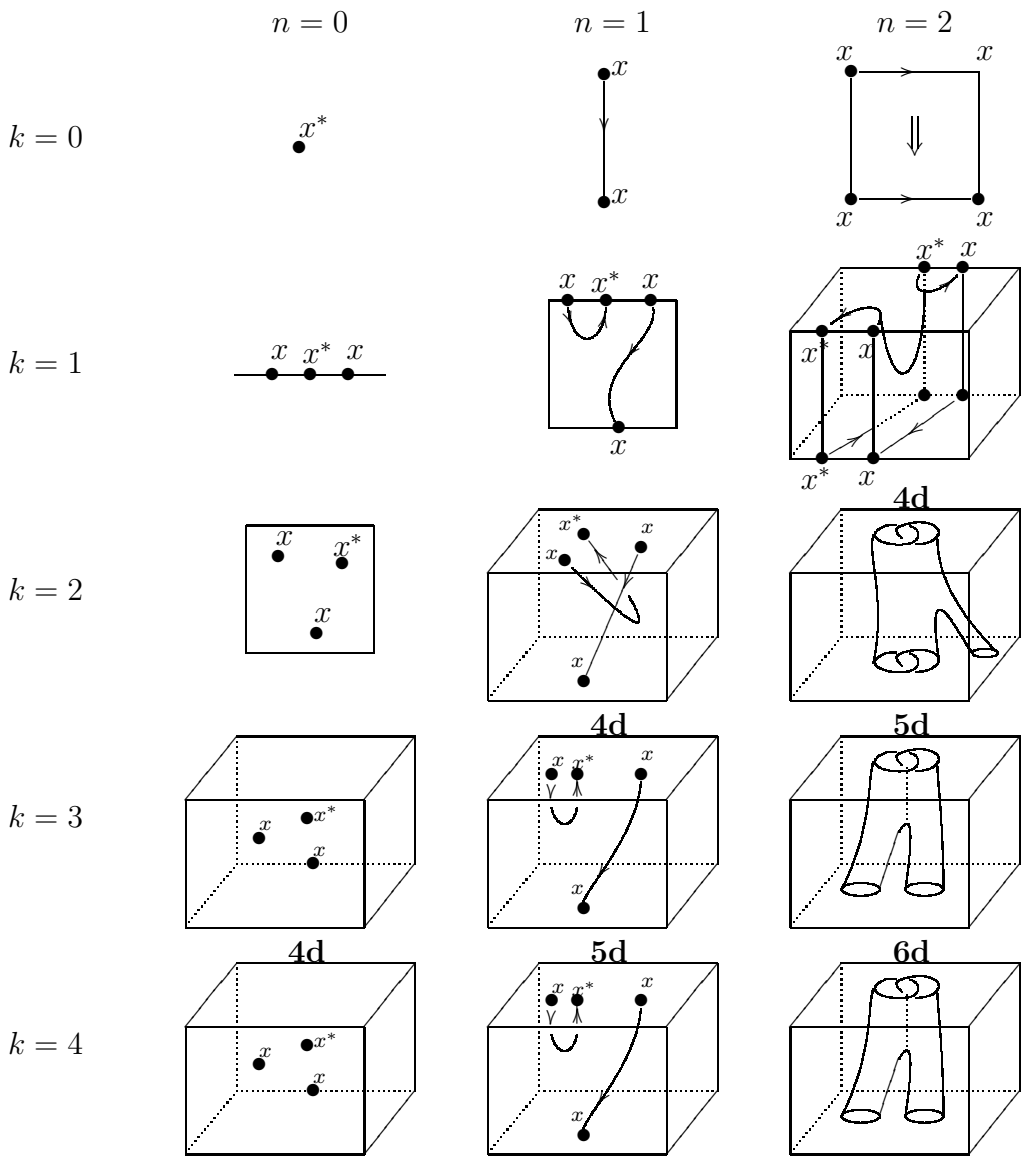
Why n-Categories?

John C. Baez



many figures by Aaron Lauda

Every Interesting Equation is a Lie!

