Space and State, Spacetime and Process

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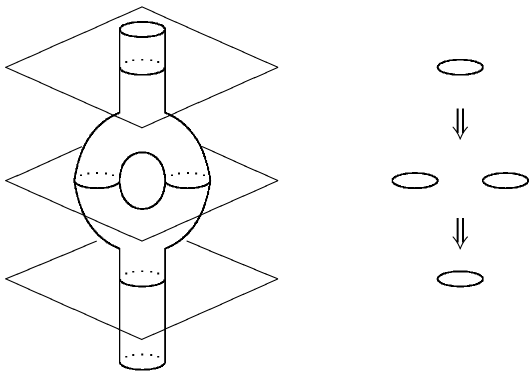


many figures by Aaron Lauda

Right now physicists are trying to reconcile general relativity (our best theory of spacetime) and quantum theory (our best theory of physical processes).

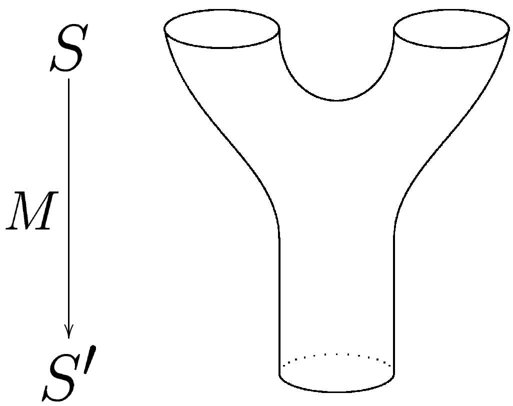
Luckily, there's an analogy to guide us:

GENERAL RELATIVITY	**QUANTUM THEORY**
$(n-1)$**-dimensional manifold (space)**	**Hilbert space (states)**
cobordism between $(n-1)$**-dimensional manifolds (spacetime)**	**operator between Hilbert spaces (process)**
composition of cobordisms	**composition of operators**
identity cobordism	**identity operator**

In both cases there is a **symmetric monoidal category with duals** at work:

- nCob: n-dimensional cobordisms between (compact framed) $(n-1)$-manifolds.
- Hilb: linear operators between (finite-dimensional) Hilbert spaces.

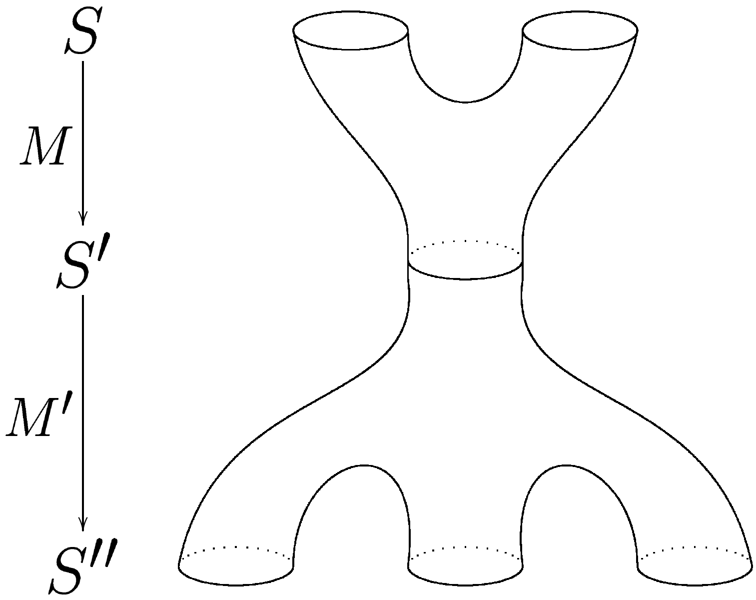
Morphisms



A cobordism

A morphism in Hilb is a linear operator $T\colon H \to H'$ from the Hilbert space H to the Hilbert space H'.

Composition



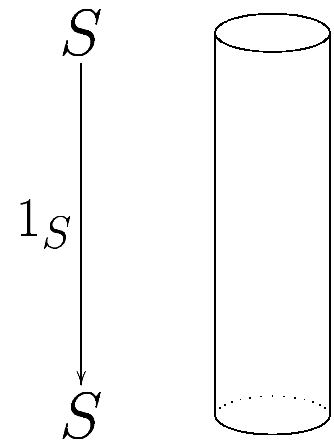
Composition of cobordisms

Composition in Hilb is defined by

$$(ST)\psi = S(T\psi)$$

for all $T\colon H \to H'$, $S\colon H' \to H''$, and $\psi \in H$.

