**An Introduction to $n$-Categories**

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May 13, 1997

**Abstract**

An $n$-category is some sort of algebraic structure consisting of objects, morphisms between objects, 2-morphisms between morphisms, and so on up to $n$-morphisms, together with various ways of composing them. We survey various concepts of $n$-category, with an emphasis on 'weak' $n$-categories, in which all rules governing the composition of $j$-morphisms hold only up to equivalence. (An $n$-morphism is an equivalence if it is invertible, while a $j$-morphism for $j < n$ is an equivalence if it is invertible up to a $(j+1)$-morphism that is an equivalence.) We discuss applications of weak $n$-categories to various subjects including homotopy theory and topological quantum field theory, and review the definition of weak $n$-categories recently proposed by Dolan and the author.

# 1 Introduction

Very roughly, an $n$-category is an algebraic structure consisting of a collection of 'objects', a collection of 'morphisms' between objects, a collection of '2-morphisms' between morphisms, and so on up to $n$, with various reasonable ways of composing these $j$-morphisms. A 0-category is just a set, while a 1-category is just a category. Recently $n$-categories for arbitrarily large $n$ have begun to play an increasingly important role in many subjects. The reason is that they let us *avoid mistaking isomorphism for equality.*

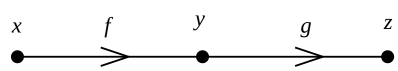
In a mere set, elements are either the same or different; there is no more to be said. In a category, objects can be different but still 'the same in a way'. In other words, they can be unequal but still isomorphic. Even better, we can explicitly keep track of the way they are the same: the isomorphism itself. This more nuanced treatment of 'sameness' is crucial to much of mathematics, physics, and computer science. For example, it underlies the modern concept of symmetry: since an object can be 'the same as itself in different ways', it has a symmetry group, its group of automorphisms. Unfortunately, in a category this careful distinction between equality and isomorphism breaks down when we study the morphisms. Morphisms in a category are either the same or different; there is no concept of isomorphic morphisms. In a 2-category this is remedied by introducing 2-morphisms between morphisms. Unfortunately, in a

2-category we cannot speak of isomorphic 2-morphisms. To remedy this we need the notion of 3-category, and so on.

The plan of this paper is as follows. We do not begin by defining $n$-categories. Many definitions have been proposed. So far, all of them are a bit complicated. Ultimately a number of them should turn out to be equivalent, but this has not been shown yet. In fact, the correct sense of 'equivalence' here is a rather subtle issue, intimately linked with $n$-category theory itself. Thus before mastering the details of any particular definition, it is important to have a sense of the issues involved. Section 2 starts with a rough sketch of various approaches to defining $n$-categories. Section 3 describes how $n$-categories are becoming important in a variety of fields, and how they should allow us to formalize some previously rather mysterious analogies between different subjects. Section 4 sketches a particular definition of 'weak $n$-categories' due to Dolan and the author [5]. In the Conclusions we discuss the sense in which various proposed definitions of weak $n$-category should be equivalent.

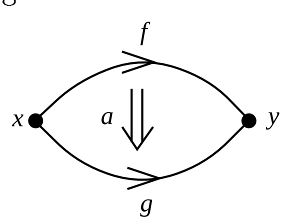
## 2 Various Concepts of $n$-Category

To start thinking about $n$-categories it is helpful to use pictures. We visualize the objects as 0-dimensional, i.e., points. We visualize the morphisms as 1-dimensional, i.e., intervals, or more precisely, arrows going from one point to another. In this picture, composition of morphisms corresponds to gluing together an arrow $f\colon x \to y$ and an arrow $g\colon y \to z$ to obtain an arrow $fg\colon x \to z$:



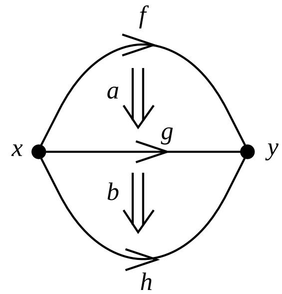

Note that while the notation $fg$ for the composite of $f\colon x \to y$ and $g\colon y \to z$ is somewhat nonstandard, it fits the picture better than the usual notation.

Continuing on in this spirit, we visualize the 2-morphisms as 2-dimensional, and compose 2-morphisms in a way that corresponds to gluing together 2-dimensional shapes. Of course, we should choose some particular shapes for our 2-morphisms. For example, we could use a 'bigon':

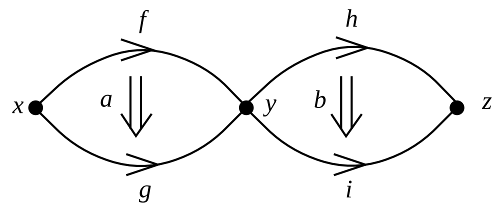



as the shape of a 2-morphism $a\colon f \Rightarrow g$ between morphisms $f, g\colon x \to y$ with the same source and target. This is the sort of 2-morphism used in the standard definitions of 'strict 2-categories' [21] — usually just called 2-categories — and the somewhat more general 'bicategories' [11]. There are two geometrically natural ways to compose

2-morphisms shaped like bigons. First, given 2-morphisms $a\colon f \Rightarrow g$ and $b\colon g \Rightarrow h$ as below, we can 'vertically' compose them to obtain a 2-morphism $a \cdot b\colon f \Rightarrow g$:



