Higher Algebra, Extended TQFTs, and 3D Quantum Gravity

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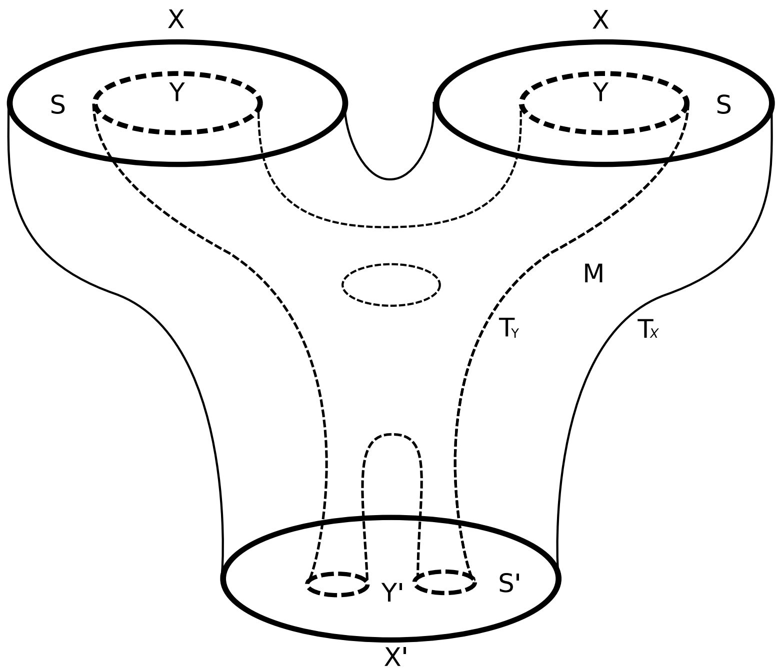
July 6, 2006

Joint work with John Baez



1 Overview

The plan of the talk is as follows:

1. TQFT's and Extended TQFT's
2. $\mathbf{nCob}_2$ and **2Vect**
3. The Fukuma-Hosono-Kawai construction and Pachner Moves
4. Restricting to a TQFT
5. The Group 2-algebra $\mathbf{Vect}[G]$
6. $G = SU(2)$: Particles in Ponzano-Regge vacuum

Topological Quantum Field Theories

A TQFT can be seen [Ati] as a "monoidal functor":

$$Z : \mathbf{nCob} \to \mathbf{Vect} \tag{1}$$

nCob: a tensor category of $(n-1)$-dimensional manifolds and n-dimensional cobordisms between them (composed by gluing along boundaries).

Vect: the tensor category of vector spaces and linear maps[1]), with the usual tensor product.

Being a monoidal functor means: Z gives a *space of states* $Z(S)$ for any $(n-1)$-manifold S, and linear maps $Z(M) : Z(S_1) \longrightarrow Z(S_2)$ for cobordisms $M : S_1 \longrightarrow S_2$, such that:

$$Z(M_2 \circ M_1) = Z(M_2) \circ Z(M_1)$$

and

$$Z(S_1 \amalg S_2) = Z(S_1) \otimes Z(S_2) \text{ and } Z(\varnothing) = \mathbb{C}$$

[1]The vector spaces given here will have a natural inner product making them Hilbert spaces.

