MOTIVATING MOTIVES

Among all the mathematical discoveries
which I've been privileged to make, the concept of the motive still
impresses me as the most fascinating,
the most charged with mystery —
indeed at the very heart of the profound identity
of geometry and arithmetic.

Alexander Grothendieck, *Récoltes et Semaille*

John C. Baez
Grothendieck's Approach to Mathematics
Chapman University
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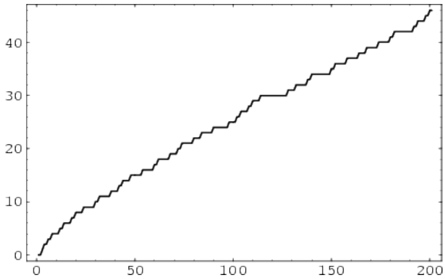
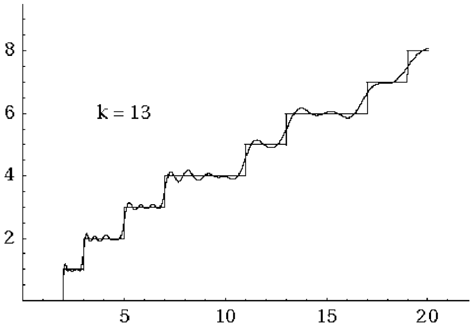
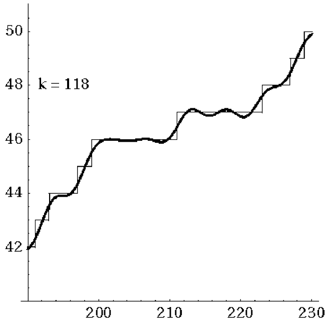
Riemann came up with a formula for the prime counting function: $\pi(n)$ is the number of primes $\leq n$:



The prime counting function can be written as a main term plus a sum of wave-like "corrections", one for each nontrivial zero of the Riemann zeta function:



So, in some very vague sense we're seeing wave-particle duality:



If primes are "particles", zeros of the Riemann zeta function correspond to "waves".

Since the prime counting function $\pi(n)$ equals the "main term"

$$\mathrm{li}(n) = \int_0^n \frac{dt}{\ln t}$$

plus corrections coming from the nontrivial Riemann zeta zeros, knowing the location of these zeros would give more information about the prime counting function.

Indeed the Riemann Hypothesis:

All nontrivial Riemann zeta zeros lie on the line $\mathrm{Re}(z) = \frac{1}{2}$

is equivalent to the claim that

$$|\pi(n) - \mathrm{li}(n)| \leq C\sqrt{n}\ln n$$

for some $C > 0$ and $n \geq 1$.

To prove the Riemann Hypothesis,
it might help to know what the Riemann zeta zeros *mean* —
— other than just providing oscillatory corrections to the prime counting function.

The Weil Conjectures give an easier context in which to study this kind of question. In this variant, the count of points on an algebraic variety over a finite field has a "main term" and some "correction terms".

Each of these terms corresponds to a "motive".

Let's look at an example.

Let's count the number of solutions of

$$y^2 + y = x^3 + x$$

over the finite field with $q = p^n$ elements, called $\mathbb{F}_q$. Take $p = 2$:

n	$q = p^n$	*number of solutions*
1	2	4
2	4	4
3	8	4
4	16	24
5	32	24
6	64	64
7	128	144
8	256	224
9	512	544
10	1024	1024
11	2048	1984
12	4096	4224

Since $y^2 + y = x^3 + x$ is one equation with two unknowns we might naively guess that over the field with p^n elements it has p^n solutions.

This is pretty close! This is the "main term". Let's subtract it off:

n	*correction term*
1	2
2	0
3	-4
4	8
5	-8
6	0
7	16
8	-32
9	32
10	0

Hasse's Theorem on Elliptic Curves (1933)

Given a cubic equation with integer coefficients in two variables that defines an elliptic curve, the number of solutions in the field with p^n elements is

$$p^n - \alpha^n - \overline{\alpha}^{\,n}$$

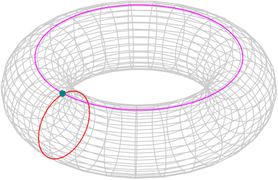
where $\alpha \in \mathbb{C}$ has $|\alpha| = \sqrt{p}$.