Identity Morphisms



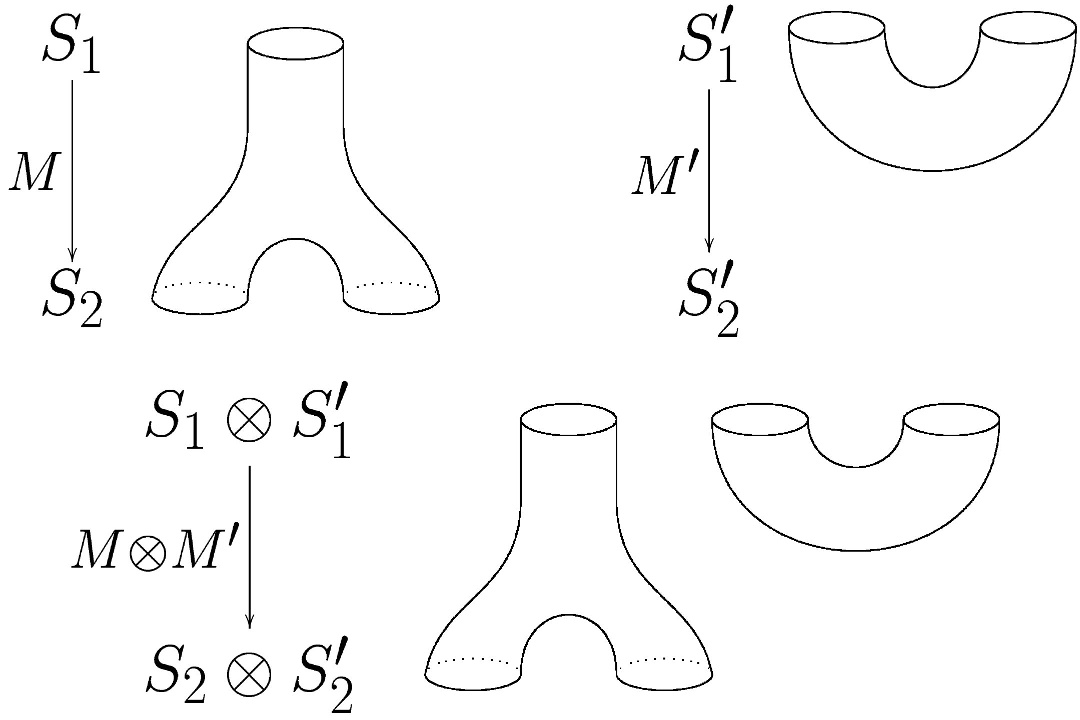
An identity cobordism

Identity morphisms in Hilb are given by

$$1_H \psi = \psi$$

for all $H \in$ Hilb and $\psi \in H$.

Tensor Product



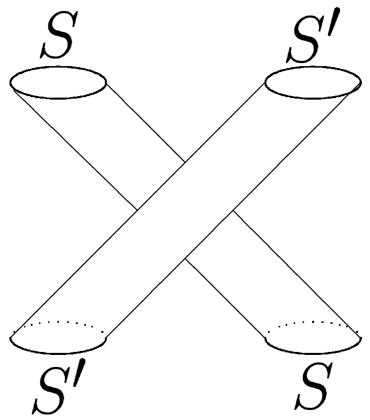
Tensor product of cobordisms

The tensor product of Hilbert spaces is their algebraic tensor product completed to become a Hilbert space; the tensor product of operators $T\colon H \to K$ and $T'\colon H' \to K'$ is given by:

$$(T \otimes T')(\psi \otimes \psi') = T\psi \otimes T'\psi'.$$

The Braiding

$B_{S,S'}$ =



Braiding in nCob

The braiding in Hilb is given by

$$B_{H,H'}(\psi \otimes \psi') = \psi' \otimes \psi$$

for all $H, H' \in$ Hilb, $\psi \in H$ and $\psi' \in H'$.