Extended TQFTs

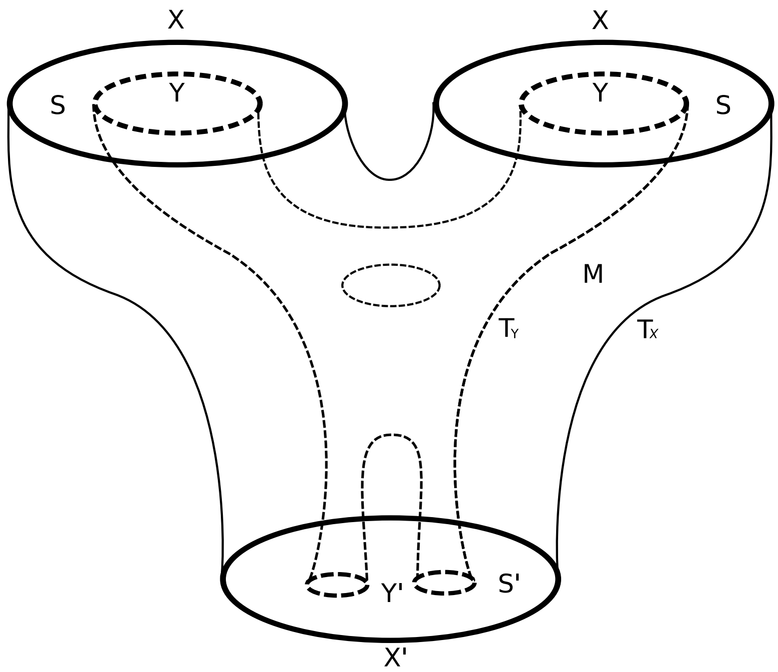
An *extended TQFT* assigns "2-algebraic" data to the codimension 0, 1, and 2 faces of a cobordism with corners. It is a "2-functor":

$$Z : \mathbf{nCob}_2 \to \mathbf{2Vect}$$

$\mathbf{nCob}_2$: a "2-category" of $(n-2)$-manifolds, $(n-1)$-cobordisms between them, and n-cobordisms between these

2Vect: the 2-category of 2-vector spaces, 2-linear maps, and natural transformations.

The *Dijkgraaf-Witten (DW) models* are "categorified" analogs of FHK, related to gauge theory with gauge group G, which give extended TQFTs using triangulations of cobordisms with corners:



2-Vector Spaces

2Vect is a 2-category with[2]:

- **objects**: *2-vector spaces*, which are all isomorphic to $\mathbf{Vect^k}$, whose objects are just k-tuples of vector spaces, morphisms are tuples of linear maps
- **morphisms**: matrices of vector spaces, acting on $\mathbf{Vect^k}$ by "matrix multiplication" using tensor product and direct sum
- **2-morphisms**: natural transformations

$$\begin{tikzcd} \mathbf{Vect^k} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & \mathbf{Vect^l} \arrow[from=U, to=D, Rightarrow, "\alpha"] \end{tikzcd} \tag{2}$$

Morphisms of 2-vector spaces are functors $F : \mathbf{Vect^k} \to \mathbf{Vect^l}$, which amount to matrices of vector spaces, acting by matrix multiplication using the direct sum and tensor product instead of operations in $\mathbb{C}$. The natural transformations between these are matrices of linear transformations:

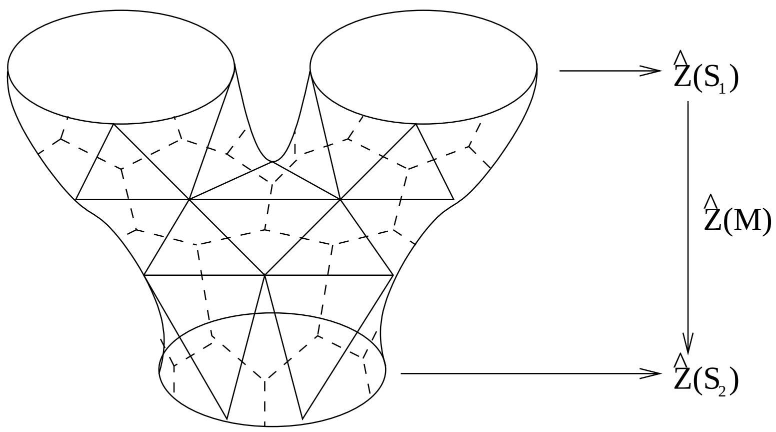
$$\alpha = \begin{pmatrix} f & g \\ h & k \end{pmatrix} : \begin{pmatrix} \mathbb{C}^7 & \mathbb{C}^2 \\ \mathbb{C} & \mathbb{C}^5 \end{pmatrix} \longrightarrow \begin{pmatrix} \mathbb{C}^3 & \mathbb{C}^6 \\ \mathbb{C}^9 & \mathbb{C}^2 \end{pmatrix}$$

[2] Objects of **2Vect** are $\mathbb{C}$-linear finitely generated semisimple abelian categories: categories $\mathbf{C}$ with $\oplus$, kernels and cokernels, where $\hom(x, y)$ for $x, y \in \mathbf{C}$ is a vector space; and such that all objects in $\mathbf{C}$ are direct sums of simple objects, which come in finitely many different kinds. Morphisms of iiV are "exact $\mathbb{C}$-linear functors".

