Higher Categories, Higher Gauge Theory – III

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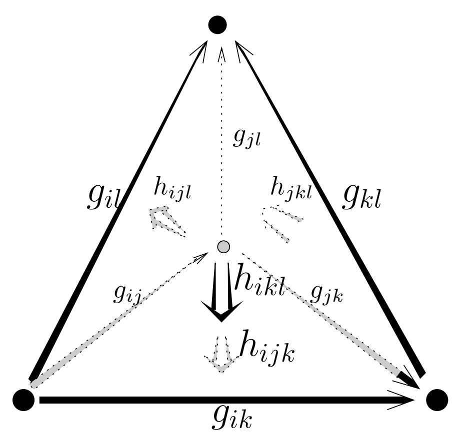


Notes and references at:

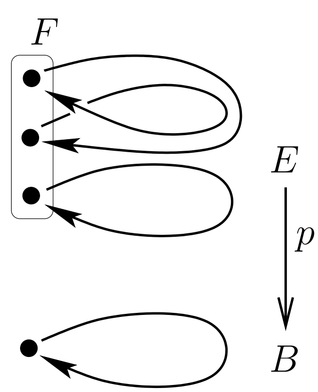
http://math.ucr.edu/home/baez/namboodiri/

From Covering Spaces To Bundles

One version of the basic principle of Galois theory:

> **Covering spaces $F \hookrightarrow E \to B$ are classified by smooth functors**
>
> $$\Pi_1(B) \to \mathrm{Aut}(F).$$

Here B is a *space* but the fiber F is just a *set*, so $\mathrm{Aut}(F)$ is a *discrete group*. We get the functor from the covering space by lifting paths:



But what if B is smooth, and F is not just a set but a *smooth space*, or more generally a *smooth category*?

Then we need to introduce *connections on bundles*, or more generally *2-connections on 2-bundles*.

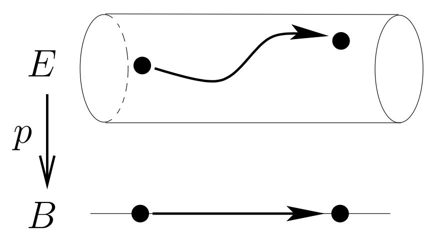
Suppose B is a smooth space, F is a smooth space, and G is a smooth group acting on F:

$$G \to \mathrm{Aut}(F).$$

Now it makes sense to demand that

$$F \hookrightarrow E \to B$$

is a bundle with gauge group G, or 'G-bundle' for short. We must choose a 'connection' to lift smooth paths:



We'll recall these notions and see:

> **G-bundles $F \hookrightarrow E \to B$ with connection are classified by smooth anafunctors**
>
> $$\mathcal{P}_1(B) \to G.$$

Now the fundamental groupoid $\Pi_1(B)$ has been replaced by the **path groupoid** $\mathcal{P}_1(B)$, defined last time. The group $\mathrm{Aut}(F)$ has been generalized to any smooth group G acting on F.

$\mathcal{P}_1(B)$ is a smooth groupoid; G is a smooth groupoid with one object. For this result the right maps between smooth groupoids are not 'smooth functors', but smooth 'anafunctors'... we'll see why.

Bundles

A **bundle** over a smooth space B is:

- a smooth space E (the **total space**),
- a smooth space F (the **fiber**),
- a smooth map $p\colon E \to B$ (the **projection**),

such that B is covered by open sets U_i equipped with diffeomorphisms

$$t_i\colon p^{-1}U_i \to U_i \times F$$

(**local trivializations**) such that

$$\begin{array}{ccc} p^{-1}U_i & \xrightarrow{\;t_i\;} & U_i \times F \\ & {\scriptstyle p}\searrow \qquad \swarrow & \\ & U_i & \end{array}$$

commutes.

In other words, E looks *locally* like the product of B and F... but perhaps not *globally*.