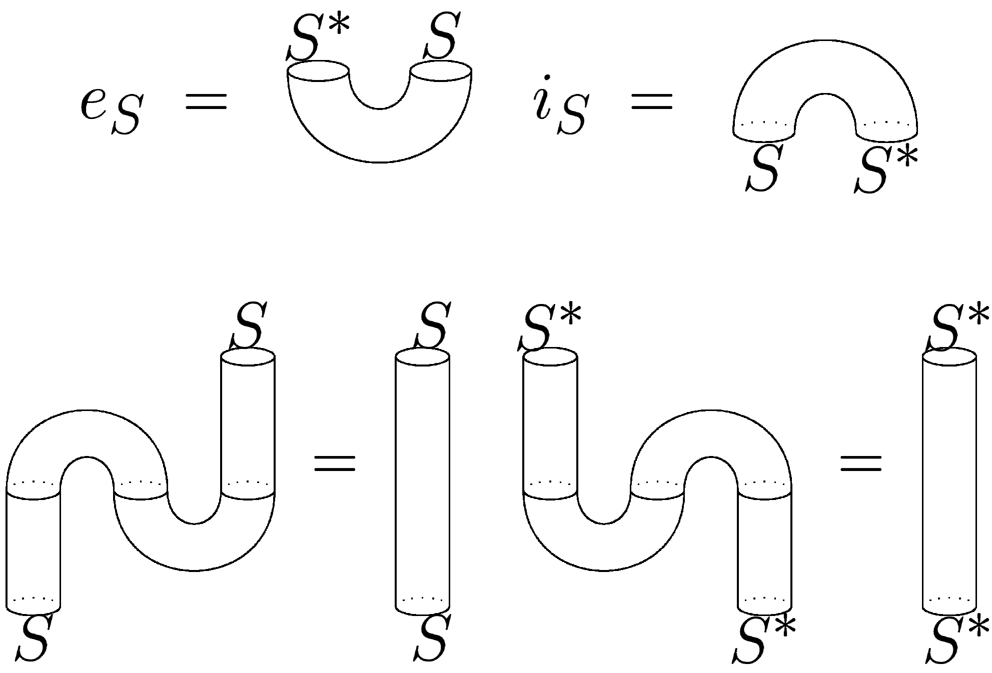
In both nCob and Hilb the braiding satisfies

$$B_{x,y} B_{y,x} = 1$$

so they are symmetric monoidal categories.

Duality for Objects

Both nCob and Hilb are **compact** symmetric monoidal categories: for every object x there is a dual x^* with counit $e_x\colon x^* \otimes x \to 1$ and unit $i_x\colon 1 \to x \otimes x^*$ satisfying the zig-zag identities.

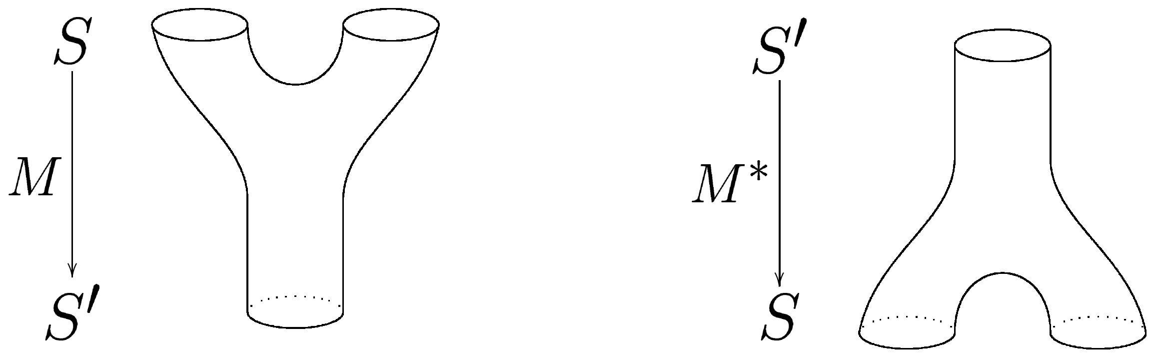


Duals for objects in nCob

In nCob, S^* has the opposite orientation from S. In Hilb, H^* is the space of continuous linear functionals on H.

Duals for morphisms

Both nCob and Hilb are **$*$-categories**: they are equipped with a contravariant endofunctor $*$ acting as the identity on objects and with $*^2 = 1$.