The Fukuma, Hosono and Kawai Construction

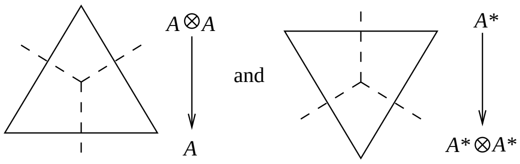
Fukuma, Hosono and Kawai show [FHK] how 2D TQFT's can be constructed by any semisimple algebra A (if A is a Lie algebra, this means its Killing form is nondegenerate, but there are other examples). Our standard example will be $A = \mathbb{C}[G]$ for some finite group G. (We can also use Lie groups such as $SU(2)$, but this needs more analysis.) It will turn out that $Z(S^1)$ is the centre of A. To get this, one first constructs a map $\hat{Z} : \mathbf{\Delta 2Cob} \to \mathbf{Vect}$, where $\mathbf{\Delta 2Cob}$ is the category of *triangulated* 1-manifolds and 2-cobordisms.

Given a triangulated cobordism M from S_1 to S_2, (so M, S_1 and S_2 are all triangulated), label the dual graph with copies of A.



Pachner Moves in 2D TQFTs

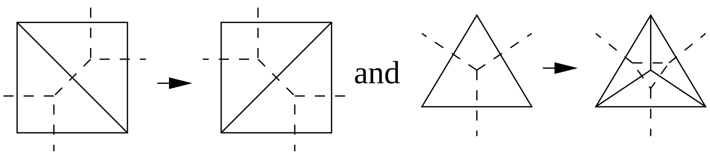
In FHK: edges of dual graph are labelled by A; vertices by an operator m.



If the triangulation of S_i consists of n edges, then we define

$$\hat{Z}(S_i) = A^{\otimes n} \tag{3}$$

So we can define $\hat{Z}(M)$ to be the composite of all the operators in the diagram, which is independent of how we order them, provided (A, m) is a semisimple algebra. This gives $\hat{Z}(M) : \hat{Z}(S_1) \to \hat{Z}(S_2)$, which depends only on the topology of the cobordism and the triangulation of the boundary: **Alexander's Theorem** says any two triangulations of the same compact 2-manifold which agree on the boundary are related by a sequence of *Pachner moves* - the **2-2 move** and the **1-3 move** (and their inverses):

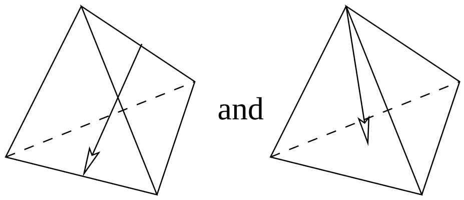


These are equations because of associativity, and semisimplicity.

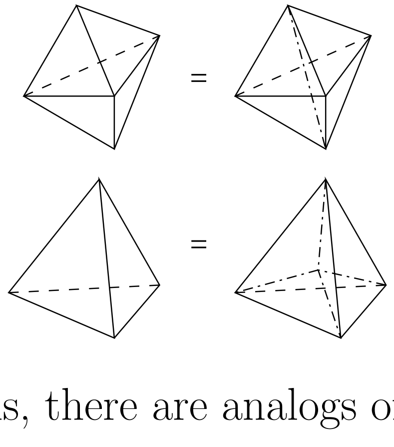
Pachner Moves in Higher Dimensions

Tetrahedrons in triangulations for the 3D DW model correspond to the Pachner moves in 2-D: attaching a tetrahedron to a triangulation along one, two, or three triangular faces gives a move by replacing the attached faces with the remaning faces of the tetrahedron.

To increase dimension, we replace the equations given by the Pachner moves with 2-morphisms. Each move gives a relation between a pair of morphisms in **2Vect**, corresponding to a tetrahedron thought of as connecting two parts of its boundary:



These obey coherence laws (equations) given by the 2-3 and 1-4 Pachner moves:



For higher dimensions, there are analogs of this fact.

