby[1] for details).

18.3. PROPOSITION. <u>The following conditions are equivalent.</u>

(i) <u>The map</u> $V_R \to \mathrm{Hom}_{k\text{-}\underline{\underline{alg}}}(\mathcal{O}(V),R)$ <u>is bijective, for all</u> $R \in k\text{-}\underline{\underline{alg}}$;

(ii) V <u>is a finitely generated and projective</u> k-<u>module</u>.

<u>If these conditions are satisfied</u> $\mathcal{O}(V)$ <u>is canonically isomorphic with the symmetric algebra over the dual of</u> V .

(Note that (i) just means that the functors $\underline{\underline{V}}$ and $\mathrm{Hom}_{k\text{-}\underline{\underline{alg}}}(\mathcal{O}(V),-)$ are isomorphic; i.e., $\underline{\underline{V}}$ is representable by $\mathcal{O}(V)$).

<u>Proof</u>. (i) $\to$ (ii): Let $A = \mathcal{O}(V)$, and let $X = \sum_{i=1}^{r} v_i \otimes f_i \in V_A$ (where $f_i \in A$, $v_i \in V$) be the unique element such that $\hat{X} = \mathrm{Id}_A$. Let $x \in V_R$. Then it follows from 18.1.3 that

$$(1) \qquad x = \sum_{i=1}^{r} v_i \otimes f_i(x) .$$

In particular, for $R = k$ we see that V is generated as a k-module by the elements $v_1,\ldots,v_r$. If we replace x by Tx in (1) where T is an indeterminate and compare coefficients at powers of T we see that (1) holds with f_i replaced by $\lambda_i = f_i^{(1)}$, the homogeneous component of degree one of f_i which by 18.2 is a linear form on V. In particular, we have $x = \Sigma\, \lambda_i(x)v_i$ for all $x \in V$, and by the dual basis lemma, V is finitely generated and projective.

(ii) $\rightarrow$ (i) : Let S be the symmetric algebra over the dual $V^\vee$ of V. If $R \in$ k-$\underline{\underline{\text{alg}}}$ then by well-known properties of finitely generated projective modules we have $V_R \cong \mathrm{Hom}_k(V^\vee,R)$ and by the universal property of S we have $\mathrm{Hom}_k(V^\vee,R) \cong \mathrm{Hom}_{k\text{-}\underline{\underline{\text{alg}}}}(S,R)$. Denoting for $x \in V_R$ the corresponding k-algebra homomorphism $S \rightarrow R$ by $\tilde{x}$ we can define a homomorphism $s \mapsto \tilde{s}$ from S into A by setting $\tilde{s}(x) = \tilde{x}(s)$, for all $x \in V_R$, $R \in$ k-$\underline{\underline{\text{alg}}}$. Conversely, consider the map $\mathrm{Hom}_{k\text{-}\underline{\underline{\text{alg}}}}(S,R) \rightarrow \mathrm{Hom}_{k\text{-}\underline{\underline{\text{alg}}}}(A,R)$ defined by the commutative diagram

$$\begin{tikzcd} V_R \arrow[rr, "\cong"] \arrow[dr] & & \mathrm{Hom}_{k\text{-}\underline{\underline{\text{alg}}}}(S,R) \arrow[dl] \\ & \mathrm{Hom}_{k\text{-}\underline{\underline{\text{alg}}}}(A,R) & \end{tikzcd}$$

For $R = S$ we obtain a homomorphism $A \rightarrow S$ corresponding to $\mathrm{Id}_S \in \mathrm{Hom}_{k\text{-}\underline{\underline{\text{alg}}}}(S,S)$. One checks that the two homorphisms $S \rightarrow A$ and $A \rightarrow S$ are inverses of each other.

18.4. Assume that V is finitely generated and projective. The unique element $X \in V \otimes \mathcal{O}(V)$ such that $\hat{X} = \mathrm{Id}_{\mathcal{O}(V)}$ is called the <u>generic point</u> of V. Thus we have

$$f = \hat{X}(f) = f(X)$$

for all $f \in \mathcal{O}(V)$, and every $x \in V_R$ ($R \in$ k-$\underline{\underline{\text{alg}}}$) is a "specialisation" of X :

$$x = \hat{x}(X) \ ,$$

if $\hat{x}\colon V_{\mathcal{O}(V)} \rightarrow V_R$ also denotes the map induced by $\hat{x}\colon \mathcal{O}(V) \rightarrow R$. Depending on the context, one or the other of the notations f or $f(X)$ is more convenient, and either one will be used.