The elliptic curve also has a point at infinity, so its number of points over the field with p^n elements is

$$p^n - \alpha^n - \overline{\alpha}^n + 1$$

where $\alpha \in \mathbb{C}$ has $|\alpha| = \sqrt{p}$.

The four terms correspond, in some profound way, to these four pieces of the elliptic curve over $\mathbb{C}$, which is a torus:



The pieces of dimension k give the terms that grow like $p^{\frac{k}{2}n}$.

With a lot of work, Weil generalized Hasse's result to curves of arbitrary genus.

Weil's Theorem (1940–1948)

Given a smooth algebraic curve of genus g defined over the field with p elements, its number of points over the field with p^n elements is

$$p^n - \alpha_1^n - \cdots - \alpha_{2g}^n + 1$$

where all the $\alpha_i \in \mathbb{C}$ have $|\alpha_i| = \sqrt{p}$.

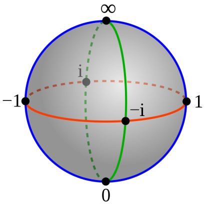
The simplest example: the projective line $\mathbb{P}^1$. Over the field with $q = p^n$ elements this has

$$q + 1$$

points.

It's easy to split $\mathbb{P}^1$ into two parts: a copy of $\mathbb{F}_q$, with q points, and a point at infinity.

Over the complex numbers $\mathbb{P}^1$ looks like this:



Over the field with q elements the projective plane $\mathbb{P}^2$ has

$$q^2 + q + 1$$

elements. It breaks up into:

- A copy of $\mathbb{F}_q^2$, with q^2 elements.
- A copy of $\mathbb{F}_q^1$, with q elements.
- A copy of $\mathbb{F}_q^0$, with 1 element.

Higher-dimensional projective spaces follow the same pattern.

Over the complex numbers, the projective space $\mathbb{P}^d$ has no cohomology in odd dimensions, and a rank-1 cohomology group in dimensions

$$1, 2, 4, \ldots, 2d$$

because it's made of chunks called "Schubert varieties" that are copies of

$$\mathbb{C}^0, \mathbb{C}^1, \mathbb{C}^2, \ldots, \mathbb{C}^d$$

Over the field with q elements, $\mathbb{P}^d$ is made of Schubert varieties that are copies of

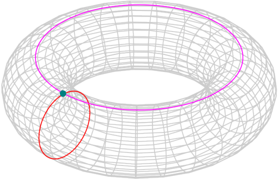
$$\mathbb{F}_q^0, \mathbb{F}_q^1, \mathbb{F}_q^2, \ldots, \mathbb{F}_q^d$$

so its number of points is

$$1 + q + q^2 + \cdots + q^d$$

Some other interesting projective varieties — “flag varieties” — are also made of chunks called Schubert varieties, and the same reasoning applies to them. But most varieties are not so simple!

Elliptic curves illustrate the extra subtleties. As we’ve seen, they have both even- and odd-dimensional cohomology over $\mathbb{C}$:



And over $\mathbb{F}_q$ their number of points is not a polynomial in $q = p^n$. It’s

$$p^n - \alpha^n - \overline{\alpha}^n + 1$$

with $|\alpha| = p^{1/2}$.

Riemann Hypothesis for Varieties over Finite Fields (Conjectured by Weil in 1949)

Given a d-dimensional smooth projective variety defined over the field with p elements, its number of points over $\mathbb{F}_{p^n}$ is

$$\sum_{k=0}^{2d} \sum_{i=1}^{\beta_k} (-1)^k \alpha_{ik}^n$$

where $|\alpha_{ik}| = p^{k/2}$ and β_k is the kth "Betti number" of the variety — that is, the rank of its kth cohomology group.

This conjecture is usually phrased in terms of a "zeta function". It claims this has zeros and poles on the lines $\mathrm{Re}(z) = \frac{k}{2}$: zeros when k is odd and poles when k is even.

Remember the formula:

$$\text{number of points over } \mathbb{F}_{p^n} \quad = \quad \sum_{k=0}^{2d} \sum_{i=1}^{\beta_k} (-1)^k \alpha_{ik}^n$$

where β_k is the kth Betti number of the variety and $|\alpha_{ik}| = p^{k/2}$.