3D TQFT's and Semisimple 2-Algebras

A 2-functor $Z : \mathbf{3Cob_2} \to \mathbf{2Vect}$ must assign higher-algebraic operators to the 3-cell of a tetrahedron, such as the associators, which correspond to Pachner moves:

$$\begin{tikzcd} A^{\otimes 3} \arrow[r, bend left, "m\circ(1\otimes m)"{name=U}] \arrow[r, bend right, "m\circ(m\otimes 1)"'{name=D}] & A \arrow[Rightarrow, from=U, to=D, "\alpha"] \end{tikzcd} \tag{4}$$

where

$$\alpha_{x,y,z} : x \otimes (y \otimes z) \to (x \otimes y) \otimes z \tag{5}$$

is unique up to "twisting". Resolved in a basis, associators give the $6j$-symbols.

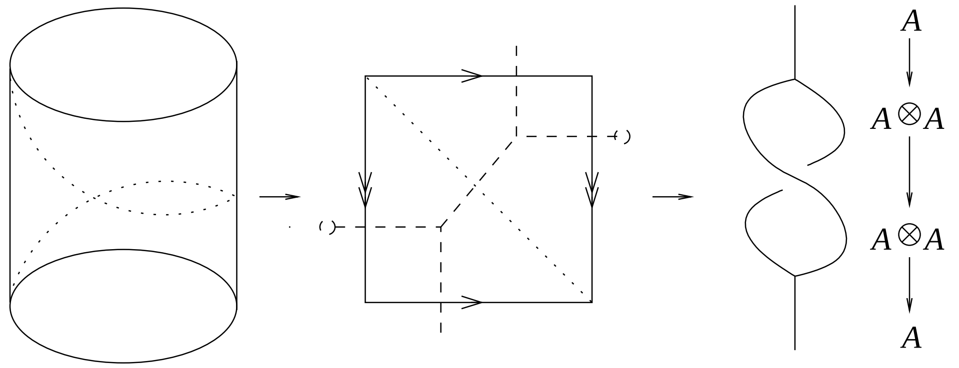
These moves correspond to the identity which associators must satisfy, the *pentagon identity*:

$$\begin{tikzcd} & (f\otimes g)\otimes(h\otimes j) \arrow[dr, "a_{f,g,h\otimes j}"] & \\ ((f\otimes g)\otimes h)\otimes j \arrow[ur, "a_{f\otimes g,h,j}"] \arrow[d, "a_{f,g,h}\otimes 1_j"'] & & f\otimes(g\otimes(h\otimes j)) \\ (f\otimes(g\otimes h))\otimes j \arrow[r, "a_{f,g\otimes h,j}"'] & f\otimes((g\otimes h)\otimes j) \arrow[ur, "1_f\otimes a_{g,h,j}"'] & \end{tikzcd} \tag{6}$$

and to the 2-algebraic property of semisimplicity.

Restricting to a TQFT

We want the identity map on $Z(S^1)$ to correspond to the cylinder, which has a triangulation giving a projection into the centre of A:



so we can define

$$Z(S^1) = \mathrm{Ran}(\hat{Z}(S^1 \times [0,1])) \tag{7}$$

To get a TQFT Z, we restrict $\hat{Z}$ to $Z(S^1)$. This is a projection operator, and its range is in $Z(A)$ (i.e. the centre of A) and say:

$$Z(M) = \hat{Z}(M)|_{Z(S^1)} \tag{8}$$

Which is well defined, since we always have $\hat{Z}(M)(Z(S_1)) = Z(S_2)$, as one can see by changing the triangulation of M to include the projection operator.

Then one can show that this Z defines a *symmetric monoidal functor* from **2Cob** to **Vect**, namely a TQFT. Next we see how this categorifies.

The Group 2-Algebra $\mathbf{Vect}[G]$

The Dijkgraaf-Witten model - a topological gauge theory - also gives a 3D "extended TQFT" from the (semisimple) group 2-algebra of a finite gauge group G. This is (approximately) the category of "vector bundles" on G.