Let V and W be finitely generated and projective k-modules. It is well known that the symmetric algebra over $V \times W$ is isomorphic with the tensor product of the symmetric algebras over V and W. Hence it follows from 18.3 that the homomorphism 18.1.2 is an isomorphism.

Now assume that V is finitely generated and free, and let $v_1,\ldots,v_n$ be a basis of V over k. Then we have an isomorphism $k[T_1,\ldots,T_n] \to \mathcal{O}(V)$ as follows: If $f(T_1,\ldots,T_n) \in k[T_1,\ldots,T_n]$ and $x = \Sigma\alpha_i x_i \in V_R$ $(\alpha_i \in R)$ we set $f(x) = f(\alpha_1,\ldots,\alpha_n)$. Under this isomorphism, the generic point X of V corresponds to the element $\Sigma\, T_i v_i$ of $V_{k[T_1,\ldots,T_n]}$. In the special case $V = k$ (with the natural basis 1) we will identify $\mathcal{O}(k)$ and $k[T]$. Thus the indeterminate T is the generic point of k.

18.5. For k-modules V and W we define $\mathcal{O}(V,W)$ as the set of all morphisms from $\underline{\underline{V}}$ to $\underline{\underline{W}}$, and call its elements <u>polynomial maps</u> from V to W (as before, these are not maps in the usual sense!). The homogeneous components of an element $f \in \mathcal{O}(V,W)$ are defined as before. If W is finitely generated and projective then the natural map $\mathcal{O}(V) \otimes_k W \to \mathcal{O}(V,W)$ is an isomorphism.

18.6. Let $f \in \mathcal{O}(V,W)$. The <u>differential of</u> f is the element $df \in \mathcal{O}(V \times V,W)$ defined by

$$f(x + \varepsilon y) = f(x) + \varepsilon df(x,y) ,$$

for all $x,y \in V_R$. Here $x + \varepsilon y \in V_{R(\varepsilon)}$ and $R(\varepsilon)$ is the algebra of dual numbers over R. To see that $df(x,y) = df(x).y$ is linear in y let $g \in \mathcal{O}(V\times V,W)$ be defined by $g(x,y) = f(x + y)$, and let $g = \Sigma\, g^{(i,j)}$ be the expansion into bihomogeneous components. Then

$$f(x + \varepsilon y) = \sum g^{(i,j)}(x,y)\varepsilon^j = f(x) + \varepsilon.\sum g^{(i,1)}(x,y) \ .$$

Thus $df(x) = df(x,-) \in \mathrm{Hom}_R(V_R,W_R)$ for all $x \in V_R$. If V is finitely generated and projective then we may consider df as an element of $\mathcal{O}(V,\mathrm{Hom}_k(V,W))$.

If $f \in \mathcal{O}(V,W)$ is homogeneous of degree n we have Euler's differential equation:

$$df(x).x = n.f(x) \ .$$

Indeed, $f(x) + \varepsilon df(x).x = f(x + \varepsilon x) = f((1+\varepsilon)x) = (1+\varepsilon)^n f(x) = (1 + n\varepsilon).f(x)$.

We remark that higher derivatives, partial derivatives, etc. may also be defined, and that all the usual results of differential calculus (chain rule, symmetry of higher derivatives,...) hold.

18.7. From now on, let k be a field, and let V be a finite-dimensional vector space over k . By 18.4, $\mathcal{O}(V)$ is isomorphic with a polynomial algebra $k[T_1,\ldots,T_n]$. From this it follows that an element $f \in \mathcal{O}(V)$ is uniquely determined by the function $f\colon V \to k$, provided that k is infinite. Also, $\mathcal{O}(V)$ is an entire (even factorial) ring, and we can form its quotient field $\mathcal{R}(V)$, called the field of rational functions of V . A choice of basis in V gives us an isomorphism $\mathcal{R}(V) = k(T_1,\ldots,T_n)$. Since we have unique factorization in $\mathcal{O}(V)$ every $f \in \mathcal{R}(V)$ has a reduced expression $f = g/h$, with g and h relatively prime, which is unique up to a non-zero factor in k . The denominator of a reduced expression of f is called an exact denominator of f .

