Lie 2-Algebras and the Geometry of Gerbes

Danny Stevenson

Department of Mathematics
University of California, Riverside
email: dstevens@math.ucr.edu

References

Breen & Messing math.AG/0106083
Baez & Schreiber hep-th/0412325
Aschieri, Cantini & Jurco hep-th/0312154

Principal G-bundles

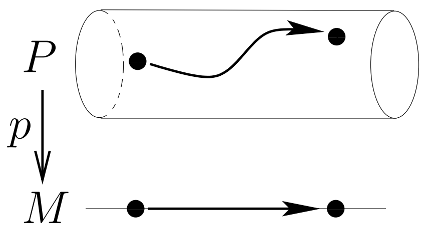
Principal bundles are ubiquitous in geometry and mathematical physics. A **principal** G-bundle consists of

- a surjective submersion $\pi\colon P \to M$
- an action
$$\begin{tikzcd} P \times G \arrow[rr] \arrow[dr] & & P \arrow[dl] \\ & M & \end{tikzcd}$$
of G on M

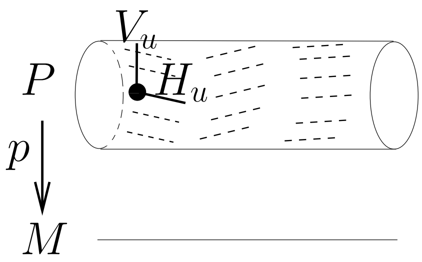
such that

- the action is strongly free in the sense that the natural map
$$P \times G \to P \times_M P$$
is a diffeomorphism

Classically, **connections** on P are understood in terms of **parallel transport**:



Another point of view is to think of a connection on P as an invariant choice of **horizontal subspace** $H_u \subset T_u P$ for each $u \in P$:



This horizontal subspace gives a **splitting** for this exact sequence:

$$0 \to V_u \to T_u P \xrightarrow{dp} T_{p(u)} M \to 0$$

where the **vertical subspace** V_u is the kernel of dp.

We would like to think about connections as splittings from a more **categorical** viewpoint …

The Atiyah Sequence of a Principal Bundle

Suppose that

$$\begin{array}{ccc} G & \longrightarrow & P \\ & & \downarrow \\ & & M \end{array}$$

is a principal G-bundle. Associated to P is an **exact sequence** of vector bundles on M:

$$0 \to \mathrm{ad}(P) \to TP/G \to TM \to 0$$

A **connection** on P is a **splitting** A of this exact sequence. Associated to this exact sequence is an **extension** of **Lie algebras**

$$0 \to \Gamma(\mathrm{ad}(P)) \to \Gamma(TP/G) \to \Gamma(TM) \to 0$$

The **curvature** F_A of A can be understood as a measure of the failure of A to be a **homomorphism** of Lie algebras:

$$F_A(X, Y) = [A(X), A(Y)] - A[X, Y] \qquad X, Y \in \Gamma(TM)$$

F_A is skew and bilinear in X, Y and is linear over $C^\infty(M)$ so defines an element

$$F_A \in \Omega^2(M, \mathrm{ad}(P))$$

Lie 2-algebras and Crossed Modules

A (**strict**) **Lie 2-algebra** is a category $\mathbb{L}$ **internal** to **LieAlg**. Thus $\mathbb{L}$ consists of

- a *Lie algebra* of objects L_0,
- a *Lie algebra* of morphisms L_1,

such that each operation is a homomorphism of Lie algebras.

There is a bijective correspondence between Lie 2-algebras and **crossed modules** of Lie algebras. A crossed module of Lie algebras consists of a homomorphism

$$t\colon L \to J$$

of Lie algebras, together with an action $\alpha\colon J \times L \to L$ of J on L by derivations, such that

$$t(\alpha(x)(\xi)) = [x, t(\xi)] \qquad \text{and} \qquad \alpha(t(\xi))(\eta) = [\xi, \eta]$$

To such a crossed module is associated a Lie 2-algebra with

$$\begin{aligned} \text{objects } &= J \\ \text{morphisms } &= J \ltimes L \end{aligned}$$

where the semidirect product structure is defined by the action $\alpha\colon J \to \mathrm{Der}(L)$.