The group algebra $\mathbb{C}[G]$ consists of the set of elements formed as formal linear combinations elements of G:

$$b = \sum_{g \in G} b_g \cdot g$$

where all but finitely many b_g are zero. We can think of these as complex functions on G.

The *group 2-algebra* $A = \mathbf{Vect}[G]$ is the *category* of vector bundles V over G. That is, direct sums of vector spaces associated to elements of G:

$$V = \bigoplus_{g \in G} V_g$$

Where $V_g \in \mathbf{Vect}$ is a vector space. We can take direct sums of these pointwise, so that $(V \oplus V')_g = V_g \oplus V'_g)$, and there is a "scalar" product with elements of $\mathbf{Vect}$ given by $(W \otimes V)_g = W \otimes V_g$.

The algebra multiplication on $\mathbb{C}[G]$ is given by the multiplication in G:

$$b \star b' = \sum_{g,g' \in G} (b_g b'_{g'}) \cdot gg'$$

This does not correspond to the multiplication of functions on G, but to *convolution*:

$$(b \star b')_g = \sum_{h \cdot h' = g} b_h \cdot b'_{h'}$$

There is also a *group 2-algebra* product of "bundles" involving a convolution on G:

$$(V \star V')_h = \bigoplus_{g \cdot g' = h} V_g \otimes V'_{g'}$$

When G is a Lie group, and the vector spaces have an inner product, we can use a direct integral instead of a direct sum. In the case of infinite groups like $G = SU(2)$, since there is no local triviality condition, one gets the category of *coherent sheaves of vectorspaces* on G, rather than vector bundles.

Categorified Centre of Group 2-Algebra

To construct a TQFT, we restricted $\tilde{Z}$ to the centre of A. To construct the 3D extended TQFT, we need to know the categorified centre $Z(\mathbf{Vect}[G])$.

To find $Z(\mathbf{Vect}[G])$, instead of specifying that its elements commute with all others, we require that they do so up to an isomorphism called the *braiding*. Equivalently, conjugation by $g \in G$ gives an isomorphism (not necessarily the identity!)

So an object of $Z(\mathbf{Vect}[G])$ is a "bundle" $E \to G$ equipped with isomorphisms

$$R_g : E \to gEg^{-1} \tag{9}$$

such that

$$R_g R_h = R_{gh} \tag{10}$$

Where gEg^{-1} has the same fibres as E, but the vector space lying over base point h in E now lies over the point ghg^{-1} in gEg^{-1}.

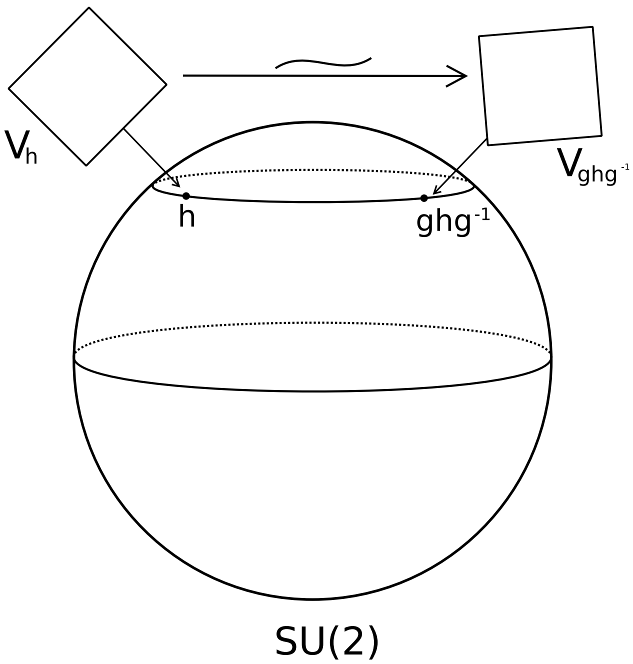
When G is a Lie group and $A = \mathbb{C}[\mathbb{G}]$, there is more analysis, but the result is similar, and one has:

$$Z(S^1) = \mathcal{L}^2(G/\operatorname{Ad}(G))$$

where $\operatorname{Ad}(G)$ is the adjoint action of G on itself by conjugation. These are functions which are constant on conjugacy classes of G, or equivalently functions on the congugacy classes.