If W is another finite-dimensional vector space we set $\mathcal{R}(V,W) = \mathcal{R}(V)\otimes_k W$ and call its elements rational maps from V to W (although, again, these are not maps from V to W in the usual sense). Let $w_1,\ldots,w_m$ be a basis of W . Then an element $f \in \mathcal{R}(V,W)$ can be written uniquely as $f = \Sigma f_i w_i$ where the f_i are rational functions on V , and we can also write $f_i = g_i/h$ where the functions $g_1,\dots,g_m,h \in \mathcal{O}(V)$ have no non-constant common divisor. Then $g = \Sigma g_i w_i$ and h are uniquely determined by f (up to a factor in k) and we call $f = g/h$ a reduced expression, and h an exact denominator of f .

Let $R \in k$-alg and $x \in V_R$. We say that $f \in \mathcal{R}(V,W)$ (with exact denominator h) is defined in x if $h(x)$ is invertible in R . In this case, we set

$$f(x) = g(x)h(x)^{-1} .$$

In particular, let $R = \mathcal{R}(V)$. Then f is defined in the generic point $X \in V_{\mathcal{O}(V)} \subset V_{\mathcal{R}(V)}$, and $f = f(X)$ as before (18.4).

18.8. Let $f = g/h \in \mathcal{R}(V,W)$ be a reduced expression, and let f be defined in $x \in V_R$. Then f is also defined in $x + \varepsilon y \in V_{R(\varepsilon)}$ for every $y \in V_R$. Indeed, $h(x+\varepsilon y)^{-1} = (h(x) + \varepsilon dh(x)y)^{-1} = h(x)^{-1}(1 - \varepsilon h(x)^{-1}dh(x)y)$. The differential of f is defined as usual by $f(x + \varepsilon y) = f(x) + \varepsilon df(x)y$. Now

$$\begin{aligned} f(x + \varepsilon y) &= g(x + \varepsilon y)h(x + \varepsilon y)^{-1} \\ &= h(x)^{-1}(1 - \varepsilon h(x)^{-1}dh(x)y)(g(x) + \varepsilon dg(x)y) \\ &= f(x) + \varepsilon h(x)^{-2}(h(x)dg(x)y - g(x)dh(x)y) \end{aligned}$$

and thus $df = (h.dg - g.dh)/h^2$ can be considered as an element of $\mathcal{R}(V,\mathrm{Hom}(V,W))$. The usual differentiation rules continue to hold for rational maps.

18.9. Let K be an extension field of k . Then we have $\mathcal{O}(V_K) = \mathcal{O}(V)\otimes_k K$; in particular, we may consider $\mathcal{O}(V)$ to be contained in $\mathcal{O}(V_K)$. Sometimes, the elements of $\mathcal{O}(V_K)$ belonging to $\mathcal{O}(V)$ are said to be defined over k . Also, we have $\mathcal{R}(V) \subset \mathcal{R}(V_K)$ (but $\mathcal{R}(V)\otimes_k K \neq \mathcal{R}(V_K)$!), and similarly for polynomial or rational maps with values in some vector space W . If $f = g/h \in \mathcal{R}(V,W)$ is a reduced expression then it remains a reduced expression in $\mathcal{R}(V_K,W_K)$. This follows from basic properties of polynomial rings and is left to the reader.

18.10. Let $\bar{k}$ be the algebraic closure of k. The elements of $\bar{V} = V_{\bar{k}}$ are also called geometric points of V. Since $\bar{k}$ is infinite every $f \in \mathcal{O}(\bar{V})$ is uniquely determined by the corresponding function $f: \bar{V} \to \bar{k}$. As usual, $\bar{V}$ carries the Zariski topology whose closed sets are the sets defined by finitely many equations $f_1(x) = \dots = f_r(x) = 0$, where $f_i \in \mathcal{O}(\bar{V})$. If $f_i \in \mathcal{O}(V)$ then the corresponding set is called k-closed. The (coarser) topology defined by the k-closed sets is called the k-topology.

Topological terminology will always refer to the Zariski topology. We recall the fundamental property that the intersection of any two open and dense (= open and non-empty) subsets is again open and dense. Also, every $f \in \mathcal{O}(\bar{V})$ defines a continuous function on $\bar{V}$. If $f = g/h$ is a reduced expression of a rational map $f \in \mathcal{R}(\bar{V},\bar{W})$ then f is defined on the open and dense subset of all $x \in \bar{V}$ where $h(x) \neq 0$.

18.11. LEMMA. <u>Let</u> k <u>be algebraically closed, and let</u> $f_i \in \mathcal{O}(V)$ <u>be non-constant</u> $(i = 1,\dots,m)$. <u>Then there exists</u> $v \in V$ <u>and a subspace</u> V' <u>of codimension one of</u> V <u>with the following properties</u>.

