The Higgins-Selkov oscillator

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Here I analyse the long-time behaviour of the Higgins-Selkov oscillator. The system is

$$\dot{x} = k_0 - k_1 x y^2, \tag{1}$$
$$\dot{y} = k_1 x y^2 - k_2 y. \tag{2}$$

The unknowns x and y, being concentrations, are non-negative. The reaction constants k_i are positive. This system corresponds to the special case $\gamma = 2$ of the system given by Selkov. Its behaviour at infinity can be investigated using a change of variables related to the Poincaré sphere. Define $\tilde{Y} = \frac{y}{x}$ and $\tilde{Z} = \frac{1}{x}$. Inverting these relations gives $x = \frac{1}{\tilde{Z}}$ and $y = \frac{\tilde{Y}}{\tilde{Z}}$. The x-axis, i.e. the set $y = 0$ gets mapped to the $\tilde{Z}$ axis, i.e. the set $\tilde{Y} = 0$. The orientation of x is reversed. The $\tilde{Y}$-axis, i.e. $\tilde{Z} = 0$ corresponds to infinity in the old coordinates. $\tilde{Y}$ increases in the same direction as y. Call the right hand sides of the evolution equations $P(x,y)$ and $Q(x,y)$. Transforming the system to the coordinates $(\tilde{Y}, \tilde{Z})$ gives

$$\frac{d\tilde{Y}}{dt} = -\tilde{Z}\left[\tilde{Y} P\left(\frac{1}{\tilde{Z}}, \frac{\tilde{Y}}{\tilde{Z}}\right) - Q\left(\frac{1}{\tilde{Z}}, \frac{\tilde{Y}}{\tilde{Z}}\right)\right], \tag{3}$$
$$\frac{d\tilde{Z}}{dt} = -\tilde{Z}^2 P\left(\frac{1}{\tilde{Z}}, \frac{\tilde{Y}}{\tilde{Z}}\right). \tag{4}$$

In the case of the Higgins-Selkov oscillator this becomes

$$\frac{d\tilde{Y}}{dt} = -\tilde{Z}\left[\tilde{Y}\left(k_0 - k_1\frac{\tilde{Y}^2}{\tilde{Z}^3}\right) - k_1\frac{\tilde{Y}^2}{\tilde{Z}^3} + k_2\frac{\tilde{Y}}{\tilde{Z}}\right], \tag{5}$$
$$\frac{d\tilde{Z}}{dt} = -\tilde{Z}^2\left(k_0 - k_1\frac{\tilde{Y}^2}{\tilde{Z}^3}\right). \tag{6}$$

This can be simplified by using the notation (y,z) instead of $(\tilde{Y}, \tilde{Z})$ and multiplying the right hand side by z^2 through the use of a new time coordinate. The result is

$$y' = +k_1 y^2 + k_1 y^3 - k_2 y z^2 - k_0 y z^3, \tag{7}$$
$$z' = +k_1 y^2 z - k_0 z^4. \tag{8}$$

For this system $y = 0$ is an invariant manifold and there z' is negative. It is also the case that $z = 0$ is an invariant manifold and there y' is positive.

The system just introduced is appropriate for analysing the behaviour of the solutions for x large. To analyse the behaviour for y large a different change of coordinates can be used. In this case let $\tilde{X} = \frac{x}{y}$ and $\tilde{Z} = \frac{1}{y}$ with inverse $y = \frac{1}{\tilde{Z}}$ and $x = \frac{\tilde{X}}{\tilde{Z}}$. The y-axis, i.e. the set $x = 0$ gets mapped to the $\tilde{Z}$ axis, i.e. the set $\tilde{X} = 0$. The orientation of y is reversed. The $\tilde{X}$-axis, i.e. $\tilde{Z} = 0$ corresponds to infinity in the old coordinates. $\tilde{X}$ increases in the same direction as x. The transformed system is

$$\frac{d\tilde{X}}{dt} = -\tilde{Z}\left[\tilde{X}Q\left(\frac{\tilde{X}}{\tilde{Z}}, \frac{1}{\tilde{Z}}\right) - P\left(\frac{\tilde{X}}{\tilde{Z}}, \frac{1}{\tilde{Z}}\right)\right], \tag{9}$$

$$\frac{d\tilde{Z}}{dt} = -\tilde{Z}^2 Q\left(\frac{\tilde{X}}{\tilde{Z}}, \frac{1}{\tilde{Z}}\right). \tag{10}$$