Ponzano-Regge Model: $G = SU(2)$

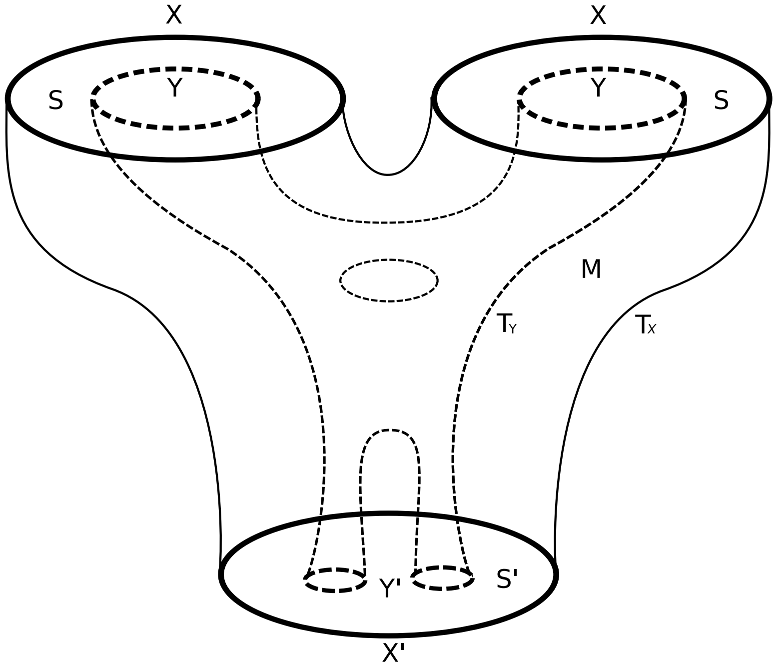
If $G = SU(2)$, the objects of $Z(\mathbf{Vect}[G])$ can be represented in terms of a basis of irreducible "bundles" (really coherent sheaves). These amount to a choice of conjugacy class in $SU(2)$ and action of $SU(2)$ on its fibres coming from the isomorphism assocated to conjugation by g.



This model reproduces the Ponzano-Regge model for 3D quantum gravity. We interpret a conjugacy class in SU(2) is an angle in $[0, \pi]$, which is a *mass* m; In the case $m = 0$ the isomorphism is just a spin - an irrep of $SU(2)$, labelled by an integer. For other m, we get a spin when we "trivialize the bundle".

3D Spacetimes With Holes As Matter

We can think of a tube removed from the space below to be a Feynman graph for one or more particles.



This extended TQFT gives "particles" - boundaries in space - labelled by a representation of a certain group. Our example was derived from a finite group, but if $G = SU(2)$ the label is a mass and spin) moving on a background described by Ponzano-Regge quantum gravity (see work by Freidel, Livine, Louapré, Krasnov et. al.). (The action is given by the choice of associator.)

The same framework will work with other groups. Quantum groups such as $SU(2)_q$ are expected to recover the Turaev-Viro model.

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Higher Dimensions: $\mathbf{nCob}_2$

$\mathbf{nCob}_2$ is a "2-category" with $(n-2)$ manifolds for objects, $(n-1)$ cobordisms for morphisms, and n cobordisms with corners for 2-cells. We draw diagrams in $\mathbf{nCob}_2$ like this:

$$\begin{tikzcd} X \arrow[r, "S"] \arrow[d, "T_X"'] \arrow[dr, phantom, "\swarrow M"] & Y' \arrow[d, "T_Y"] \\ X' \arrow[r, "S'"'] & Y' \end{tikzcd} \tag{11}$$