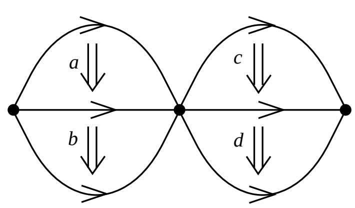

Second, given 2-morphisms $a\colon f \Rightarrow g$ and $b\colon h \Rightarrow i$ as below, we can 'horizontally' compose them to obtain a 2-morphism $ab\colon fh \Rightarrow gi$:




The definition of a strict 2-category is easy to state. The objects and morphisms must satisfy the usual rules holding in a category, while horizontal and vertical composition satisfy some additional axioms: vertical and horizontal composition are associative, for each morphism $f$ there is a 2-morphism $1_f\colon f \Rightarrow f$ that is an identity for vertical composition, and for each identity morphism $1_x$ the 2-morphism $1_{1_x}$ is also an identity for horizontal composition. Finally, we require the following 'interchange law' relating vertical and horizontal composition:

$$(a \cdot b)(c \cdot d) = (ac) \cdot (bd)$$

whenever either side is well-defined. This makes the following composite 2-morphism unambiguous:




We can think of it either as the result of first doing two vertical composites and then one horizontal composite, or as the result of first doing two horizontal composites and then one vertical composite.

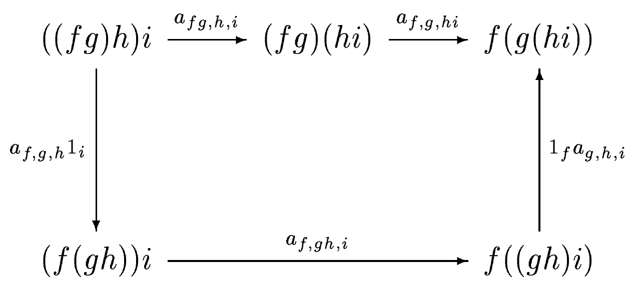
The definition of a bicategory is similar, but instead of requiring that the associativity and unit laws for morphisms hold 'on the nose' as equations, one requires merely that they hold up to isomorphism. Thus one has invertible 'associator' 2-morphisms

$$a_{f,g,h}\colon (fg)h \Rightarrow f(gh)$$

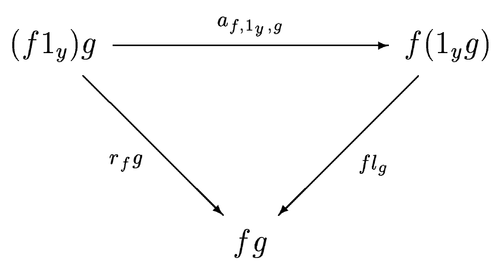
for every composable triple of morphisms, as well as invertible 2-morphisms called 'left and right identity constraints'

$$l_f \colon 1_x f \Rightarrow f, \qquad r_f \colon f 1_y \Rightarrow f$$

for every morphism $f \colon x \to y$. These must satisfy some equations of their own. For example, repeated use of associator lets one go from any parenthesization of a product of morphisms to any other parenthesization, but one can do so in many ways. To ensure that all these ways are equal, one imposes the Stasheff 'pentagon identity', which says that the following diagram commutes:




Mac Lane's coherence theorem [29] says that this identity suffices. Similarly, given morphisms $f \colon x \to y$ and $g \colon y \to z$, one requires that the following triangle commute:




One also requires that the associators and unit constraints are natural with respect to their arguments. Also, as with strict 2-categories, one requires that vertical composition be associative, that vertical and horizontal composition satisfy the interchange law, and that the morphisms $1_x$ are identities for vertical composition.

While bicategories at first seem more clumsy than strict 2-categories, they are applicable to a wider range of problems. The reason is that frequently 'everything is only true up to something'. In a sense, the whole point of introducing $(n+1)$-morphisms is to allow $n$-morphisms to be isomorphic rather than merely equal. From this point of view, it was inappropriate to have imposed equational laws between 1-morphisms in the definition of a strict 2-category, and the definition of bicategory corrects this problem. This is known as 'weakening'.

To see some bicategories that are not strict 2-categories, consider bicategories with one object. Given a bicategory $C$ with one object $x$, we can form a category $\tilde{C}$ whose objects are the morphisms of $C$ and whose morphisms are the 2-morphisms of $C$.

This is a special sort of category: we can 'multiply' the objects of $\tilde{C}$, since they are really just morphisms in $C$ from $x$ to itself. We call this sort of category — one that is really just a bicategory with one object — a 'weak monoidal category'. We can do the same thing starting with a strict 2-category and get a 'strict monoidal category'.

The category Set becomes a weak monoidal category if we multiply sets using the Cartesian product. However, it is not a strict monoidal category! The reason is that the Cartesian product is not strictly associative:

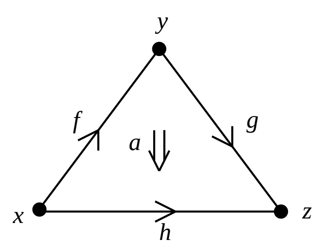
$$(X \times Y) \times Z \neq X \times (Y \times Z).$$

To see this, one needs to pry into the set-theoretic definition of ordered pairs. The usual von Neumann definition is $(x, y) = \{\{x\}, \{x, y\}\}$, and using this, we clearly do not have strict associativity for the Cartesian product. Instead, we have associativity *up to a specified isomorphism*, the associator:

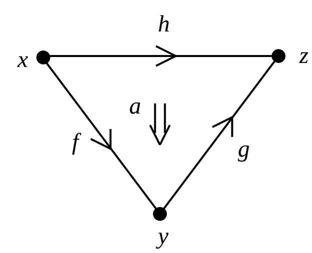
$$a_{X,Y,Z}\colon (X \times Y) \times Z \to X \times (Y \times Z)$$

which satisfies the pentagon identity. This is a typical example of how the bicategories found 'in nature' tend not to be strict 2-categories.