In the case of the Higgins-Selkov oscillator this becomes

$$\frac{d\tilde{X}}{dt} = -\tilde{Z}\left[\tilde{X}\left(k_1\frac{\tilde{X}}{\tilde{Z}^3} - \frac{k_2}{\tilde{Z}}\right) - \left(k_0 - k_1\frac{\tilde{X}}{\tilde{Z}^3}\right)\right], \tag{11}$$

$$\frac{d\tilde{Z}}{dt} = -\tilde{Z}^2 Q\left(k_1\frac{\tilde{X}}{\tilde{Z}^3} - \frac{k_2}{\tilde{Z}}\right). \tag{12}$$

Consolidating the notation as in the previous case leads to the system

$$x' = -k_1 x - k_1 x^2 + k_2 x z^2 + k_0 z^3, \tag{13}$$

$$z' = -k_1 x z + k_2 z^3. \tag{14}$$

In this case $z = 0$ is invariant and there $x' < 0$. On $x = 0$ we have inflow. Call the origin in this coordinate system P_1. The linearization at P_1 has rank one and there is a one-dimensional centre manifold. The centre manifold is of the form $x = \phi(z) = O(z^2)$. Differentiating this relation with respect to t gives $x' = \phi'(x)z'$ and substituting in the evolution equations shows that $x = (k_0/k_1)z^3 + \ldots$. This implies that z' is positive for small z and the stationary point is a saddle.

Next the behaviour of solutions of the system for y and z near the origin will be examined. The linearization at the origin is identically zero and to obtain more information the system will be transformed to polar coordinates as in section 2.10 of Perko. In polar coordinates all terms in these equations contain a factor r because the linearization of the original system at the origin vanishes. Thus we can change the time coordinate so as to reduce the powers of r by one. The notation for the time coordinate will not be changed. This results in the system

$$r' = r[k_1 \cos^3\theta] + r^2[k_1 \cos^4\theta - k_2 \sin^2\theta\cos^2\theta + k_1 \sin^2\theta\cos^2\theta]$$
$$+ r^3[-k_0 \sin^3\theta\cos^2\theta - k_0 \sin^5\theta], \tag{15}$$

$$\theta' = [-k_1 \sin\theta\cos^2\theta] + r k_2 \sin^3\theta\cos\theta. \tag{16}$$

Note that $\theta = 0$ corresponds to $z = 0$. We are only interested in the case that the variables y and z are non-negative, in other words that $0 \leq \theta \leq \pi/2$. The set $r = 0$ is an invariant manifold as are the sets $\theta = 0$ and $\theta = \pi/2$. For $\theta = 0$ and $\theta = \pi/2$ we have $r' > 0$ and $r' < 0$, respectively. The stationary points at $r = 0$ occur when $\sin\theta = 0$ or $\cos\theta = 0$ and between them $\theta' < 0$. The stationary point at $\theta = 0$, call it P_2, is a hyperbolic saddle. In the case of the stationary point with $\theta = 0$ the lowest order terms are two quadratic terms arising from the terms in the equation for θ'. Let $w = (w_1, w_2) = (r, \pi/2 - \theta))$. Then the leading terms in the last set of equations can be expressed as follows.

$$w_1' = -k_0 w_1^3 + O(|w|^4) \qquad (17)$$

$$w_2' = k_1 w_2^2 - k_2 w_1 w_2 + O(|w|^4) \qquad (18)$$

The order of the error term in the equation for w_2' relies on the fact that only every second term in the Taylor expansion of sin and cos is non-zero. In particular the linearization of this system at the origin is identically zero. Next we transform to polar coordinates (r_1, θ_1) in the (w_1, w_2) plane. This gives

$$r_1' = r_1^2(k_1 \sin\theta_1 - k_2 \cos\theta_1)\sin^2\theta_1 - r_1^3 k_0 \cos^4\theta_1 + O(r_1^4) \qquad (19)$$

$$\theta_1' = r_1(k_1 \sin\theta_1 - k_2 \cos\theta_1)\sin\theta_1 \cos\theta_1 + O(r_1^3) \qquad (20)$$

Note that $\theta_1 = 0$ corresponds to $\theta = \pi/2$. Again we only need to look at θ_1 in the interval $[0, \pi/2]$. A factor r_1 can be removed from the left hand side by yet another change of time coordinate. The result is

$$\frac{dr_1}{ds} = r_1(k_1 \sin\theta_1 - k_2 \cos\theta_1)\sin^2\theta_1 - r_1^2 k_0 \cos^4\theta_1 + O(r_1^3) \qquad (21)$$

$$\frac{d\theta_1}{ds} = (k_1 \sin\theta_1 - k_2 \cos\theta_1)\sin\theta_1 \cos\theta_1 + O(r_1^2) \qquad (22)$$