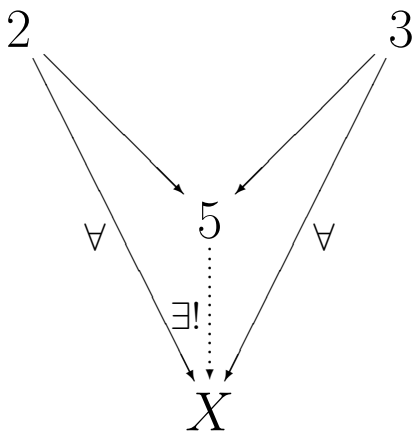
$x = x$ true, but boring

$x = y$ potentially interesting– but says two *different* things are *the same*!

Any interesting equation is really a summary of an interesting *process*. For example:

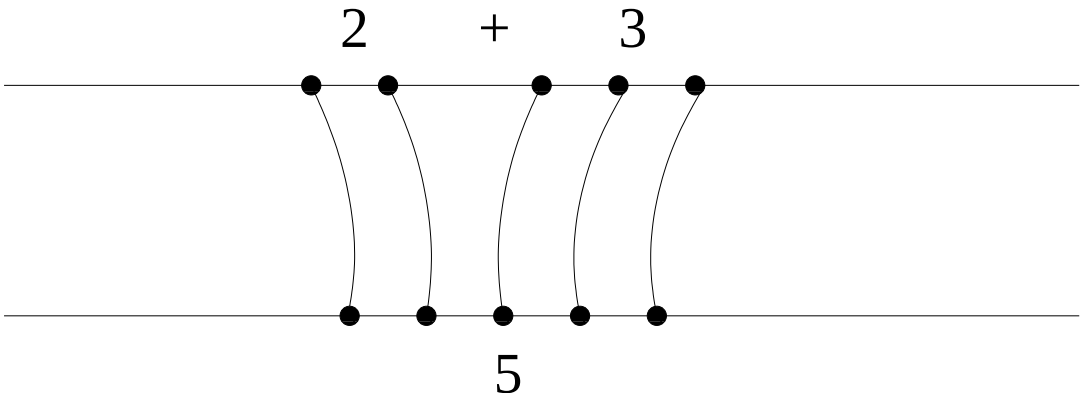
$$2 + 3$$
$$\|$$
$$5$$

is short for:

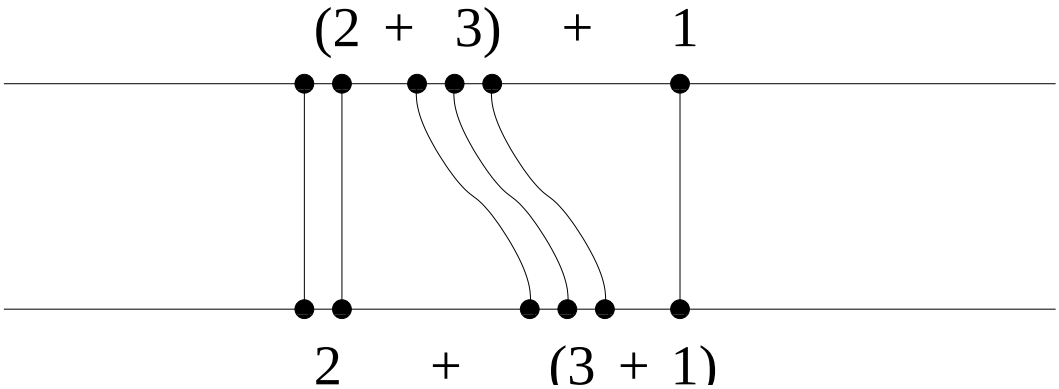


Codimension 1: Composition, Associator,...

We add by putting 0-dimensional rocks in a 1-dimensional line:

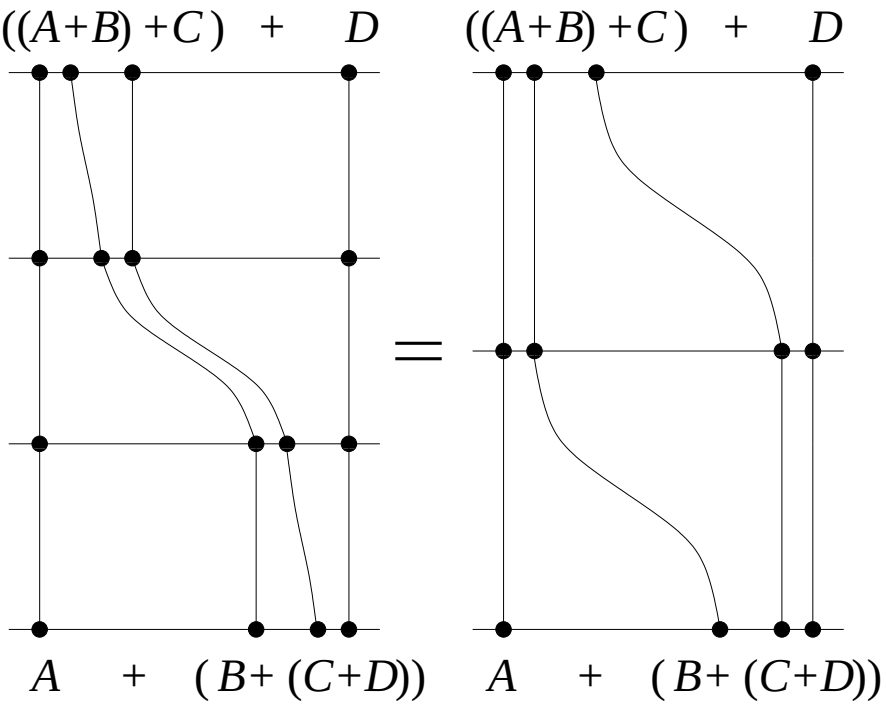


Proving associativity takes time:



We call this proof the **associator**: note 1-dimensional 'worldlines' in 2-dimensional 'spacetime', hence again codimension 2-1 = 1.

The associator satisfies the **pentagon identity**:

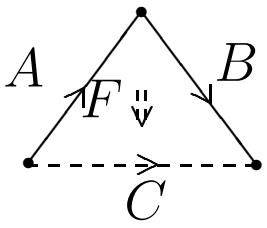


But the process of proving this traces out a 2d surface in 3 dimensions: the **pentagonator**!

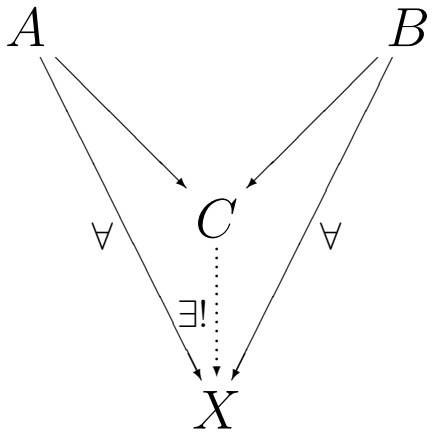
And so on....

Higher Associative Laws: a Simplicial Viewpoint

The hierarchy of 'higher associative laws' can also be formalized using *simplices*:



If every way of filling the triangular 'horn' factors through F, we may call it *a process of composing* A and B, and call C *a composite*. This applies to addition of sets:



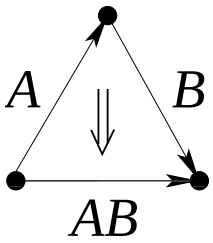
and many other examples, especially *composition of paths in a topological space.*

Considering higher-dimensional horns, we get this hierarchy:

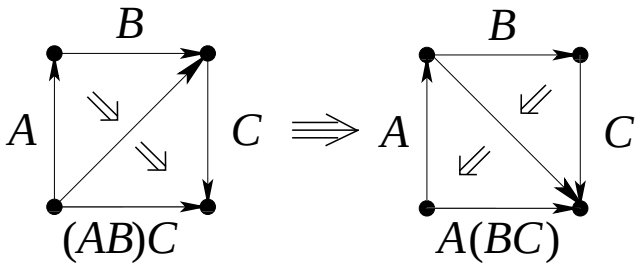
Object:

Morphism:

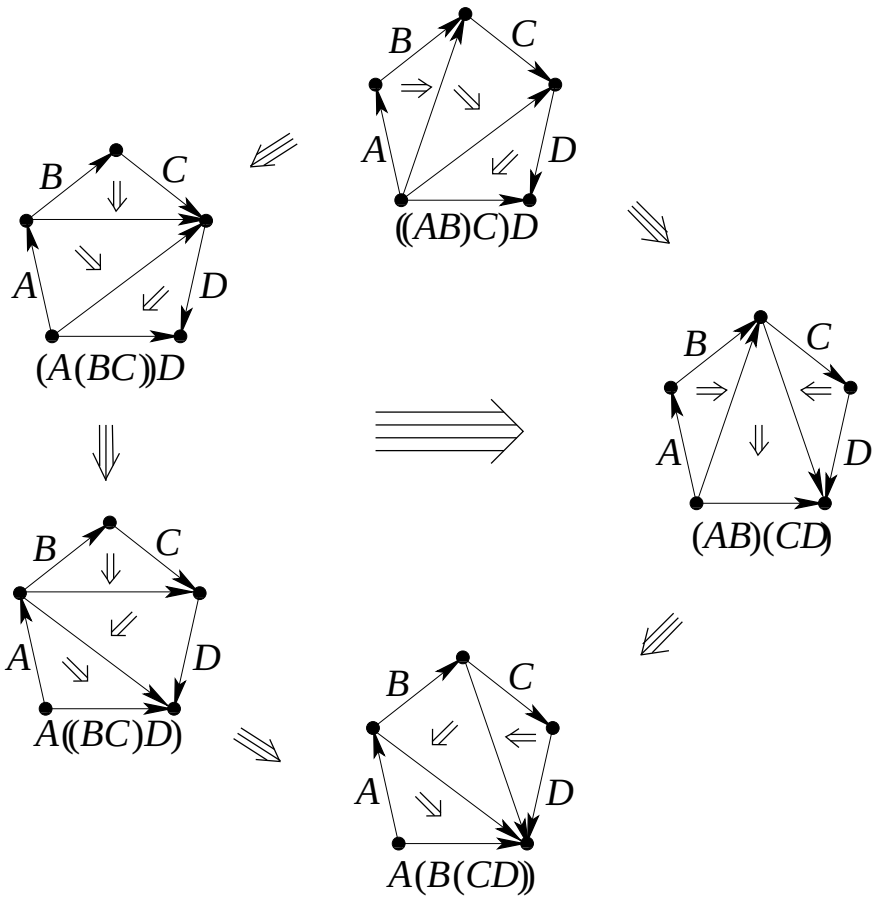
Composition:



Associator:



Pentagonator:



... and so on forever: the Stasheff associahedra!

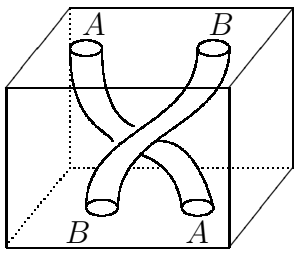
(But there's a subtlety in higher dimensions, appearing already in the next associahedron: the *commutative* law gets involved!)

Codimension 2: Braiding, Yang–Baxterator,...

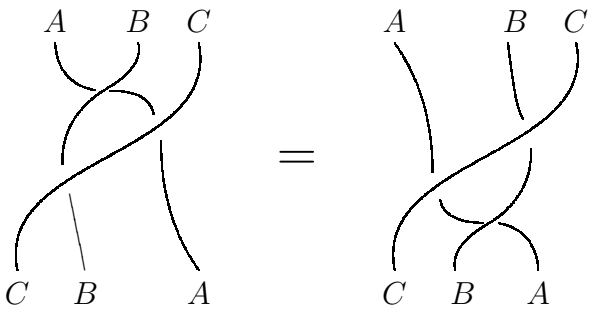
If space is at least 2-dimensional, we can prove the commutative law:

$$A + B = B + A$$

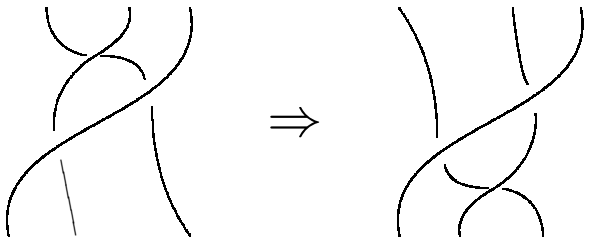
by sliding two piles of rocks around each other. But the proof takes time:



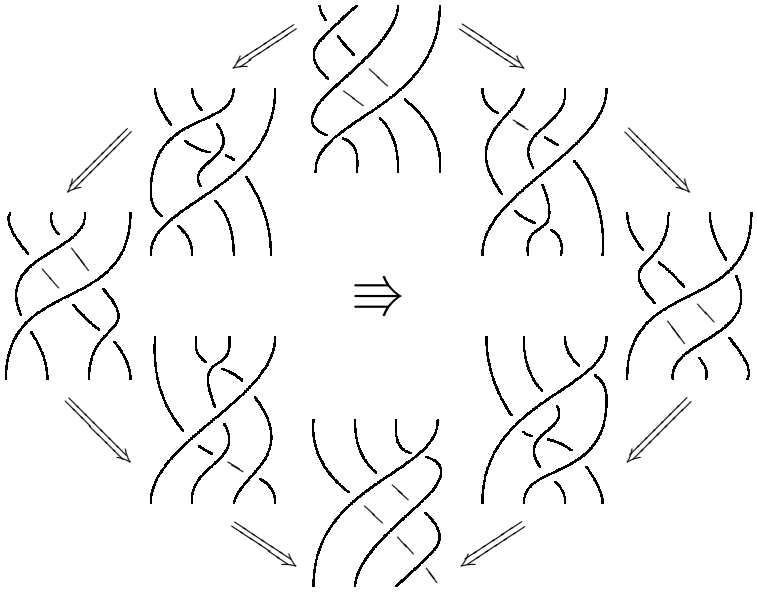
Note 1-dimensional 'worldlines' in 3-dimensional 'spacetime': the **braiding**. The braiding in turn satisfies the **Yang–Baxter equation**:



The process of proving the Yang–Baxter equation traces out a 2d surface in 4 dimensions, the **Yang–Baxterator**:



This in turn satisfies the **Zamolodchikov tetrahedron equation**:

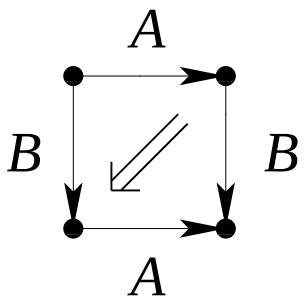


but the proof of this traces out a 3d surface in 5 dimensions... *and so on!*

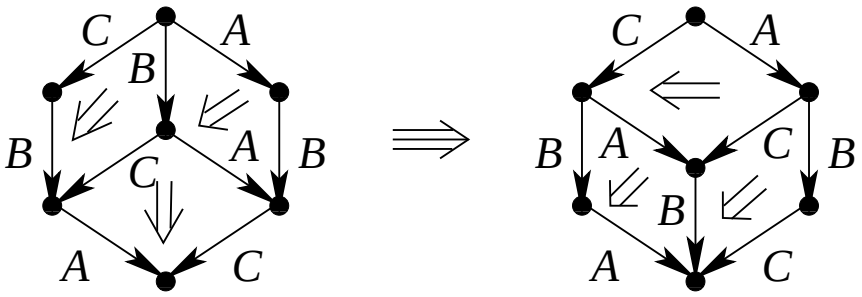
Higher Commutative Laws: a Cubical Viewpoint

The hierarchy of 'higher commutative laws' can also be formalized using *cubes.*

Braiding:



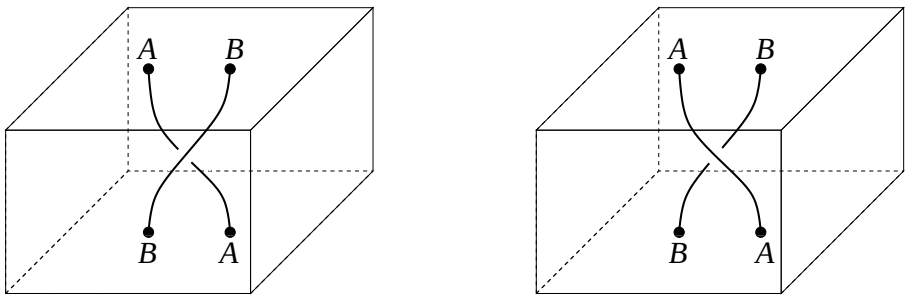
Yang-Baxterator:



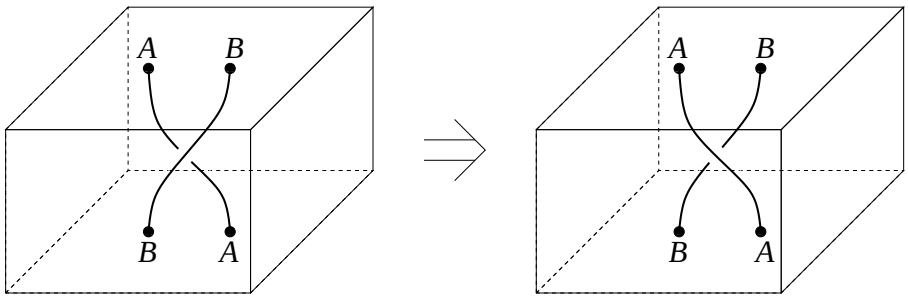
Similarly, the Zamolodchikov tetrahedron equation relates the 'front' and 'back' of a 4-cube, each of which is built from 4 Yang–Baxterator 3-cubes... *and so on!*

Codimension 3: Syllepsis,...

If space is 2-dimensional, there are two fundamentally different proofs that $A + B = B + A$: the **braiding** versus the **inverse braiding**:



since these are nonisotopic braids in 3d spacetime. But if space is at least 3-dimensional, one proof can be continuously deformed to the other:



since all braids in 4d spacetime are isotopic! This process traces out a 2d surface in 5 dimensions: the **syllepsis**.

The syllepsis satisfies a law of its own... but the proof of this traces out a 3d surface in 6 dimensions... *and so on!*

And so on for higher codimensions!

For example, in codimension 4 we get an isotopy between the syllepsis and the 'inverse syllepsis', which are 2d surfaces in 6 dimensions. This isotopy traces out a 3d surface in 7 dimensions, and satisfies a law whose proof traces out a 4d surface in 8 dimensions, etc....

In short: a hierarchy of 'higher braidings', one for each codimension $k \geq 2$, each satisfying a hierarchy of laws.

Warning: this is a drastically **simplified** version of the story!

Why n-Categories?

We've seen how beautiful but overwhelmingly complex structures arise when we *treat every equation as a summary of a process*. I've only *begun* to describe these structures! To keep track of them, we need n-categories - and not just a definition, but a detailed theory of them.

In particular:

Let a **k-tuply monoidal n-category** be an $(n+k)$-category that is trivial below dimension k - viewed as an n-category with k ways to multiply objects.

Everything I've said so far should be summarized by some theorem relating k-tuply monoidal n-categories to 'n-braids in codimension k'. A bit more precisely...

The Braid Hypothesis: The free k-tuply monoidal n-category on one object is $\boldsymbol{n}\mathbf{Braid}_{\boldsymbol{k}}$, where:

- objects are finite collections of points in $\mathbb{R}^k$, i.e. elements of

$$X_k = \bigsqcup_{j=0}^{\infty} \frac{\{(x_1, \ldots, x_j) \colon \ x_i \in \mathbb{R}^k, \ x_i \text{ distinct}\}}{S_j}$$

- morphisms are paths in X_k,
- 2-morphisms are paths of paths in X_k,
- etc...
- n-morphisms are *homotopy classes* of paths of paths of paths... in X_k.

THE PERIODIC TABLE

We expect k-tuply monoidal n-categories go like this:

	$n = \mathbf{0}$	$n = \mathbf{1}$	$n = \mathbf{2}$
$k = \mathbf{0}$	**sets**	**categories**	**2-categories**
$k = \mathbf{1}$	**monoids**	**monoidal categories**	**monoidal 2-categories**
$k = \mathbf{2}$	**commutative monoids**	**braided monoidal categories**	**braided monoidal 2-categories**
$k = \mathbf{3}$	"	**symmetric monoidal categories**	**sylleptic monoidal 2-categories**
$k = \mathbf{4}$	"	"	**symmetric monoidal 2-categories**
$k = \mathbf{5}$	"	"	"
$k = \mathbf{6}$	"	"	"

- n acts like the *dimension*.
- k acts like the *codimension*.

The Braid Hypothesis: Examples

The free monoid on one generator is $0\mathrm{Braid}_1$, the **natural numbers**: isotopy classes of collections of points on the line.