Lie Schreier Theory

Let J be a Lie algebra. J acts by derivations on itself and so defines a **crossed module** of Lie algebras

$$\mathrm{ad}\colon J \to \mathrm{Der}(J)$$

Associated to this crossed module is a **Lie** 2-**algebra** $\mathrm{DER}(J)$ as explained above with objects $\mathrm{DER}(J) = \mathrm{Der}(J)$ and morphisms $\mathrm{DER}(J) = \mathrm{Der}(J) \ltimes J$. The bracket on $\mathrm{Der}(J) \ltimes J$ is defined as usual by

$$[(f,\xi),(g,\eta)] = ([f,g],[\xi,\eta] + g(\xi) - f(\eta))$$

Suppose we are given an arbitrary extension of Lie algebras

$$0 \to J \to K \to L \to 0$$

A splitting σ of this exact sequence induces a linear map

$$\begin{aligned} \sigma\colon L &\to \mathrm{Der}(J) \\ x &\mapsto \mathrm{ad}_{\sigma(x)}|_J \end{aligned}$$

together with a skew bilinear map

$$\begin{aligned} &\omega\colon L \times L \to J \\ &\omega(x,y) = [\sigma(x),\sigma(y)] - \sigma[x,y] \end{aligned}$$

We would like the pair (σ,ω) to define a **homomorphism** of semistrict Lie 2-algebras.

So we would like to think of $\omega(x,y)$ as defining a morphism

$$[\sigma(x),\sigma(y)] \xrightarrow{\omega(x,y)} \sigma[x,y]$$

We need this morphism to satisfy a coherence law: the diagram

$$\begin{tikzcd}
{[\sigma(x),[\sigma(y),\sigma(z)]]} \arrow[r, "="] \arrow[d, "{[\sigma(x),\omega(y,z)]}"'] & {[[\sigma(x),\sigma(y)],\sigma(z)] + [\sigma(y),[\sigma(x),\sigma(z)]]} \arrow[d, "{[\omega(x,y),\sigma(z)]+[\sigma(y),\omega(x,z)]}"] \\
{[\sigma(x),\sigma[y,z]]} \arrow[d, "{\omega(x,[y,z])}"'] & {[\sigma[x,y],\sigma(z)] + [\sigma(y),\sigma[x,z]]} \arrow[d, "{\omega([x,y],z)+\omega(y,[x,z])}"] \\
{\sigma[x,[y,z]]} \arrow[r, "="'] & {\sigma[[x,y],z] + \sigma[y,[x,z]]}
\end{tikzcd}$$

should commute. This is equivalent to the **Bianchi Identity**

$$d_A\omega(x,y,z) = 0$$

for ω. Here d_A is the linear map

$$d_A \colon \wedge^p L^* \otimes J \to \wedge^{p+1} L^* \otimes J$$

defined by

$$d_A(\omega)(x_1,\ldots,x_{p+1}) = \sum_{i=0}^{p+1}(-1)^i[\sigma(x_i),\omega(x_1,\ldots,\hat{x}_i,\ldots,x_{p+1})]$$
$$+\sum_{i<j}(-1)^{i+j}\omega([x_i,x_j],x_1,\ldots,\hat{x}_i,\ldots,\hat{x}_j,\ldots,x_{p+1})$$

Theorem: *There is a bijective correspondence*

$$\mathrm{Ext}(L,J) \simeq \pi_0[L, \mathrm{DER}(J)]$$

where $\mathrm{Ext}(L,J)$ *denotes isomorphism classes of split extensions of Lie algebras*

$$0 \to J \to K \to L \to 0$$

$\pi_0[L, \mathrm{DER}(J)] =$ "*nonabelian Lie algebra cohomology*".