G-Bundles

If F is a smooth space, $\mathrm{Aut}(F)$ is a smooth group. If $E \to B$ is a bundle with fiber F, the local trivializations over open sets U_i covering B give smooth maps called **transition functions**:

$$g_{ij} \colon U_i \cap U_j \to \mathrm{Aut}(F)$$

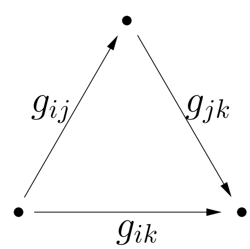
via:

$$t_j t_i^{-1}(x, f) = (x, g_{ij}(x)(f)).$$

These satisfy the **1-cocycle condition**

$$g_{ij}(x) g_{jk}(x) = g_{ik}(x)$$

for any $x \in U_i \cap U_j \cap U_k$. In other words, this diagram commutes:



For any smooth group G, we say the bundle $E \to B$ has G as its **gauge group** when the maps g_{ij} factor through an action $G \to \mathrm{Aut}(F)$. We then call $E \to B$ a G-**bundle**.

Connections

Last time we treated holonomies as smooth functors

$$\mathrm{hol}\colon \mathcal{P}_1(B) \to G$$

and showed these correspond to $\mathfrak{g}$-valued 1-forms A on B. Now this only works *locally!*

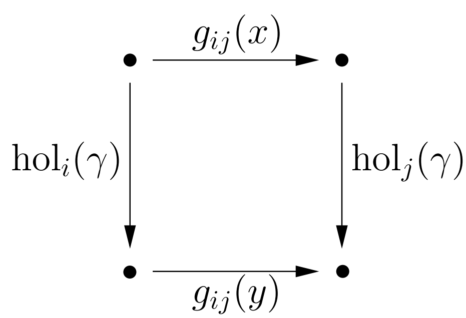
Suppose $E \to B$ is a G-bundle with local trivializations over neighborhoods U_i covering B. Define a **connection** to be a smooth functor

$$\mathrm{hol}_i\colon \mathcal{P}_1(U_i) \to G$$

for each i, such that the transition function g_{ij} defines a smooth natural isomorphism:

$$g_{ij}\colon \mathrm{hol}_i|_{\mathcal{P}_1(U_i \cap U_j)} \to \mathrm{hol}_j|_{\mathcal{P}(U_i \cap U_j)}$$

for all i, j. In other words, this diagram commutes:



for any path $\gamma\colon x \to y$ in $U_i \cap U_j$.

Theorem. There is a one-to-one correspondence between connections on the G-bundle $E \to B$ and $\mathfrak{g}$-valued 1-forms A_i on the open sets U_i satisfying

$$A_i = g_{ij}\, A_j\, g_{ij}^{-1} + g_{ij}\, dg_{ij}^{-1}$$

on the intersections $U_i \cap U_j$.

So, our definition of connection is secretly the usual one!

Smooth Anafunctors

Given smooth categories X and Y, the obvious sort of map

$$F\colon X \to Y$$

is a functor that is smooth on objects and on morphisms. Alas, many interesting functors are naturally isomorphic to a smooth one *locally*, but not *globally*. The right maps are 'smooth anafunctors' — defined by Toby Bartels in his thesis. He calls them '2-maps' between '2-spaces'.

The holonomy of a connection is an example. For a trivial bundle, this is a smooth functor

$$\mathrm{hol}\colon \mathcal{P}_1(B) \to G.$$

For a nontrivial bundle, we only get smooth functors *locally:*

$$\mathrm{hol}_i\colon \mathcal{P}_1(U_i) \to G,$$

but they are related by smooth natural isomorphisms g_{ij} on double intersections $U_i \cap U_j$, satisfying the 1-cocycle condition on triple intersections $U_i \cap U_j \cap U_k$. This is precisely a smooth anafunctor! So:

> **G-bundles $F \hookrightarrow E \to B$ with connection are classified by smooth anafunctors**
>
> $$\mathcal{P}_1(B) \to G.$$

2-Bundles

Now let's categorify all the above and get *higher gauge theory!* First we categorify the concept of *bundle*, following the thesis of Toby Bartels.