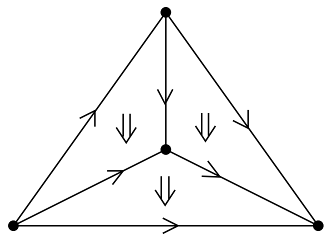
So far we have considered only bigons as possible shapes for 2-morphisms, but there are many other choices. For example, we might wish to use triangles going from a pair of morphisms $f\colon x \to y$, $g\colon y \to z$ to a morphism $h\colon x \to z$:




There is no good way to glue together triangles of this type to form other triangles of this type, but if we also allow the 'reverse' sort of triangle going from a single morphism to a pair:

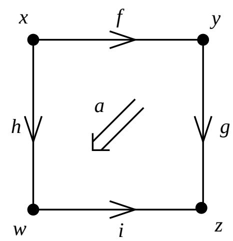



then there are various ways to glue together 3 triangles to form a larger one. For example:



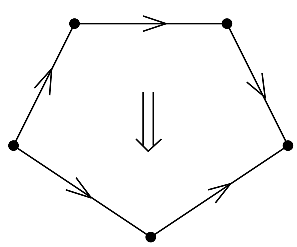
One can do quite a bit of topology in a purely combinatorial way using triangles and their higher-dimensional analogues, called simplices [31]. In these applications one often assumes that all $j$-morphisms are invertible, at least in some weakened sense. In the 2-dimensional case, this motivates the idea of 'reversing' a triangle.

Another approach would be to use squares going from a pair of morphisms $f\colon x \to y$, $g\colon y \to z$ to a pair $h\colon x \to w$, $i\colon w \to z$:



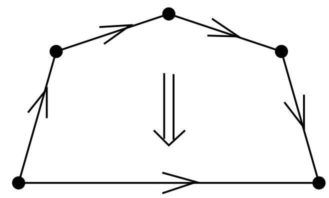

Much as with bigons, one can compose squares vertically and horizontally and require an 'interchange law' relating these two types of composition. This is the idea behind the definition of 'double categories' [16, 21], where the vertical arrows are treated as of a different type as the horizontal ones. If one treats the vertical and horizontal arrows as the same type, one obtains a theory equivalent to that of strict 2-categories.

Alternatively, one might argue that the business of picking a particular shape of 2-morphism as 'basic' is somewhat artificial. One might instead allow all possible polygons as shapes for 2-morphisms. The idea would be to use polygons whose boundary is divided into two parts having the arrows consistently oriented:



There are many ways to compose such polygons. However, while this approach might seem more general, one can actually define and work with these more general polygons within the theory of strict 2-categories [25, 34, 35].

Yet another approach would be to use only polygons having many 'infaces' but only one 'outface', like this:



As we shall see, this has certain advantages of its own.

What about $n$-categories for higher $n$? In general, $j$-morphisms can be visualized as $j$-dimensional solids, and part of their boundary represents the source while the rest represents the target. However, as $n$ increases one can imagine more and more definitions of '$n$-category', because there are more and more choices for the shapes of the $j$-morphisms. The higher-dimensional analogues of bigons are called 'globes'. Globes are the basic shape in the traditional approach to 'strict $n$-categories', often simply called '$n$-categories' [17, 15]. There has also been a lot of work on globular weak $n$-categories, such as Gordon, Power and Street's 'tricategories' [23], Trimble's 'tetracategories' [43], and Batanin's 'weak $\omega$-categories' [8], which can have $j$-morphisms of arbitrarily high dimension. The higher-dimensional analogues of triangles, namely simplices, are used in the 'Kan complexes' favored by topologists [31], as well as in Street's 'simplicial weak $\omega$-categories' [39] and Lawrence's '$n$-algebras' [28]. The higher-dimensional analogues of squares, namely cubes, are used in Ehresmann's '$n$-tuple categories' [16], as well as the work of Brown and his collaborators [13]. Finally, Dolan and the author [5] have given a definition of 'weak $n$-categories' based on some new shapes called 'opetopes'. We describe these in Section 4.

In addition to the issue of shapes for $j$-morphisms, there is the issue of the laws that composition operations should satisfy. Most importantly, there is the distinction between 'strict' and 'weak' approaches. In the 'strict' approach, composition of $j$-morphisms satisfies equational laws for all $j$. The philosophy behind the 'weak' approach is that equations should hold only at the top level, between $n$-morphisms. Laws concerning $j$-morphisms for $j < n$ should always be expressed as $(j+1)$-morphisms, or more precisely, 'equivalences'. Roughly, the idea here is that an equivalence between $(n-1)$-morphisms is an invertible $n$-morphism, while an equivalence between $j$-morphisms for lesser $j$ is recursively defined as a $(j+1)$-morphism that is invertible *up to equivalence.*

Strict $n$-categories are fairly well-understood [15], but the interesting and challenging sort of $n$-categories are the weak ones. Weak $n$-categories are interesting because these are the ones that tend to arise naturally in applications. The reason for this is simple yet profound. Equations of the form $x = x$ are completely useless. All interesting equations are of the form $x = y$. Equations of this form can always be viewed as asserting the existence of a reversible sort of computation transforming $x$ to $y$. In $n$-categorical terms, they assert the existence of an equivalence $f: x \to y$. To face up to this fact, it is helpful to systematically avoid equational laws and work explicitly with equivalences, instead. This leads naturally to working with weak $n$-categories, and eventually weak $\omega$-categories.