Duals for morphisms in nCob

Duals for morphisms in Hilb: given $T\colon H \to H'$, define $T^*\colon H' \to H$ via

$$\langle T^*\phi, \psi\rangle = \langle \phi, T\psi\rangle$$

for all $\psi \in H$ and $\phi \in H'$.

A morphism f in a $*$-category is **unitary** if it has f^* as its inverse.

A symmetric monoidal category **has duals** if it is compact and also a $*$-category, and all structural isomorphisms are unitary:

- the associators $a_{x,y,z}$,
- the left and right unit laws ℓ_x and r_x,
- the braidings $B_{x,y}$.

A symmetric monoidal functor $F\colon C \to D$ **preserves duals** if it is also a **$*$-functor**:

$$F(x^*) = F(x)^*, \qquad x \in C$$

for which all structural isomorphisms are unitary:

- the unit preservation isomorphism $F_1\colon 1_C \to F(1_D)$
- the tensor product preservation isomorphisms $(F_2)_{x,y}\colon F(x) \otimes F(y) \to F(x \otimes y)$.

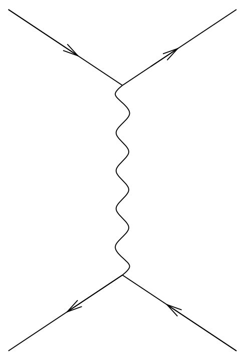
A **unitary TQFT** is a symmetric monoidal functor preserving duals,

$$Z\colon n\mathrm{Cob} \to \mathrm{Hilb}.$$

TQFTs describe *background-free quantum field theories with no local degrees of freedom*: all their interesting aspects are 'global'.

TQFTs exploit the analogy between *space* and *state*, *spacetime* and *process* in a very clear way. More 'physical' theories also do this... but in more mysterious ways! Understanding TQFTs with the help of n-categories is a 'warmup exercise' for understanding more physical theories this way.

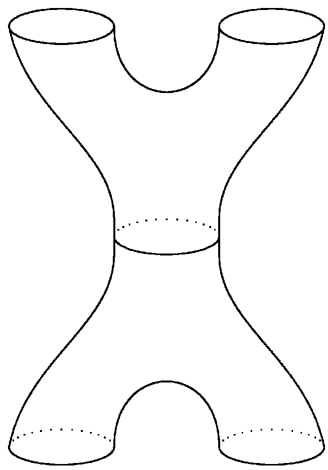
In quantum field theory, we use Feynman diagrams to calculate inside the category $\mathrm{Rep}(G)$ of unitary representations of G, the symmetry group of our theory:



- Edges are labelled with unitary representations of G: 'particles'
- Vertices are labelled with intertwining operators: 'interactions'

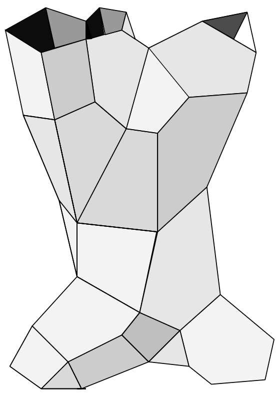
Typically G includes symmetries of *spacetime* (e.g. $\mathbb{R}^4$), and its unitary representations are described as spaces of solutions of linear PDE on spacetime. Then it can be useful to work with Feynman diagrams *living in spacetime.*

In string theory, we replace Feynman diagrams with 2-dimensional 'string worldsheets':