(i) $V = V' \oplus k.v$.

(ii) <u>If we identify</u> V <u>with</u> $V' \times k$ <u>and correspondingly identify the generic point</u> X <u>of</u> V <u>with</u> (X',T) (<u>where</u> X' <u>and</u> T <u>are the generic points of</u> V' <u>and</u> k) <u>then</u> $f_i = f_i(X',T) \in \mathcal{O}(V' \times k) \cong \mathcal{O}(V')[T]$ <u>is, up to a non-zero constant, a monic polynomial of positive degree in</u> T .

This is just a restatement of a classical lemma (cf. van der Waerden II, §74).

18.12. PROPOSITION. Let X and Y be the generic points of the finite-dimensional vector spaces V and W over k. Let $f_i = f_i(X,Y) \in \mathcal{O}(V \times W)$ $(i = 1,\ldots,n)$ and let $f_0 = f_0(X,Y)$ be their greatest common divisor. Let $F = \mathcal{R}(W)$, and let $\phi_i \in \mathcal{O}(V_F)$ be defined by $\phi_i(X) = f_i(X,Y)$ $(i = 0,\ldots,n)$. Then ϕ_0 is the greatest common divisor of $\phi_1,\ldots,\phi_n$. Also there exists a dense open subset U of $\overline{W}$ such that for all $y \in U$ the greatest common divisor of the functions $f_i(X,y) \in \mathcal{O}(\overline{V})$ $(i = 1,\ldots,n)$ is $f_0(X,y)$.

Proof. Since F is the quotient field of the ring $A = \mathcal{O}(W)$ and $\mathcal{O}(V \times W) = \mathcal{O}(V)\otimes A$ and $\mathcal{O}(V_F) = \mathcal{O}(V)\otimes F$, the first statement follows from Gauss's lemma. For the second, an easy induction shows that it suffices to prove this for $n = 2$. Also we may assume that $k = \bar{k}$ is algebraically closed. Let $f_0 g_i = f_i$ $(i = 1,2)$ so that g_1 and g_2 are relatively prime in $\mathcal{O}(V \times W)$. We may assume that $g_1(X,Y)$ really depends on X. Write $g_1(X,Y) = \Sigma\, g_j'(X)g_j''(Y)$ with g_j' non-constant, and choose a decomposition $V \cong V' \times k$ as in 18.11. Then $g_1(X,Y) = g_1(X',T,Y) = h_1(T) \in \mathcal{O}(V' \times W)[T]$ is a polynomial in T whose highest coefficient belongs to $\mathcal{O}(W)$, i.e., it depends only on Y and not on X'. Let $h_2(T) = g_2(X,Y) = g_2(X',T,Y)$. Then the resultant $r = r(X',Y)$ of h_1 and h_2 which is an element of $\mathcal{O}(V' \times W)$ is not zero. Consequently, the set

$$U_1 = \{y \in W \mid r(X',y) \neq 0\}$$

is open and dense in W. Since $g_1(X',T,y)$ is monic in T this means that $g_1(X,y) = g_1(X',T,y)$ and $g_2(X,y) = g_2(X',T,y)$ are relatively prime in $\mathcal{O}(V)$, for all $y \in U_1$. It follows that the greatest common divisor of $f_1(X,y)$ and $f_2(X,y)$ is $f_0(X,y)$, for all $y \in U = U_1 \cap U_0$ where

$$U_0 = \{y \in W \mid f_0(X,y) \neq 0\}.$$

18.13. PROPOSITION. Let V, W, Z be finite-dimensional vector spaces over k, and let f be a rational map from $V \times W$ into Z with reduced expression $f = g/h$.

(a) There exists an open dense subset U of $\overline{W}$ such that, for all $y \in U$, $f(X,y)$ is a rational map from $\overline{V}$ to $\overline{Z}$ and $g(X,y)/h(X,y)$ is a reduced expression for it.

(b) Let $F = \mathcal{R}(W)$. Then $\phi(X) = f(X,Y)$ is a rational map from V_F into Z_F, and $\phi = \gamma/\eta$ is a reduced expression for ϕ where $\gamma(X) = g(X,Y) \in \mathcal{O}(V_F, Z_F)$ and $\eta(X) = h(X,Y) \in \mathcal{O}(V_F)$.

This follows immediately from 18.12.