The reason $n$-categories are challenging is that when equational laws are replaced by equivalences, these equivalences need to satisfy new laws of their own, called 'coherence laws', so one can manipulate them with some of the same facility as equations. The main problem of weak $n$-category theory is: how does one systematically determine these coherence laws? A systematic approach is necessary, because in general these coherence laws must themselves be treated not as equations but as equivalences, which satisfy further coherence laws of their own, and so on! This quickly becomes very bewildering if one proceeds on an ad hoc basis.

For example, suppose one tries to write down definitions of 'globular weak $n$-categories', that is, weak $n$-categories in the approach where the $j$-morphisms are shaped like globes. These are usually called categories, bicategories, tricategories, tetracategories, and so on. The definition of a category is quite concise; the most complicated axiom is the associative law $(fg)h = f(gh)$. As we have seen, in the definition of a bicategory this law is replaced by a 2-morphism, the associator, which in turn satisfies the pentagon identity. In the definition of a tricategory, the pentagon identity is replaced by a 3-isomorphism satisfying a coherence law which is best depicted using a 3-dimensional commutative diagram in the shape of the 3-dimensional 'associahedron'. In the definition of a tetracategory, this becomes a 3-morphism which satisfies a coherence law given by the 4-dimensional associahedron. In fact, the associahedra of all dimensions were worked out by Stasheff [38] in 1963 using homotopy theory. However, there are other sequences of coherence laws to worry about, spawned by the equational laws of the form $1f = f = f1$, and also the interchange laws governing the various higher-dimensional analogues of 'vertical' and 'horizontal' composition.

At this point the reader can be forgiven for wondering if the rewards of setting up a theory of weak $n$-categories really justify the labor involved. Before proceeding, let us describe some of the things $n$-categories should be good for.

# 3 Applications of $n$-Categories

One expects $n$-categories to show up in any situation where there are things, processes taking one thing to another, 'meta-processes' taking one process to another, 'meta-meta-processes', and so on. Clearly computer science is deeply concerned with such situations. Unfortunately, the author is not competent to discuss applications to this subject! Some other places where applications are evident include:

1. $n$-category theory
2. homotopy theory
3. topological quantum field theory

The first application is circular, but not viciously so. The point is that the study of $n$-categories leads to applications of $(n+1)$-category theory. The other two applications may sound abstruse and specialized, but there is a good reason for discussing them here. Pure $n$-category theory treats the most general iterated notion of process. Homotopy theory limits its attention to processes that are 'invertible', at least up to equivalence. Topological quantum field theory focuses attention on processes which have 'adjoints' or 'duals'. While generally not invertible even up to equivalence, such processes are reversible in a broader sense (the classic examples from category theory being adjoint functors). In what follows we briefly summarize all three applications in turn.

## 3.1 $n$-Category Theory

While self-referential, this application is perhaps the most fundamental. A 0-category is just a *set*. When one studies sets one is naturally led to consider the set of all sets. However, this turns out to be a bad thing to do, not merely because of Russell's paradox (which is easily sidestepped), but because one is interested not just in sets but also in the functions between them. What is interesting is thus the *category* of all sets, Set.

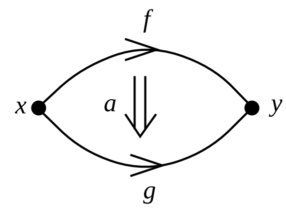
This category is in some sense the primordial category. Indeed, the Yoneda embedding theorem shows how every category can be thought of as a category of 'sets with extra structure'. However, when we study categories of sets with extra structure, it turns out to be worthwhile to develop category theory as a subject in its own right. In addition to categories and functors, natural transformations play a crucial role here. Thus one is led to study the *2-category* of all categories, Cat. This 2-category has categories as objects, functors between categories as morphisms, and natural transformations between functors as 2-morphisms.

The ladder of $n$-categories continues upwards in this way. For each $n$ there is an $(n+1)$-category of all $n$-categories, $n$Cat. To really understand $n$-categories we need to understand this $(n+1)$-category. Eventually this requires an understanding of $(n+1)$-categories in general, which then leads us to define $(n+1)$Cat.

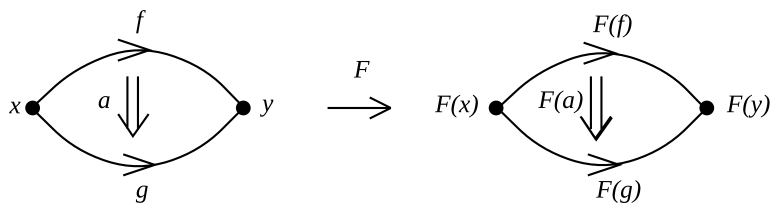
There are some curious subtleties worth noting here, though. The 2-category Cat happens to be a strict 2-category. We could think of it as a bicategory if we wanted, but weakening happens not to be needed here, since functors compose associatively 'on the nose', not just up to a natural transformation. Using the fact that Cat is the primordial 2-category, one can show that every bicategory is equivalent to a strict 2-category in a certain precise sense. Technically speaking, one proves this using the Yoneda embedding for bicategories [23].