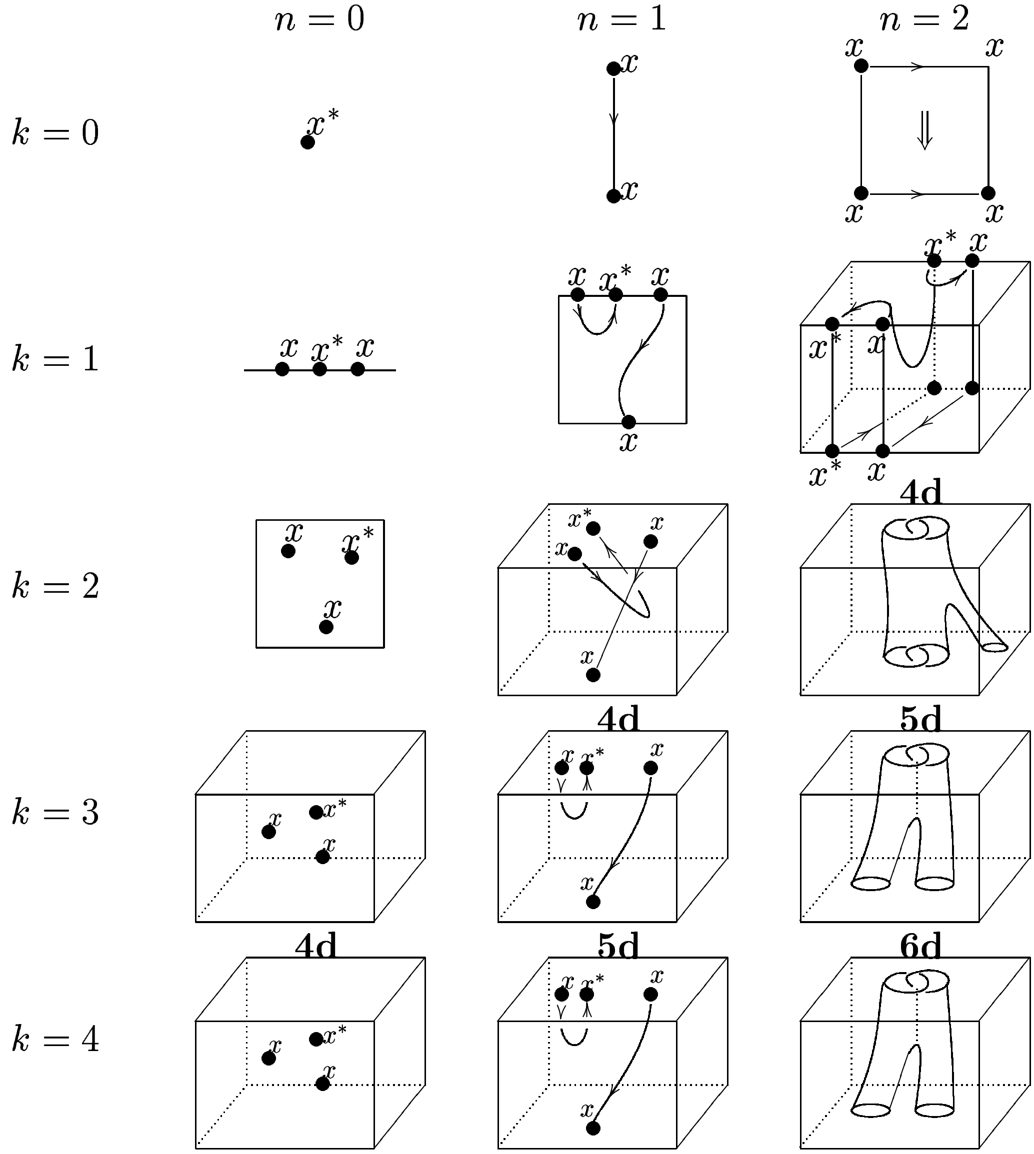
Again we may view these 'abstractly' as tools for computing in a theory — a *conformal field theory* — or as mapped into spacetime.

In spin foam models of quantum gravity we replace Feynman diagrams with 2-dimensional 'spin foams':



Both string theory and spin foam models hint at some sort of 'categorification' of Feynman diagrams! However, categorification has been developed much more extensively for topological quantum field theories.

The Tangle Hypothesis: The free k-tuply monoidal n-category with duals on one generator is $n\mathrm{Tang}_k$: top-dimensional morphisms are n-dimensional framed tangles in $n+k$ dimensions.



The Stabilization Hypothesis:

$$S\colon n\mathrm{Cat}_k \to n\mathrm{Cat}_{k+1}$$

is an equivalence of $(n+k+2)$-categories if $k \geq n+2$.

Combining these hypotheses, we obtain:

The Cobordism Hypothesis: The free stable n-category with duals on one generator is $\boldsymbol{n}$**COB**: n-morphisms here are n-dimensional framed cobordisms between framed manifolds with corners.

An **unitary extended TQFT** should be a stable n-functor preserving duals,

$$Z\colon n\mathrm{COB} \to n\mathrm{Hilb}$$

for some stable n-category with duals 'nHilb'. Z should therefore be determined by $Z(\bullet)$, where $\bullet$ is the positively oriented point.