The set $r_1 = 0$ is an invariant manifold. It follows from the origin of the system that the sets $\theta_1 = 0$ and $\theta_1 = \pi/2$ are invariant but this is not obvious from the formulae. In addition to the new stationary points at $\theta_1 = 0$ and $\theta_1 = \pi/2$ there is one more in the interior of the interval, where $\tan\theta_1 = k_2/k_1$. The stationary point at $\theta_1 = \pi/2$, call it P_3, is a hyperbolic saddle. There are one-dimensional centre manifolds at both the other points. At the point with $\theta_1 = 0$, call it P_4, the r_1-axis is a centre manifold. Thus this point is a sink. At the point with $\tan\theta_1 = k_2/k_1$, call it P_5, the r_1 coordinate line is the centre subspace. Thus the centre manifold at the latter point is of the form $\theta_1 = \theta_* + \phi(r_1) = \theta_* + O(r_1^2)$. The leading order term in the equation for dr_1/ds is $-k_0 r_1^2$. Thus the stationary point is a saddle point.

It has been shown elsewhere that there is exactly one positive stationary solution of this system and that the linearisation is such that either both eigenvalues are positive, both are negative or both are non-real. The stability is determined by the quantity $k_2^3 - k_0^2 k_1$. When this quantity is negative the solution is a hyperbolic sink. When it is positive the solution is a hyperbolic source. When it is zero there are purely imaginary eigenvalues. Consider now what the ω-limit set of a positive solution may be, assuming that the interior stationary point is a source and that there are no periodic solutions. The part of the boundary lying on the x-axis and ending at P_4 cannot contain any ω-limit points. This implies that the same is true for the part of the boundary connecting P_4 with P_5. The part of the boundary joining P_3 to P_5 is also not possible since P_5 is a sink. Of course P_5 is the ω-limit set of all solutions starting in a certain open set. The point P_3 is the ω-limit set of a solution which starts on its centre manifold and of no other solution. If P_3 belongs to the ω-limit set of another solution then P_1 and P_2 must also belong to it, together with the connecting orbits and the centre manifolds of P_1 and P_3. By the Poincaré-Bendixson theory the rest of this set consists of orbits connecting these stationary solutions. The only possibility is that the centre manifolds of P_1 and P_3 coincide. Then the ω-limit set is a heteroclinic cycle and the positive stationary solution is contained in its interior. Any stationary solution which starts in its interior must converge to the heteroclinic cycle for $t \to \infty$. Any solution which starts in its exterior must remain in its exterior and converge to P_4. To sum up, when the stationary solution is unstable there are three possible cases. In the first case there are no periodic solutions and there are solutions which approach the heteroclinic cycle at infinity. In the second case there are no periodic solutions and all positive solutions except the stationary solution approach P_5. In the third case there exists at least one periodic solution. It remains to investigate which of these three cases can actually occur for some values of the parameters.

Selkov writes the equations in dimensionless variables. Suppose we define rescaled variables by $\tilde{t} = \frac{k_1 k_0^2}{k_2^2} t$, $\tilde{x} = \frac{k_0 k_1}{k_2^2} x$ and $\tilde{y} = \frac{k_2}{k_0} y$. Then, dropping the tildes, the system becomes

$$\dot{x} = 1 - xy^2, \tag{23}$$

$$\dot{y} = \alpha(xy^2 - y). \tag{24}$$

where $\alpha = \frac{k_2^3}{k_0^2 k_1}$. Then the stability of the positive stationary solution with coordinates $(1, 1)$ is lost when α increases through one. Let $u = x - 1$ and $v = y - 1$. Then $x = u + 1$, $y = v + 1$ and the system can be rewritten as

$$\dot{u} = -u - 2v - 2uv - v^2 - uv^2, \tag{25}$$

$$\dot{v} = \alpha(u + v + 2uv + v^2 + uv^2). \tag{26}$$

The determinant of the linearization at the origin is α and the trace is $-1+\alpha$. We see that there is a transition from stability to instability as α increases through one. There are also quadratic and cubic terms. Next the Lyapunov coefficient will be calculated. In the notation used by Perko the non-vanishing coefficients are $a = -1$, $b = -2$, $c = 1$, $d = 1$, $a_{11} = -2$, $a_{02} = -1$, $b_{11} = 2$, $b_{02} = 1$, $a_{12} = -1$ and $b_{12} = 1$. I am only interested in the sign of the coefficient and so I ignore the numerical factors. I ignore the factor involving the determinant, which is positive. The factor involving b is negative and so the sign of σ the equation on p. 353 of Perko is equal to the sign of the expression in curly brackets. Let us look first at the contribution from the third order terms. The factor in front of the square brackets is equal to one. The total contribution is -3. Consider next the quadratic terms. The first, third, fifth and sixth terms make no contribution. It remains to compute the second, fourth and seventh terms, which make contributions 4, 2 and -8. Thus the value of the total expression is -5. The Lyapunov coefficient is negative and for α slightly greater than one there is a unique stable periodic orbit close to the point $(1,1)$. Thus case 3 does occur for some parameter values.