We have seen:

- A connection on a principal G-bundle is a **splitting** A of an extension of Lie algebras
- The curvature of the connection measures the **failure** of A to be a homomorphism of Lie algebras
- The Bianchi Identity for the curvature can be understood as a **coherence law**.

All of this can be neatly encoded in a homomorphism

$$\nabla \colon \Gamma(TM) \to \mathrm{DER}(\mathrm{ad}(P))$$

of semistrict Lie 2-algebras.

We want to **categorify** this picture.

Gerbes and Categorified Bundles

Suppose that $\mathbb{G} = (G_0, G_1)$ is a Lie 2-group. For example $\mathbb{G} = \mathrm{AUT}(G)$ for a compact Lie group G. A **principal** $\mathbb{G}$**-bundle** on a smooth groupoid $\mathbb{M}$ consists of

- a **surjective submersion** $\pi\colon \mathbb{P} \to \mathbb{M}$,
- an **action**

  $$\begin{tikzcd} \mathbb{P} \times \mathbb{G} \arrow[rr] \arrow[dr] & & \mathbb{P} \arrow[dl] \\ & \mathbb{M} & \end{tikzcd}$$

  of $\mathbb{G}$ on $\mathbb{M}$,

such that

- the natural functor

  $$\mathbb{P} \times \mathbb{G} \to \mathbb{P} \times_{\mathbb{M}} \mathbb{P}$$

  is a **diffeomorphism**

Note that $P_0 \to M_0$ and $P_1 \to M_1$ are principal G_0 and G_1 bundles respectively. $\mathbb{P}$ is an example of a category internal to the category **PrinBund** of principal bundles. Principal $\mathbb{G}$-bundles are closely related to **gerbes**.

Examples

If $\mathbb{M} = X \times_M X \rightrightarrows X$ is the groupoid associated to a surjective submersion $\pi\colon X \to M$ we say that $\mathbb{P}$ is a $\mathbb{G}$-gerbe on M.

Example 1: Suppose that $P \to M$ is a principal K-bundle where K forms part of a central extension

$$1 \to S^1 \to \hat{K} \to K \to 1$$

Let $\mathbb{M} = P \times K \rightrightarrows P$ and $\mathbb{P} = P \times \hat{K} \rightrightarrows P$ be transformation groupoids. Then $\mathbb{P} \to \mathbb{M}$ is a gerbe on M for the 2-group $S^1[1]$ with one object and morphisms S^1.

Example 2: Let G be a compact, simple and simply connected Lie group, and take $K = \Omega G$, $P = P_0 G$. Then

$$\hat{\mathbb{G}} = P_0 G \times \widehat{\Omega G} \rightrightarrows P_0 G$$

is a gerbe on G — this is the **string** 2-**group**.

Example 3: Suppose that M is a 2-connected spin manifold such that a certain characteristic class $c \in H^4(M;\mathbb{Z})$, twice which is p_1, vanishes. Then there is a gerbe $\mathbb{P}$ on M for the string 2-group $\hat{\mathbb{G}}$ — the **string gerbe**.

Local description of G-gerbes with connection

If $\mathbb{M}$ is the groupoid $\sqcup U_{ij} \rightrightarrows \sqcup U_i$ associated to an open cover of M, then an $\mathrm{AUT}(G)$-**gerbe** with connection and curving can be described locally by the following data

$$(\lambda_{ij}, g_{ijk}, \gamma_{ijk}, m_{ij}, \nu_i, \delta_{ij}, B_i, \omega_i)$$

where $g_{ijk} \colon U_{ijk} \to G$, $\lambda_{ij} \colon U_{ij} \to \mathrm{Aut}(G)$ and the remaining fields are described in the following table:

	1-forms	2-forms	3-forms
$\mathfrak{g}$-valued	γ_{ijk}	δ_{ij}, B_i	ω_i
$\mathrm{Der}(\mathfrak{g})$-valued	m_i	ν_i	

The 2-form ν_i is called the '**fake curvature**'; the $\mathfrak{g}$-valued 3-form ω_i is called the '**3-curvature**'. These fields are required to satisfy the following equations:

$$\lambda_{ij}(g_{jkl})g_{ijl} = g_{ijk}g_{ikl}$$
$$\lambda_{ij}\lambda_{jk} = \mathrm{Ad}_{g_{ijk}}\lambda_{ik}$$

$\boxed{\text{\ldots plus 10 other even more complicated ones}}$

One of these equations is the **Higher Bianchi Identity** which says that

$$d\omega_i + m_i(\omega_i) = \nu_i(B_i)$$

Is there a more conceptual way to understand this??

Connections on Gerbes

Suppose that $\mathbb{P} \to \mathbb{M}$ is a gerbe for the 2-group $\mathbb{G}$. The Atiyah sequences for the principal bundles P_0 and P_1 combine to form a diagram of groupoids and functors between them

$$\begin{tikzcd}
\mathrm{ad}(P_1) \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & TP_1/G_1 \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & TM_1 \arrow[d, shift right] \arrow[d, shift left] \\
\mathrm{ad}(P_0) \arrow[r] & TP_0/G_0 \arrow[r] & TM_0
\end{tikzcd}$$

We can think of this as an analogue of the Atiyah sequence:

$$0 \to \mathrm{ad}(\mathbb{P}) \to T\mathbb{P}/\mathbb{G} \to T\mathbb{M} \to 0$$

The individual groupoids in this sequence one can think of as 2-*vector bundles* on $\mathbb{M}$, i.e groupoids internal to **VectBund**.

A **connection** on $\mathbb{P}$ is a splitting A of this exact sequence, i.e a smooth functor $A\colon T\mathbb{M} \to T\mathbb{P}/\mathbb{G}$ such that $p \circ A = 1$.

Associated to the exact sequence above we get an **exact sequence** of Lie 2-algebras.

$$0 \to \Gamma(\mathrm{ad}(\mathbb{P})) \to \Gamma(T\mathbb{P}/\mathbb{G}) \to \Gamma(TM) \to 0$$

Here if $p\colon \mathbb{E} \to \mathbb{M}$ is a 2-vector bundle, $\Gamma(\mathbb{E})$ denotes the functors $s\colon \mathbb{M} \to \mathbb{E}$ such that $p \circ s = 1$ and natural transformations between these.

The Lie 3-algebra of Derivations

Let $\mathbb{L}$ be a strict Lie 2-algebra. A (strict) **derivation** of $\mathbb{L}$ is a linear functor $f\colon \mathbb{L} \to \mathbb{L}$ such that

$$f_0[x,y] = [f_0(x), y] + [x, f_0(y)]$$
$$f_1[u,v] = [f_1(u), v] + [u, f_1(v)]$$

for all objects x, y in L_0 and all morphisms u, v in L_1. A **morphism** of derivations is a linear natural transformation $\alpha\colon f \Rightarrow g$ such that

$$\alpha[x,y] = [\alpha(x), y] + [x, \alpha(y)]$$

The derivations of $\mathbb{L}$ and morphisms between them form a 2-vector space $\mathrm{Der}(\mathbb{L})$. We can equip $\mathrm{Der}(\mathbb{L})$ with a **bracket** functor $[\ ,\ ]\colon \mathbb{L} \times \mathbb{L} \to \mathbb{L}$ by defining

$$[f, f'] = f \circ f' - f' \circ f$$
$$[\alpha, \beta] = f'_1\beta + \alpha g_0 - g'_1\alpha - \beta f_0$$