We can think of a smooth space M as a smooth category with only identity morphisms. A **2-bundle** over M consists of:

- a smooth category E (the **total space**),
- a smooth category F (the **fiber**),
- a smooth functor $p\colon E \to M$ (the **projection**),

such that M is covered by open sets U_i equipped with smooth equivalences

$$t_i\colon p^{-1}U_i \to U_i \times F$$

(**local trivializations**) such that

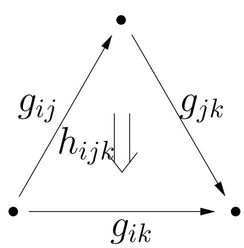
$$\begin{array}{ccc} p^{-1}U_i & \xrightarrow{\;f\;} & U_i \times F \\ & {\scriptstyle p}\searrow \qquad \swarrow & \\ & U_i & \end{array}$$

commutes.

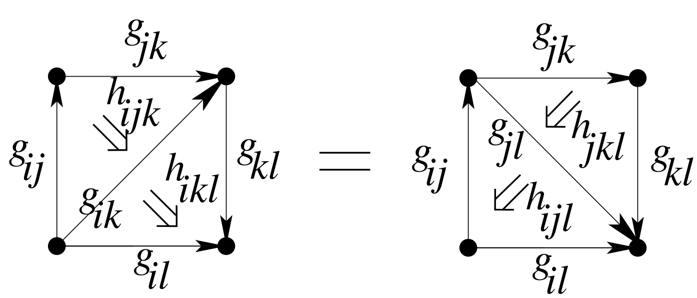
$\mathcal{G}$-2-Bundles

Theorem. Let F be a smooth category and let AUT(F) be its automorphism 2-group, which is a smooth 2-group. Given a 2-bundle $E \to B$ with fiber F, the local trivializations over open sets U_i covering B give:

- smooth maps
$$g_{ij} \colon U_i \cap U_j \to \mathrm{Ob}(\mathrm{AUT}(F))$$
- smooth maps
$$h_{ijk} \colon U_i \cap U_j \cap U_k \to \mathrm{Mor}(\mathrm{AUT}(F))$$
with
$$h_{ijk}(x) \colon g_{ij}(x) g_{jk}(x) \to g_{ik}(x)$$

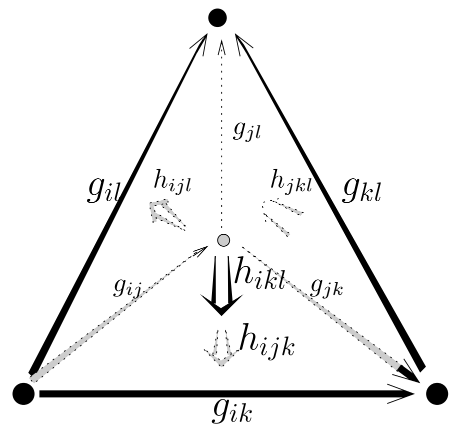


satisfying the **nonabelian 2-cocycle condition:**



on any quadruple intersection $U_i \cap U_j \cap U_k \cap U_\ell$.

In other words, this diagram commutes:



For any smooth 2-group $\mathcal{G}$, we say a 2-bundle $E \to B$ has $\mathcal{G}$ as its **gauge 2-group** when g_{ij} and h_{ijk} factor through an action $\mathcal{G} \to \mathrm{AUT}(F)$. We then call $E \to B$ is a **$\mathcal{G}$-2-bundle**.

In general, we expect that $\mathcal{G}$-n-bundles will be classified by the nth nonabelian Čech cohomology with coefficients in the smooth n-group $\mathcal{G}$. This is well-known for $n = 1$. Toby Bartels is writing up the proof for $n = 2$. As spinoffs, one obtains:

Theorem. Let $\mathcal{G}$ be the smooth 2-group with one object and U(1) as morphisms. Then equivalence classes of $\mathcal{G}$-2-bundles over B are in one-to-one correspondence with $H^3(B, \mathbb{Z})$.