The fact that every weak 2-category can be 'strictified' seems to have held back work on weak $n$-categories: it raised the hope that every weak $n$-category might be equivalent to a strict one. It turned out, however, that the strict and weak approaches diverge as we continue to ascend the ladder of $n$-categories. On the one hand, we can always construct a strict $(n+1)$-category of strict $n$-categories. On the other hand, we can construct a weak $(n+1)$-category of weak $n$-categories. The latter is *not* equivalent to a strict $(n+1)$-category for $n \geq 2$.

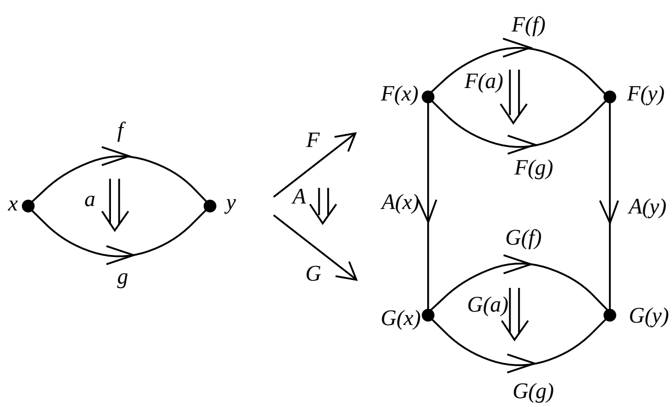
Consider the case $n = 2$. On the one hand, we can form a strict 3-category 2Cat whose objects are strict 2-categories. We can visualize a strict 2-category as a bunch of points, arrows and bigons. For simplicity, let us consider a very small 2-category $C$ with just one interesting 2-morphism:



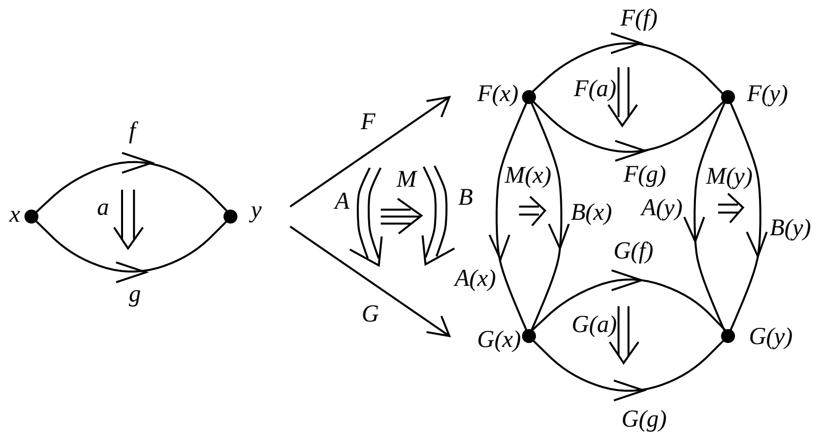

(We have not drawn the identity morphisms and 2-morphisms.) The morphisms in 2Cat are called '2-functors'. A 2-functor $F\colon C \to D$ sends objects to objects, morphisms to morphisms, and 2-morphisms to 2-morphisms, strictly preserving all structure. We can visualize $F$ as creating a picture of the 2-category $C$ in the 2-category $D$:




The 2-morphisms in 2Cat are called 'natural transformations'. A natural transformation $A\colon F \Rightarrow G$ between 2-functors $F, G\colon C \to D$ sends each object in $C$ to a morphism in $D$ and each morphism in $C$ to a 2-morphism in $D$, and satisfies some conditions similar to those in the definition of a natural transformation between functors. We can visualize $A$ as a prism going from one picture of $C$ in $D$ to another, built using commutative squares:




Finally, the 3-morphisms are called 'modifications'. A modification $M$ from a natural transformation $A\colon F \Rightarrow G$ to a natural transformation $B\colon F \Rightarrow G$ sends each object $x \in C$ to a 2-morphism $M(x)\colon A(x) \Rightarrow B(x)$ in $D$, in a manner satisfying some naturality conditions. We can visualize a modification $M$ as follows:




Note how the $n$-dimensionality of an $n$-category leads naturally to the $(n+1)$-dimensionality of $n$Cat.

If instead we adopt the weak approach, we can form a tricategory Bicat whose objects are bicategories [23]. The morphisms in Bicat are called 'pseudofunctors'. A pseudofunctor $F: C \to D$ need not strictly preserve all the structure. For example, given morphisms $f: x \to y$ and $g: y \to z$ in $C$, we do not require that $F(fg) = F(f)F(g)$. Instead, we require only that the two sides are isomorphic by a specified 2-morphism, which in turn must satisfy some coherence laws. The 2-morphisms in Bicat are called 'pseudonatural transformations'. In these, the squares that had to commute in the definition of a natural transformation need only commute up to a 2-isomorphism satisfying certain coherence laws. The 3-morphisms in Bicat are called 'modifications'. Here there is no room for weakening, since a modification sends each object in $C$ to a 2-morphism in $D$, and the only sort of laws that 2-morphisms can satisfy in a bicategory are equational laws.