$[\ ,\ ]$ is skew, bilinear and satisfies the Jacobi identity on the nose. $\mathrm{Der}(\mathbb{L})$ is a strict Lie 2-algebra. Define a homomorphism

$$\begin{aligned} \mathrm{ad}\colon \mathbb{L} &\to \mathrm{Der}(\mathbb{L}) \\ x &\mapsto \mathrm{ad}(x) \\ u &\mapsto \mathrm{ad}(u) \end{aligned}$$

where $\mathrm{ad}(x)$ is the derivation of $\mathbb{L}$ defined on objects by $\mathrm{ad}(x)(y) = [x,y]$ and similarly for morphisms.

The homomorphism $\mathrm{ad}\colon \mathbb{L} \to \mathrm{Der}(\mathbb{L})$ is an example of a (strict) *crossed module* of Lie 2-algebras.

A **crossed module** of Lie 2-algebras consists of a homomorphism $t\colon \mathbb{L} \to \mathbb{J}$ together with an action $\alpha\colon \mathbb{J} \times \mathbb{L} \to \mathbb{L}$ of $\mathbb{J}$ on $\mathbb{L}$ by **derivations** such that the following diagrams commute:

$$\begin{tikzcd} \mathbb{J} \times \mathbb{L} \arrow[r, "\alpha"] \arrow[d, "1\times t"'] & \mathbb{L} \arrow[d, "t"] \\ \mathbb{J} \times \mathbb{J} \arrow[r, "\mathrm{ad}"'] & \mathbb{J} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{L} \times \mathbb{L} \arrow[rr, "t\times 1"] \arrow[dr, "\mathrm{ad}"'] & & \mathbb{J} \times \mathbb{L} \arrow[dl, "\alpha"] \\ & \mathbb{L} & \end{tikzcd}$$

If we think of $\mathbb{L}$ and $\mathbb{J}$ themselves as crossed modules $d\colon L_1 \to L_0$ and $\partial\colon J_1 \to J_0$ then we have a commutative square

$$\begin{tikzcd} L_1 \arrow[r, "t_1"] \arrow[d, "d"'] & J_1 \arrow[d, "\partial"] \\ L_0 \arrow[r, "t_0"'] & J_0 \end{tikzcd}$$

in which each arrow, and each composite arrow, is a crossed module, together with some extra conditions.

Given such a crossed module $t\colon \mathbb{L} \to \mathbb{J}$ we can form a J_0-equivariant complex of Lie algebras

$$L_1 \xrightarrow{(t_1,d)} J_1 \ltimes L_0 \xrightarrow{t_0-\partial} J_0$$

where each arrow is a crossed module.

Here the bracket on $J_1 \ltimes L_0$ is defined by

$$[(x_1, \xi_1), (x_2, \xi_2)] = ([x_1, x_2], -[\xi_1, \xi_2] + \\ \alpha(\partial(x_1))(\xi_2) - \alpha(\partial(x_0))(\xi_1))$$

We associate a **Lie 3-algebra**, i.e a 2-category in **LieAlg**, to this complex with

$$\begin{aligned} &\text{objects} = J_0 \\ &\text{1-morphisms} = J_0 \ltimes (J_1 \ltimes L_0) \\ &\text{2-morphisms} = J_0 \ltimes ((J_1 \ltimes L_0) \ltimes L_1) \end{aligned}$$

We denote by $\mathrm{DER}(\mathbb{L})$ the Lie 3-algebra associated in this way to the crossed module $\mathrm{ad}\colon \mathbb{L} \to \mathrm{Der}(\mathbb{L})$.

Towards Higher Lie Schreier Theory

Suppose that

$$0 \to \mathbb{J} \to \mathbb{K} \to \underline{L} \to 0$$

is an **exact sequence** of Lie 2-algebras, where L is an ordinary Lie algebra and $\underline{L}$ denotes the corresponding discrete Lie 2-algebra.