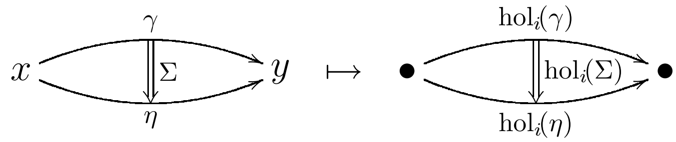
Theorem. Let $\mathcal{G} = \mathrm{AUT}(H)$ for some smooth group H. Then the 2-category of $\mathcal{G}$-2-bundles over B is equivalent to the 2-category of nonabelian H-gerbes over B.

2-Connections

Let $\mathcal{G}$ be a smooth 2-group and let $E \to B$ be a $\mathcal{G}$-2-bundle equipped with local trivializations over open sets U_i covering B. Then a **2-connection** on E consists of the following data:

- For each i a smooth 2-functor:

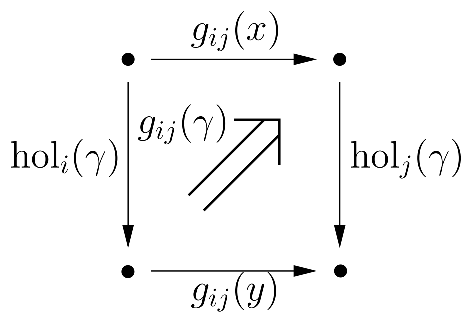
$$\mathrm{hol}_i \quad : \quad \mathcal{P}_2(U_i) \quad \to \quad \mathcal{G}$$



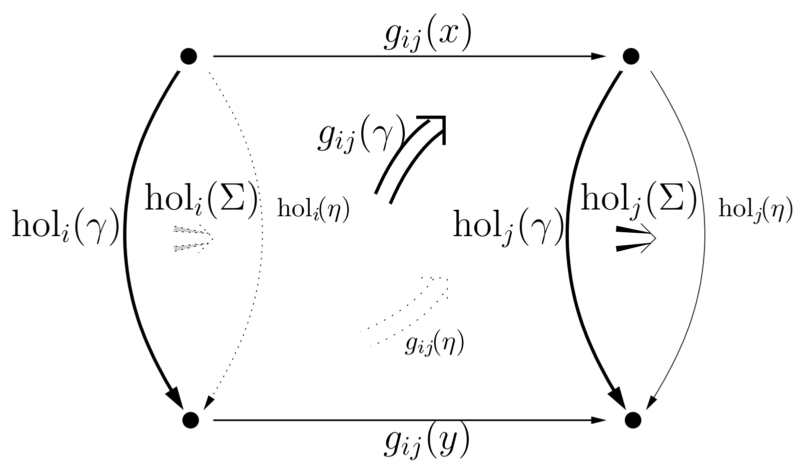
- For each i, j a pseudonatural isomorphism:

$$g_{ij}\colon \mathrm{hol}_i|_{\mathcal{P}(U_i \cap U_j)} \to \mathrm{hol}_j|_{\mathcal{P}(U_i \cap U_j)}$$

  extending the transition function g_{ij}. In other words, for each path $\gamma\colon x \to y$ in $U_i \cap U_j$ a morphism in $\mathcal{G}$:

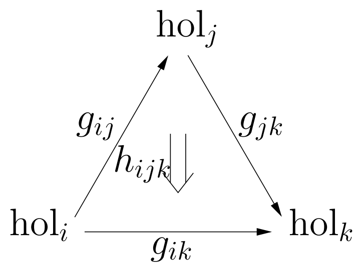


  depending smoothly on γ, such that this diagram commutes for any surface $\Sigma\colon \gamma \Rightarrow \eta$ in $U_i \cap U_j$:

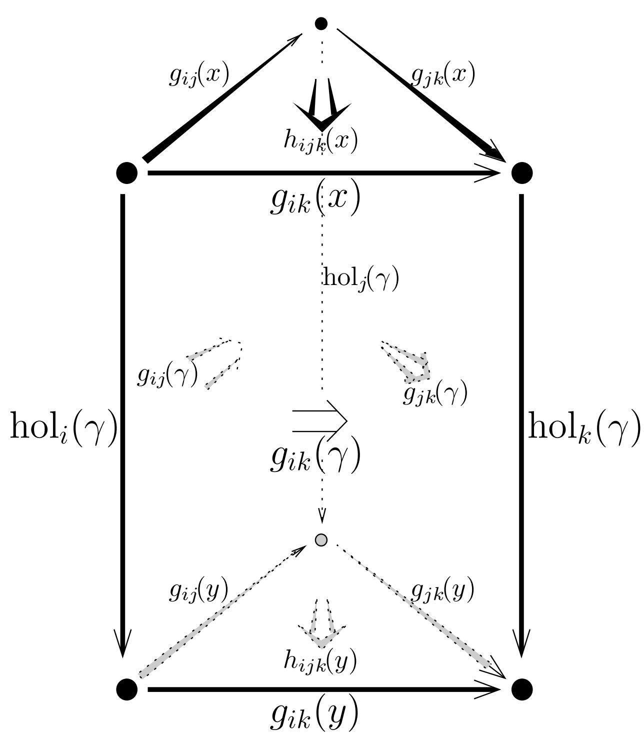


And, we require that:

- for each i, j, k the function h_{ijk} defines a modification:



In other words, this diagrams commutes for any path $\gamma\colon x \to y$ in $U_i \cap U_j \cap U_k$:



Theorem. Suppose that $E \to B$ is a $\mathcal{G}$-2-bundle with local trivializations over open sets U_i covering B. Then there is a one-to-one correspondence between 2-connections on E and Lie-algebra-valued differential forms (A_i, B_i, a_{ij}) satisfying certain equations:

- The holonomy 2-functor hol_i is specified by an $\mathfrak{g}$-valued 1-form A_i and an $\mathfrak{h}$-valued 2-form B_i on U_i, satisfying the fake flatness condition:

$$dA_i + A_i \wedge A_i + dt(B_i) = 0$$

- The pseudonatural isomorphism $\mathrm{hol}_i \xrightarrow{g_{ij}} \mathrm{hol}_j$ is specified by the transition functions g_{ij} together with $\mathfrak{h}$-valued 1-forms a_{ij} on $U_i \cap U_j$, satisfying the equations:

$$A_i = g_{ij} A_j g_{ij}^{-1} + g_{ij} d g_{ij}^{-1} - dt(a_{ij})$$

$$B_i = \rho(g_{ij})(B_j) + da_{ij} + a_{ij} \wedge a_{ij} + d\rho(A_i) \wedge a_{ij}$$

- For $g_{ij} \circ g_{jk} \xrightarrow{h_{ijk}} g_{ik}$ to be a modification, the functions h_{ijk} must satisfy the equation:

$$a_{ij} + \rho(g_{ij}) a_{jk} =$$

$$h_{ijk}\, a_{ik}\, h_{ijk}^{-1} + h_{ijk}\, d\rho(A_i)\, h_{ijk}^{-1} + h_{ijk}\, dh_{ijk}^{-1}$$

Punchline. Except for fake flatness, these weird-looking equations show up already in Breen and Messing's definition of a *connection on a nonabelian gerbe!* So, 2-bundles and nonabelian gerbes give closely related approaches to higher gauge theory.

Ultimately we expect to find:

$\mathcal{G}$-2-bundles $F \hookrightarrow E \to B$ with 2-connection are classified by smooth 2-anafunctors

$$\mathcal{P}_2(B) \to \mathcal{G}.$$

And why stop at 2? The basic principle of Galois theory keeps growing....