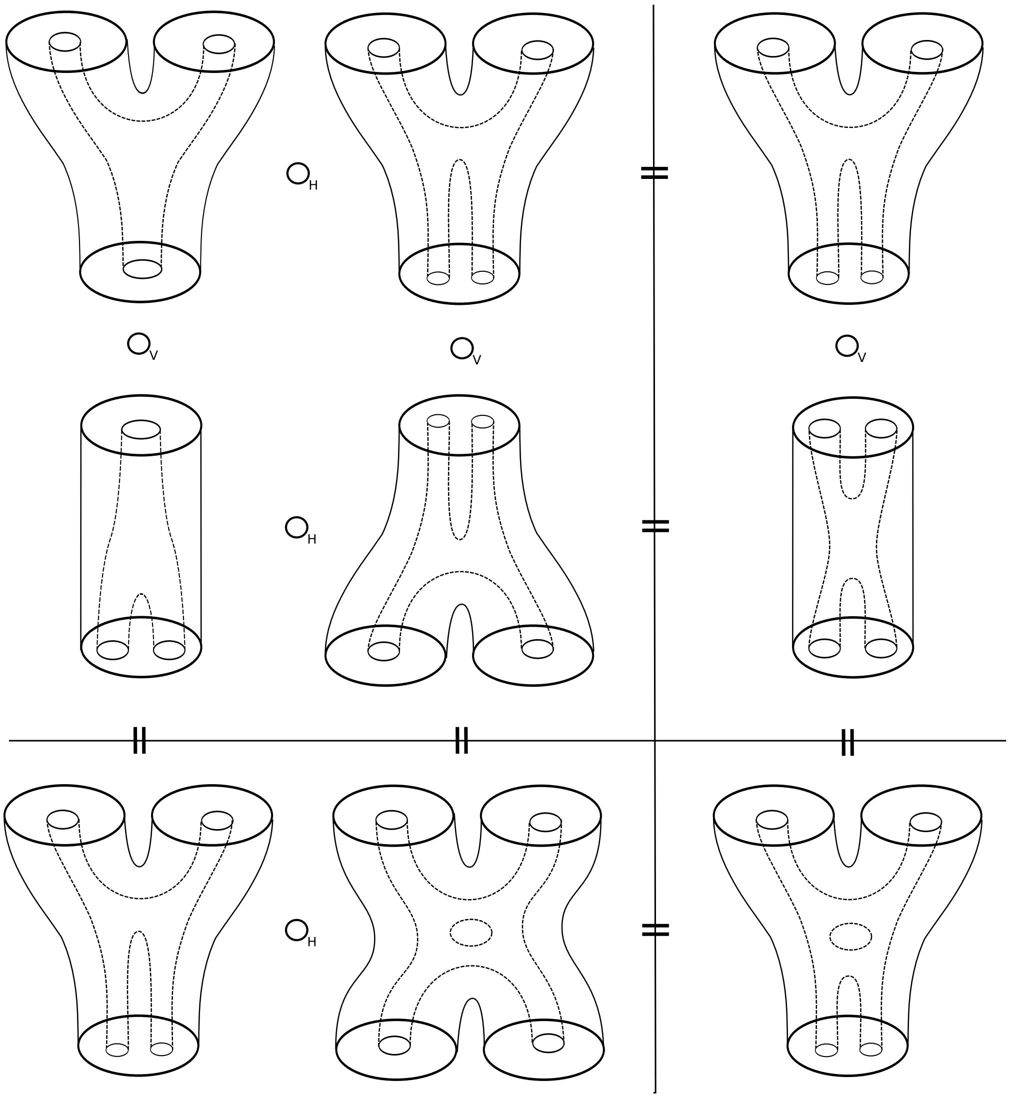
One can simplify these diagrams, producing ones like this:

$$\begin{tikzcd} X \arrow[r, bend left, "T_Y \circ S"] \arrow[r, bend right, "S' \circ T_X"'] \arrow[r, phantom, "\Downarrow M"] & Y' \end{tikzcd} \tag{12}$$

The most important operations in $\mathbf{nCob}_2$ are the tensor product (disjoint union) and the composition operations. There are "horizontal" and "vertical" composition operations for both morphisms and 2-cells. These satisfy various axioms, including the interchange law:

$$(F \otimes_V F') \otimes_H (G \otimes_V G') = (F \otimes_H G) \otimes_V (F' \otimes_H G') \tag{13}$$

Interchange Law in $\mathbf{nCob_2}$



This shows that horizontal and vertical composition can be interchanged. The convention in this picture is that horizontal composition pastes the “outside” surface of the right hand cobordism to the “inside” surface of the left.