One can show that Bicat is not equivalent to a strict 3-category, so we really need the more general notion of tricategory. Or do we? One might argue that we needed tricategories only because we made the mistake of not strictifying our bicategories. After all, every bicategory is equivalent to a strict 2-category. Perhaps if we replaced every bicategory with an equivalent strict 2-category, we could work in the strict 3-category 2Cat and never need to think about Bicat.

Alas, while superficially plausible, this line of argument is naive. We have said that every bicategory $C$ is equivalent to a strict 2-category $C'$. But what does 'equivalent' mean here, precisely? It means 'equivalent, as an object of Bicat'. In other words, for every bicategory $C$ there is a strict 2-category $C'$ and a pseudofunctor $F: C \to C'$ that is invertible up to a pseudonatural transformation that is invertible up to a modification that is invertible! In practice, therefore, the business of strictifying bicategories requires a solid understanding of Bicat as the full-fledged tricategory it is.

There is much more to say about this subject, but the basic point is that, like it or not, sets are weak 0-categories, and a deep understanding of weak $n$-categories requires an understanding of weak $(n+1)$-categories. For this reason mathematics has been forced, over the last century, to climb the ladder of weak $n$-categories. To see how computer science is repeating this climb, try for example the paper by Power [36] entitled 'Why tricategories?'.

The actual history of this climb is quite interesting, but the details are quite complicated, so we content ourselves here with a thumbnail sketch. While the formalization of the notion of set was a slow process, the now-standard Zermelo-Fraenkel axioms reached their final form in 1922. Categories were defined by Eilenberg and MacLane later in their 1945 paper [18]. Strict 2-categories were developed by Ehresmann [16] by 1962, and reinvented by Eilenberg and Kelly [17] in a paper appearing in the proceedings of a conference held in 1965. Gray [20] discussed Cat as a strict 2-category in the same conference proceedings, and Bénabou's [11] bicategories appeared in 1967. Gordon, Power and Street's definition of tricategories [23] was published in 1995, and about this time Trimble formulated the definition of tetracategories [43].

Subsequent work has concentrated on radically accelerating this process by defining weak $n$-categories for all $n$ simultaneously. Actually, Street [39] proposed a simplicial definition of weak $n$-categories for all $n$ in 1987, but this appears not to have been seriously studied, perhaps in part because it came too early! Starting in 1995, Dolan and the author gave a definition of weak $n$-categories using 'opetopic sets' [4, 5], Tamsamani gave a definition using 'multisimplicial sets' [41, 42], and Batanin gave definition of globular weak $\omega$-categories [8, 9, 40]. Dolan and the author have constructed the weak $(n+1)$-category of their $n$-categories, and Simpson [37] has constructed the weak $(n+1)$-category of Tamsamani's $n$-categories. Now the focus is turning towards working with these different definitions and seeing whether they are equivalent. We return to this last issue in the Conclusions.

## 3.2 Homotopy Theory

A less inbred application of $n$-category theory is to the branch of algebraic topology known as homotopy theory. In fact, many of our basic insights into $n$-categories come from this subject. The reason is not far to seek. Topology concerns the category Top whose objects are topological spaces and whose morphisms are continuous maps. Unfortunately, there is no useful classification of topological spaces up to isomorphism — an isomorphism in Top being called a 'homeomorphism'. When topologists realized this, they retreated to the goal of classifying spaces up to various coarser equivalence relations. Homotopy theory is all about properties that are preserved by continuous deformations. More precisely, given spaces $X, Y \in \mathrm{Top}$ and maps $F, G\colon X \to Y$, one defines a 'homotopy' from $F$ to $G$ to be a map $H\colon [0,1] \times x \to y$ with

$$H(0,\cdot) = F, \qquad H(1,\cdot) = G.$$

Homotopy theory studies properties of maps that are preserved by homotopies. Thus two spaces $X$ and $Y$ are 'the same' for the purposes of homotopy theory, or more precisely 'homotopy equivalent', if there are maps $F\colon X \to Y$, $G\colon Y \to X$ which are inverses *up to homotopy.*

In fact, what we have done here is made Top into a 2-category whose objects are spaces, whose morphisms are maps between spaces, and whose 2-morphisms are homotopies between maps. This allows us to replace the categorical concept of isomorphism between spaces by the more flexible 2-categorical concept of equivalence. However, work on homotopy theory soon led to the study of 'higher homotopies'. Since a homotopy is itself a map, the concept of a homotopy between homotopies makes perfect sense, and we may iterate this indefinitely. This amounts to treating Top as an $n$-category for arbitrarily large $n$, or for that matter, as an $\omega$-category.

It is worthwhile pondering how the seemingly innocuous category Top became an $\omega$-category. The key trick was to use the unit interval $[0, 1]$ to define higher-level morphisms. The reason this trick works is that the unit interval resembles an *arrow* going from 0 to 1. One could say that the abstract arrow we use in category theory is a kind of metaphor for the unit interval — or conversely, that the unit interval we use in topology is a kind of metaphor for the process of going from 'here' to 'there'. However, unlike the most general sort of abstract arrow, the unit interval has a special feature: we can go from 1 to 0 as easily as we can go from 0 to 1.

Taking advantage of this insight, Grothendieck [24] proposed thinking of homotopy theory as a branch of $n$-category theory, as follows. We should be able to associate to any space $X$ a weak $\omega$-category $\Pi(X)$ whose objects are points $x \in X$, whose morphisms are paths (maps $F\colon [0, 1] \to X$) going from one point to another, whose 2-morphisms are certain paths of paths, and so on. Due to the special feature of the unit interval, every $j$-morphism in this $\omega$-category should be an equivalence. We call this special sort of $\omega$-category an '$\omega$-groupoid', since a category with all morphisms invertible is called a groupoid.