NOTES

For universal envelopes of Jordan algebras and Jordan triple systems see Jacobson[2], McCrimmon[7], Loos[5]. By classification it is known that the universal envelope of a finite-dimensional separable Jordan algebra over a field of characteristic $\neq 2$ is separable. It would be interesting to have an a priori proof of this and also extend it to the Jordan triple system resp. Jordan pair case.

In the case of a linear Jordan pair, the concept of V-solvability (14.3) was first introduced by Meyberg[1,2]. It is closely related to solvability in the Koecher-Tits algebra constructed from V, and is also similar to Penico-solvability for Jordan algebras. 14.10 and 14.11 were first proved in Loos[2](for char $k \neq 2$) and Loos[5] for Jordan triple systems, using at a crucial point the algebraic group associated with the Koecher-Tits algebra of V . Here we give an elementary proof which requires only Engel's theorem (in 14.9). The proof of 14.11 follows the lines of the corresponding one for Jordan algebras (McCrimmon[10]) but is actually somewhat simpler since one doesn't have to worry about squares in a Jordan pair.

For the theory of Cartan subalgebras of Jordan algebras we refer to Jacobson[2], see also Loos[8]. This theory seems to be restricted to linear Jordan algebras over fields of characteristic $\neq 2$. In contrast, the theory of Cartan subpairs of Jordan pairs developed here works equally well in all characteristics. The important "density theorem" (15.15) was inspired by the corresponding result for Cartan subalgebras of Lie algebras of algebraic groups (Demazure and Grothendieck[1]). Also, the characterization of Cartan subpairs as centralizers of tori of maximum dimension and the existence proof for Cartan subpairs (15.20) is similar to the algebraic group case (see Borel[1]).

The approach to the generic minimum polynomial as the exact denominator of a suitable rational map is due to Springer[1] for the case of a Jordan algebra. For a different approach which also works in infinite dimensions see Jacobson[4]. The polynomial function $N(x,y) = N(x)N(x^{-1} - y)$ (cf. 16.3) which plays a considerable role in the Jordan algebra case (McCrimmon[9]) appears here as the generic norm of the associated Jordan pair. It should be noted that the generic minimum polynomial as defined here differs from the one in Loos[2] by a factor T .

It is a remarkable fact that the classification of finite-dimensional simple Jordan pairs over algebraically closed fields is independent of the characteristic of the base field. Indeed, from the list it is obvious that these Jordan pairs are obtained from Jordan pairs over the integers by reduction modulo p and extending to the algebraic closure. The classification over arbitrary fields reduces (by general principles, cf. Demazure and Gabriel[1]) to a problem in Galois cohomology since the automorphism group of a separable Jordan pair is smooth in the sense of group schemes (Loos[7]). All this is no longer true for quadratic Jordan algebras which again indicates that Jordan pairs are a more natural concept. The classification also resembles that of bounded symmetric domains. This is no accident since there is a natural correspondence between semisimple Jordan pairs over the complex numbers and bounded symmetric domains under which Jordan pairs with invertible elements correspond to tube domains. The genus plays an important role in the theory of these domains (see Koranyi[1] where it is denoted by p).

BIBLIOGRAPHY

Artin, E.

1. Geometric algebra. Interscience, New York, 1957.

Borel, A.

1. Linear algebraic groups. W. A. Benjamin, New York, 1969.

Cohn, P.

1. Universal algebra. Harper&Row, New York, 1965.

Demazure, M. and P. Gabriel

1. Groupes algebriques I. Masson, Paris, 1970.

Demazure, M. and A. Grothendieck

1. Schemas en groupes II. Springer lecture notes, Berlin-Heidelberg-New York, 1970.

Hestenes, M. R.

1. A ternary algebra with applications to matrices and linear transformations. Arch. Rat. Mech. Anal. 11 (1962), 138 - 194.

Jacobson, N.

1. Lie algebras. Interscience, New York, 1962.
2. Structure and representations of Jordan algebras. Amer. Math. Soc. Colloq. Publ. 39, Providence, 1968.
3. Lectures on quadratic Jordan algebras. Lecture notes, Tata Institute, Bombay, 1969.
4. (with J. Katz) Generically algebraic quadratic Jordan algebras. Scripta Math. 29 (1973), 215 - 227.

Kleinfeld, E.

1. Simple alternative rings. Ann. of Math. 58 (1953), 544 - 547.

Koecher, M.

1. Imbedding of Jordan algebras into Lie algebras. Am. J. Math. 89 (1967), 787 - 816, and 90 (1968), 476 - 510.

2. Über eine Gruppe von rationalen Abbildungen. Inv. Math. 3 (1967), 136 - 171.