Grothendieck's dream: we can always break the variety into abstract chunks called *motives* of dimension $k = 0, 1, \ldots, 2d$.

The k-dimensional motives contribute terms of the form $(-1)^k \alpha_{ik}^n$ to the number of points.

So, each chunk contributes to the number of points in our variety. And it can contribute a negative number of points!

Indeed, there is a category Var of smooth projective varieties and regular maps over $\mathbb{F}_p$, and a functor

$$h\colon \mathrm{Var}^{\mathrm{op}} \to \mathrm{Mot}$$

where Mot, the category of "pure Chow motives", resembles categories we know from linear algebra:

- It is a "linear category": the hom-sets are vector spaces, and composition is bilinear.
- It is "Cauchy complete", or "Karoubian": it has direct sums, and any $p\colon X \to X$ with $p^2 = p$ is projection onto Y for some direct sum decomposition $X \cong Y \oplus Z$.
- It is "symmetric monoidal": it has a well-behaved tensor product $\otimes$, coming from the cartesian product of smooth projective varieties.

We can map all of $\mathbb{P}^1$ to a single point:

$$p\colon \mathbb{P}^1 \to \mathbb{P}^1$$

and this map clearly has $p^2 = p$.
Thus, in the category Mot we have

$$h(\mathbb{P}^1) = h(1) \oplus \mathbb{L}$$

where 1 corresponds to the point and $\mathbb{L}$ is some motive called the **Lefschetz motive**.

Similarly, we have

$$h(\mathbb{P}^n) \;\cong\; h(1) \,\oplus\, \mathbb{L} \,\oplus\, \mathbb{L}^{\otimes 2} \,\oplus\, \cdots \,\oplus\, \mathbb{L}^{\otimes n}$$

and this corresponds to the formula we saw for the number of points of $\mathbb{P}^n$ over $\mathbb{F}_q$:

$$1 \,+\, q \,+\, \cdots \,+\, q^n$$

But curves and other varieties typically decompose into motives that are *not* just tensor powers of the Lefshetz motive $\mathbb{L}$.

We can construct the category Mot of pure Chow motives by hand starting from Var.

We can also characterize it via a univeral property: every “Weil cohomology theory”

$$H\colon \mathsf{Var}^{\mathrm{op}} \to \mathsf{Vect}$$

factors uniquely through

$$h\colon \mathsf{Var}^{\mathrm{op}} \to \mathsf{Mot}$$

Weil cohomology theories were introduced by Grothendieck and others to help prove the Weil conjectures.

Grothendieck showed the Riemann Hypothesis for finite fields would follow from the so-called "Standard Conjectures".

Among other things, these conjectures would imply:

- Every variety X has $h(X) \cong X_0 \oplus \cdots \oplus X_n$ where the motive X_k has "weight" k, meaning it contributes terms of the form α^n with $|\alpha| = p^{k/2}$ to the count of points of X over $\mathbb{F}_{p^n}$.
- The category Mot is "abelian": it has well-behaved kernels, cokernels, subobjects and quotient objects.
- The category Mot is "semisimple": every motive is a finite direct sum of motives that have only two subobjects, 0 and that motive itself.

Alas, Grothendieck was unable to prove the Standard Conjectures.

They remain unproved to this day!

It's still not known if Mot has the desirable properties I just listed.

Deligne proved the Riemann Hypothesis for varieties over finite fields in 1974 in a way that sidestepped these questions.

Thus, motives remain deeply mysterious.

In 1995, Manin outlined a dream of proving the actual Riemann Hypothesis by generalizing motives to the mystical "field with one element":

- Yuri Manin, Lectures on zeta functions and motives.

Connes has been trying to prove the Riemann Hypothesis using these ideas. An early report, also from 1995, is his book:

- Alain Connes and Matilde Marcolli, *Noncommutative Geometry, Quantum Field Theory and Motives*.

Pure motives are just a special case of "mixed motives", which aim to handle varieties that are not smooth, or not projective.

These are also mysterious —
but Voevodsky essentially managed to define the derived category of mixed motives without defining mixed motives themselves!
In 1996 he used these ideas to solve the Milnor conjecture, triggering a burst of new work.

So, motives continue to tantalize and inspire!