Grothendieck also argued that conversely, we should be able to obtain a topological space $N(G)$ from any weak $\omega$-groupoid $G$, essentially by taking seriously the picture we can draw with points for objects of $G$, intervals for morphisms of $G$, and so on. By this means we should be able to obtain weak $\omega$-functors $\Pi\colon \mathrm{Top} \to \omega\mathrm{Gpd}$ and $N\colon \omega\mathrm{Gpd} \to \mathrm{Top}$. Using these, we should be able to show that that the weak $\omega$-categories Top and $\omega$Gpd are equivalent, as objects of $\omega$Cat. In short, homotopy theory is another word for the study of $\omega$-groupoids!

There many ways to try to realize this program, a number of which have already obtained results. It is well-known that all of homotopy theory can be done purely combinatorially using 'Kan complexes' [31], which may be regarded as simplicial weak $\omega$-categories. Brown, Higgins, Loday, and collaborators have developed a variety of approaches using cubes [13]. Kapranov and Voevodsky [26] have shown that homotopy theory is in principle equivalent to the study of their '$\infty$-groupoids'. Tamsamani has also shown that his approach to weak $n$-categories reduces to homotopy theory in the $n$-groupoid case [42].

Many homotopy theorists might doubt the importance of seeing homotopy theory as a branch of $n$-category theory. In a sense, they already implicitly know many of the lessons $n$-category theory has to offer: the idea of replacing equations by equivalences, the importance of 'homotopies between homotopies', and the crucial importance of coherence laws. Eventually $n$-category should be able to help homotopy theory in its treatment of morphisms that are not equivalences. In the short term, however, the question is not what $n$-categories can do for homotopy theory, but what homotopy theory can do for $n$-categories.

In fact, many ideas in $n$-category theory have already had their origin in homotopy theory. A good example is Stasheff's work on the associahedron [38]. Recall that a 'monoid' is a set equipped with an associative product and multiplicative unit, while a 'topological monoid' is a monoid equipped with a topology for which the product is continuous. Stasheff wanted to uncover the homotopy-invariant structure contained in a topological monoid. Suppose $X$ is a topological monoid and $Y$ is a space equipped with a homotopy equivalence to $X$. What sort of structure does $Y$ inherit from $X$? Clearly we can use the homotopy equivalence to transport the product and unit from $X$ to $Y$, obtaining a product and unit on $Y$ satisfying the laws of a monoid *up to homotopy.* For example, the two maps

$$F, G: Y \times Y \times Y \to Y$$

given by

$$\begin{aligned} F(y_1, y_2, y_3) &= (y_1 y_2) y_3 \\ G(y_1, y_2, y_3) &= y_1 (y_2 y_3) \end{aligned}$$

need not be equal, but there is a homotopy between them, the 'associator'. Stasheff showed that this associator satisfies the pentagon identity up to homotopy, and that this homotopy satisfies a coherence law of its own, again up to homotopy, and so on ad infinitum. By working out these coherence laws in detail, he discovered the associahedron. Later the associahedron turned out to be relevant to weak $\omega$-categories in general. Part of the reason is that we can think of the space $Y$ above as a special sort of $\omega$-category. A monoid can be thought of as a category with one object, by viewing the monoid elements as morphisms from this object to itself. Similarly, we can view $Y$ as a weak $\omega$-category with one object, points of $Y$ as morphisms from this object to itself, paths between these as 2-morphisms, and so on.

## 3.3 Topological Quantum Field Theory

In physics, interest in $n$-categories was sparked by developments in relating topology and quantum field theory [22]. One can roughly date the beginning of this story to 1985, when Jones came across a wholly unexpected invariant of knots while studying some operator algebras invented by von Neumann in his work on the mathematical foundations of quantum theory. Soon this 'Jones polynomial' was generalized to a family of knot invariants. It was then realized that these generalizations could be systematically derived from algebraic structures known as 'quantum groups', first invented by Drinfel'd and collaborators in their work on exactly soluble 2-dimensional field theories [14, 30]. This approach involved 2-dimensional pictures of knots. The story became even more exciting when Witten came up with a manifestly 3-dimensional approach to the new knot invariants, deriving them from a quantum field theory in 3-dimensional spacetime now known as Chern-Simons-Witten theory. This approach also gave invariants of 3-dimensional manifolds.

These developments exposed a deep but mysterious unity in what at first might seem like disparate branches of algebra, topology, and quantum physics. Interestingly, it appears that the roots of this unity lie in certain aspects of $n$-category theory. In fact, this is the main reason for the author's interest in $n$-categories: it seems that a good theory of weak $n$-categories is needed as a framework for the mathematics that will be able to reduce the currently rather elaborate subject of 'topological quantum field theory' to its simple essence. Having explained this at length elsewhere [3], we limit our remarks here to a few key points.