3. Gruppen und Lie-Algebren von rationalen Funktionen. Math. Z. 109 (1969), 349 - 392.

4. An elementary approach to bounded symmetric domains. Lecture notes, Rice University, Houston, 1969.

Koranyi, A.

1. Analytic invariants of bounded symmetric domains. Proc. Amer. Math. Soc. 19 (1968), 279 - 284.

Lewand, R.

1. Hereditary radicals in Jordan rings. Proc. Amer. Math. Soc. 33 (1972), 302 - 306.

Lister, W. G.

1. Ternary rings. Trans. Amer. Math. Soc. 154 (1971), 37 - 55.

Loos, O.

1. Jordan triple systems, R-spaces, and bounded symmetric domains. Bull. Amer. Math. Soc. 77 (1971), 558 - 561.

2. Lectures on Jordan triples. Lecture notes, The University of British Columbia, Vancouver, 1971.

3. Alternative Tripelsysteme. Math. Ann. 198 (1972), 205 - 238.

4. Assoziative Tripelsysteme. manuscr. math. 7 (1972), 103 - 112.

5. Representations of Jordan triples. Trans. Amer. Math. Soc. 185 (1973), 199 - 211.

6. A structure theory of Jordan pairs. Bull. Amer. Math. Soc. 80 (1974), 67 - 71.

7. On algebraic groups defined by Jordan structures. To appear.

8. Existence and conjugacy of Cartan subalgebras of Jordan algebras. To appear.

McCrimmon, K.

1. The radical of a Jordan algebra. Proc. Nat. Acad. Sc. 62 (1969), 671 - 678.

2. Non-degenerate Jordan rings are von Neumann regular. J. of Alg. 11 (1969), 111 - 115.

3. The Freudenthal-Springer-Tits constructions revisited. Trans. Amer. Math. Soc. 148 (1970), 293 - 314.

4. A characterisation of the radical of a Jordan algebra. J. Alg. 18 (1971), 103 - 111.

5. Homotopes of alternative algebras. Math. Ann. 191 (1971), 253 - 262.

6. Quadratic Jordan algebras and cubing operations. Trans. Amer. Math. Soc. 153 (1971), 265 - 278.

7. Representations of quadratic Jordan algebras. Trans. Amer. Math. Soc. 153 (1971), 279 - 305.

8. Inner ideals in quadratic Jordan algebras. Trans. Amer. Math. Soc. 159 (1971), 445 - 468.

9. The generic norm of an isotope of a Jordan algebra. Scripta Math. 29 (1973), 229 - 236.

10. Solvability and nilpotence for quadratic Jordan algebras. Scripta Math. 29 (1973), 467 - 483.

Meyberg, K.

1. Jordan-Tripelsysteme und die Koecher-Konstruktion von Lie-Algebren. Habilitationsschrift, München, 1969.

2. Jordan-Tripelsysteme und die Koecher-Konstruktion von Lie-Algebren. Math. Z. 115 (1970), 58 - 58.

3. Zur Konstruktion von Lie-Algebren aus Jordan-Tripelsystemen. manuscr. math. 3 (1970), 115 - 132.

4. Identitäten und das Radikal in Jordan-Tripelsystemen. Math. Ann. 197 (1972), 203 - 220.

5. Von Neumann regularity in Jordan triple systems. Arch. Math. 23 (1972), 589 - 593.

6. Lectures on algebras and triple systems. Lecture notes, The University of Virginia, Charlottesville, 1972.

7. A characterization of von Neumann regular Jordan triple systems. To appear in Proc. Amer. Math. Soc.

Roby, N.

1. Lois polynomes et lois formelles en theorie des modules. Ann. Sci. Ecole Norm. Sup. 3^e ser. t. 80 (1963), 213 - 348.

Schafer, R. D.

1. An introduction to non-associative algebras. Academic Press, New York-London, 1966.

Slin'ko, A. M.

1. On radical Jordan rings. Alg. i Logika 2 (1972), 206 - 215.

Springer, T. A.

1. Jordan algebras and algebraic groups. Springer Verlag, Berlin-Heidelberg-New York, 1973.

Stephenson, R. A.

1. Jacobson structure theory for Hestenes ternary rings. Trans. Amer. Math. Soc. 177 (1973), 91 - 98.

van der Waerden, B. L.

1. Moderne Algebra II. Springer Verlag, Berlin, 1931.