10 KINDS OF MATTER

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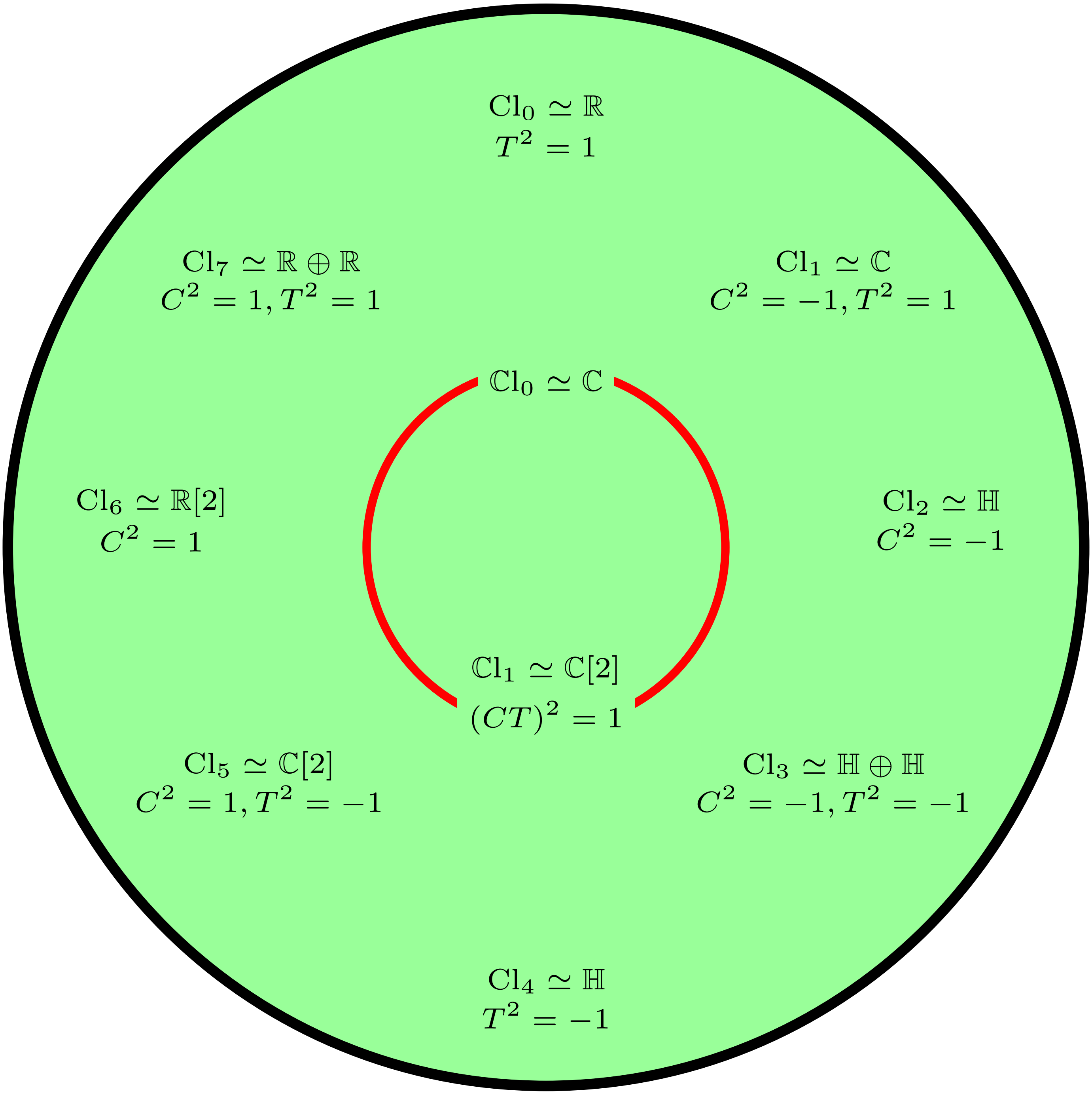

In 1996, Altland and Zirnbauer classified phases of matter into 10 kinds based on charge conjugation (C) and time reversal (T) symmetry. They can have an antiunitary C symmetry operator with $C^2 = \pm 1$ or lack this symmetry. They can have an antiunitary T operator with $C^2 = \pm 1$ or lack this. This gives $3 \times 3 = 9$ kinds. They can also lack C and T symmetry but still have a unitary operator corresponding to CT; by adjusting the phase we may assume $(CT)^2 = 1$.

Kitaev, Moore and Freed showed these 10 kinds of matter correspond to the 10 division superalgebras. There are 3 (associative) division algebras: the real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, and quaternions $\mathbb{H}$. A 'division superalgebra' is a $\mathbb{Z}_2$-graded generalization, with a bosonic and fermionic part. 8 of these have the same categories of graded representations as the real Clifford algebras called $\mathrm{Cl}_0, \dots, \mathrm{Cl}_7$ in the chart. The other 2 have the same categories of graded representations as the complex Clifford algebras $\mathbb{C}\mathrm{l}_0$ and $\mathbb{C}\mathrm{l}_1$.

The chart shows how the 10 kinds of matter correspond to the 10 division superalgebras. For example, consider matter with only T symmetry obeying $T^2 = -1$. Since T is antiunitary we have $iT = -Ti$. Thus i, T and iT are 3 anticommuting square roots of one, giving a representation of the quaternions $\mathbb{H}$. The category of representations of $\mathbb{H}$ is equivalent to the category of representations of $\mathrm{Cl}_4 = \mathbb{H}[2]$, meaning 2×2 matrices with entries in $\mathbb{H}$. So, we write $\mathrm{Cl}_4 \simeq \mathbb{H}$. An algebra can sometimes be made into a division superalgebra in more than one way. For example, $\mathrm{Cl}_4 \simeq \mathbb{H}$ where $i, j, k \in \mathbb{H}$ are even, while $\mathrm{Cl}_2 \simeq \mathbb{H}$ where $i, j \in \mathbb{H}$ are odd.

When we combine two physical systems we tensor their Hilbert spaces, and tensor the algebras these Hilbert spaces are representations of. In fact:

$$\mathrm{Cl}_i \otimes_{\mathbb{R}} \mathrm{Cl}_{i'} \simeq \mathrm{Cl}_{i+i' \bmod 8} \qquad \mathbb{C}\mathrm{l}_j \otimes_{\mathbb{C}} \mathbb{C}\mathrm{l}_{j'} \simeq \mathbb{C}\mathrm{l}_{j+j' \bmod 2} \qquad \mathrm{Cl}_i \otimes_{\mathbb{R}} \mathbb{C}\mathrm{l}_j \simeq \mathbb{C}\mathrm{l}_{i+j \bmod 2}$$

where $\simeq$ means they have equivalent categories of graded representations. This gives the 10-element set $\mathbb{Z}_8 \cup \mathbb{Z}_2$ an associative product and multiplicative identity, but not inverses — so we say it is a 'monoid' but not a group.

We have proved that for any field k, the associative division superalgebras over k form a monoid, the 'super Brauer monoid' of k. When $k = \mathbb{R}$ this gives the monoid $\mathbb{Z}_8 \cup \mathbb{Z}_2$, and the rules for combining the 10 kinds of matter.