The free braided monoidal category on one generator is $1\mathrm{Braid}_2$, the **braid groupoid**: collections of points in the plane and isotopy classes of braids in 3d going between these.

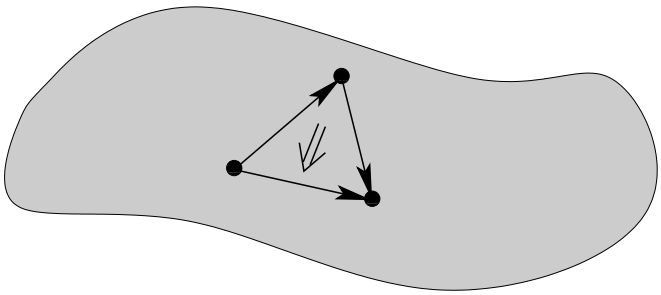
The free symmetric monoidal category on one generator is $1\mathrm{Braid}_3$, the **groupoid of finite sets**: collections of points in $\mathbb{R}^3$ and isotopy classes of braids in 4d going between these. *We live here!*

The free braided monoidal 2-category on one generator is $2\mathrm{Braid}_4$, the **2-braid 2-groupoid**: colllections of points in the plane, braids in 3d between these, and isotopy classes of 2-braids in 4d between these.

All these are just different *views* of a single concept: 'the true natural numbers'.

How To Understand n-Categories

Topology lights the way, since every space X has a 'fundamental ω-groupoid', $\Pi_\infty(X)$. In the simplicial framework:



it's the simplicial set whose j-cells are just maps

$$F\colon \Delta^j \to X.$$

Technically this is a **Kan complex**: every horn has a filler!

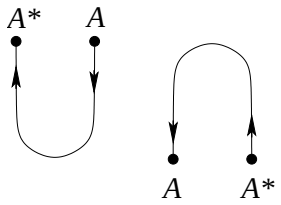
The Homotopy Hypothesis (baby version): equivalence classes of ω-groupoids are the same as **homotopy types**: homotopy equivalence classes of locally nice spaces (e.g. CW complexes).

$n\mathrm{Braid}_k$ corresponds to the homotopy type where we take X_k, the space of finite collections of points in $\mathbb{R}^k$, and 'kill homotopy groups above π_n' by attaching cells.

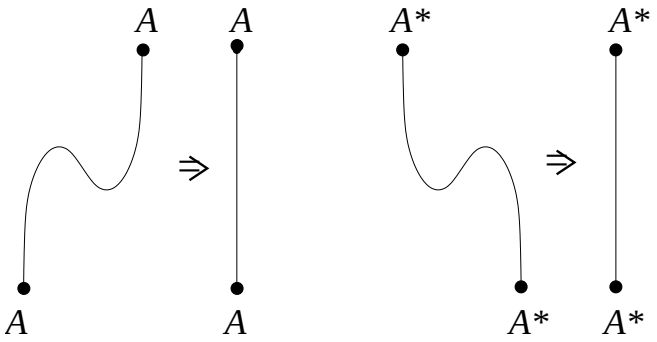
This is Just the Beginning...

... though not of my talk, you'll be glad to know.

More interesting than n-groupoids are 'n-categories with duals', where all j-morphisms have, not weak inverses, but 'duals' or 'adjoints':