Suppose that $A\colon \underline{L} \to \mathbb{K}$ is a splitting of this exact sequence. In analogy with the previous discussion we measure the failure of A to be a homomorphism of Lie 2-algebras. In general there exist morphisms in $\mathbb{J}$

$$[A(x), A(y)] \stackrel{B(x,y)}{\longrightarrow} A[x,y] + \nu(x,y)$$

natural, skew, and bilinear in x and y, where ν is a skew bilinear functor

$$\nu\colon \underline{L} \times \underline{L} \to \mathbb{J}.$$

This is the origin of the **fake curvature**. Under the homomorphism $\mathrm{ad}\colon \mathbb{J} \to \mathrm{Der}(\mathbb{J})$ the morphisms $B(x,y)$ become morphisms of derivations

$$[\mathrm{ad}_{A(x)}, \mathrm{ad}_{A(y)}] \stackrel{\mathrm{ad}_{B(x,y)}}{\longrightarrow} \mathrm{ad}_{A[x,y]} + \mathrm{ad}_{\nu(x,y)}$$

If we denote the derivation $\mathrm{ad}_{A(x)}$ of $\mathbb{J}$ by ∇_x then we can combine $\mathrm{ad}_{B(x,y)}$ and $\nu(x,y)$ into a 1-morphism in DER($\mathbb{J}$):

$$[\nabla_x, \nabla_y] \stackrel{\{\mathrm{ad}_{B(x,y)},\nu(x,y)\}}{\longrightarrow} \nabla_{[x,y]}$$

Define a bilinear functor

$$d_A \colon \bigwedge^p \underline{L}^* \otimes \mathbb{J} \to \bigwedge^{p+1} \underline{L}^* \otimes \mathbb{J}$$

using a similar formula to that above. We find that there is a skew, trilinear morpism $\omega(x,y,z)$, natural in x, y and z, such that

$$d_A\,\nu(x,y,z) = t\,\omega(x,y,z)$$
$$\omega(x,y,z) = d_A\,B(x,y,z)$$

ω satisfies a coherence law — the '**Higher Bianchi Identity**' — which can be interpreted as the equation

$$d_A\,\omega = [\nu, B]$$

This data can be neatly encoded as a homomorphism of semistrict Lie 3-algebras

$$\nabla\colon L \to \mathrm{DER}(\mathbb{J})$$

We have

- a homomorphism $L \to \mathrm{DER}(\mathbb{J})$

  $$L \ni x \mapsto \nabla_x$$

  where ∇_x is the derivation $\mathrm{ad}_{A(x)}$.

- a skew, bilinear natural transformation

  $$f_{x,y}\colon [\nabla_x, \nabla_y] \Rightarrow \nabla_{[x,y]}$$

  defined by

  $$f_{x,y} = \{B(x,y), \nu(x,y)\}$$

- a skew, trilinear modification ω as in the diagram

$$\begin{tikzcd}
{[\nabla_x, [\nabla_y, \nabla_z]]} \arrow[r, "="] \arrow[d, "{[\nabla_x, f_{y,z}]}"'] & {[[\nabla_x, \nabla_y], \nabla_z] + [\nabla_y, [\nabla_x, \nabla_z]]} \arrow[d, "{[f_{x,y}, \nabla_z] + [\nabla_y, f_{x,z}]}"] \\
{[\nabla_x, \nabla_{[y,z]}]} \arrow[d, "{f_{x,[y,z]}}"'] & {[\nabla_{[x,y]}, \nabla_z] + [\nabla_y, \nabla_{[x,z]}]} \arrow[d, "{f_{[x,y],z} + f_{y,[x,z]}}"] \\
\nabla_{[x,[y,z]]} \arrow[r, "="'] & \nabla_{[[x,y],z]} + \nabla_{[y,[x,z]]}
\end{tikzcd}$$

- ω is required to satisfy a coherence law, making a certain diagram commute