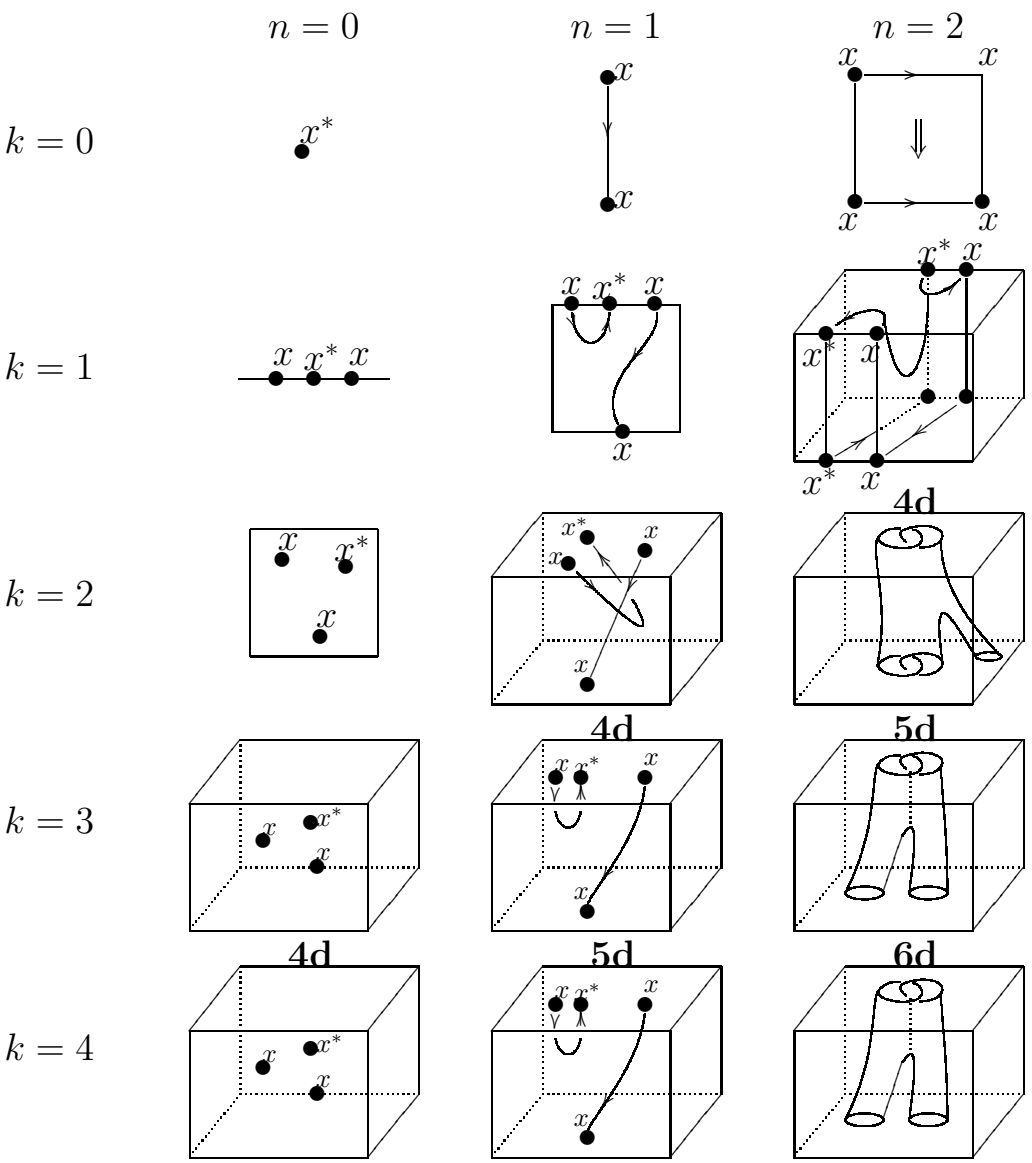
satisfying weakened 'zigzag identities':



which satisfy laws of their own... and so on.

This allows a form of 'subtraction', so it gives us some new views of 'the true integers'....

The Tangle Hypothesis: The free k-tuply monoidal n-category with duals on one generator is $n\mathrm{Tang}_k$: top-dimensional morphisms are n-dimensional framed tangles in $n+k$ dimensions.



Algebraic Structures and the Free Such Structures on One Generator

sets	**1**
monoids	$\mathbb{N}$
groups	$\mathbb{Z}$
k**-tuply monoidal** n**-categories**	$n\mathrm{Braid}_k \simeq \Pi_{n+k}(X_k)$
k**-tuply monoidal** ω**-categories**	X_k
k**-tuply groupal** n**-groupoids**	$\Pi_{n+k}(S^k)$
k**-tuply groupal** ω**-groupoids**	S^k
strict k**-tuply groupal** ω**-groupoids**	$K(\mathbb{Z}, k)$
k**-tuply monoidal** n**-categories with duals**	$n\mathrm{Tang}_k$

Thom-Pontryagin map:

$$n\mathrm{Tang}_k \to \Pi_{n+k}(S^k)$$

From homotopy to homology:

$$\Pi_{n+k}(S^k) \to K(\mathbb{Z}, k)$$