Quantum physics relies crucially on the theory of Hilbert spaces. For simplicity, we limit our attention here to the finite-dimensional case, defining a 'Hilbert space' to be a finite-dimensional complex vector space $H$ equipped with an 'inner product'

$$\langle \cdot, \cdot \rangle \colon H \times H \to \mathbb{C}$$

which is linear in the second argument, conjugate-linear in the first, and satisfies $\langle \psi, \phi \rangle = \overline{\langle \phi, \psi \rangle}$ for all $\psi, \phi \in H$ and $\langle \psi, \psi \rangle > 0$ for all nonzero $\psi \in H$. The inner product allows us to define the norm of a vector $\psi \in H$ by

$$\|\psi\| = \langle \psi, \psi \rangle^{1/2},$$

but its main role in physics is to compute amplitudes. States of a quantum system are described by vectors with norm 1. If one places a quantum system in the state $\psi$, and then does an experiment to see if it is in some state $\phi$, the probability that the answer is 'yes' equals

$$|\langle \phi, \psi \rangle|^2.$$

This is automatically a real number between 0 and 1. However, when one delves deeper into the theory, it appears that even more fundamental than the probability is the 'amplitude'

$$\langle \phi, \psi \rangle,$$

which is of course a complex number.

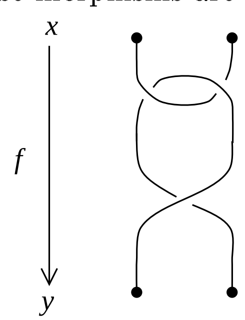
The role of the inner product in quantum physics has always been a source of puzzles to those with an interest in the philosophical foundations of the subject. Complex amplitudes lack the intuitive immediacy of probabilities. From the category-theoretic point of view, part of the problem is to understand the category of Hilbert spaces. The objects are Hilbert spaces, but what are the morphisms? Typically morphisms are required to preserve all the structure in sight. This suggests taking the morphisms to be linear operators preserving the inner product. However, other linear operators are also important. Particularly in topological quantum field theory, there are good reasons to take *all* linear operators as morphisms. However, if we define a category Hilb this way, then Hilb is equivalent to the category Vect of complex vector spaces. This then raises the question: how does Hilb really differ from Vect, if as categories they are equivalent?

Luckily, quantum theory suggests an answer to this question. Given any linear operator $F: H \to H'$ between Hilbert spaces, we may define the 'adjoint' $F^*: H' \to H$ to be the unique linear operator with

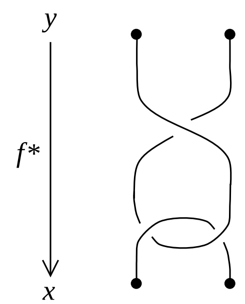
$$\langle F\phi, \psi \rangle = \langle \phi, F^*\psi \rangle$$

for all $\phi \in H$, $\psi \in H'$. This sort of adjoint is basic to quantum theory. From the category-theoretic point of view, the role of the adjoint is to make Hilb into a '$*$-category': a category $C$ equipped with a contravariant functor $*: C \to C$ fixing objects and satisfying $*^2 = 1_C$. While Hilb and Vect are equivalent as categories, only Hilb is a $*$-category.

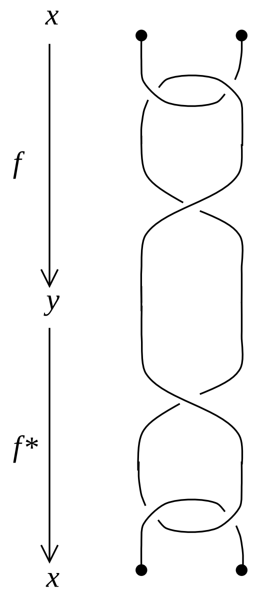
This is particularly important in topological quantum field theory. The mysterious relationships between topology, algebra and physics exploited by this subject amount in large part to the existence of interesting functors from various topologically defined categories to the category Hilb. These topologically defined categories are always $*$-categories, and the really interesting functors from them to Hilb are always '$*$-functors', functors preserving the $*$-structure. Physically, the $*$ operation corresponds to *reversing the direction of time*. For example, there is a $*$-category whose objects are collections of points and whose morphisms are 'tangles':




We can think of this morphism $f: x \to y$ as representing the trajectories of a collection of particles and antiparticles, where particles and antiparticles can be created or annihilated in pairs. Reversing the direction of time, we obtain the 'dual' morphism $f^*: y \to x$:




This morphism is not the inverse of $f$, since the composite $ff^*$ is a nontrivial tangle:




Any groupoid becomes a $*$-category if we set $f^* = f^{-1}$ for every morphism $f$, but the most interesting $*$-categories in topological quantum field theory are not groupoids.

The above example involves 1-dimensional curves in 3-dimensional spacetime. More generally, topological quantum field theory studies $n$-dimensional manifolds embedded in $(n + k)$-dimensional spacetime, which in the $k \to \infty$ limit appear as 'abstract' $n$-dimensional manifolds. It appears that these are best described using certain '$n$-categories with duals', meaning $n$-categories in which every $j$-morphism $f$ has a dual $f^*$. Unfortunately, so far the details have only been worked out in certain low-dimensional cases [2, 6]. The main problem is that the notion of '$n$-category with duals' is only beginning to be understood.

One class of $n$-categories with duals should be the $n$-groupoids; this would explain many relationships between topological quantum field theory and homotopy theory [33]. However, the novel aspects of topological quantum field theory should arise from $n$-categories with duals that are not $n$-groupoids. Indeed, this explains why the Jones polynomial and other new knot invariants were not discovered earlier using traditional techniques of algebraic topology.

The idea that duals are subtler and thus more interesting than inverses is already familiar from category theory. Given a functor $F: C \to D$, the correct sort of weakened 'inverse' to $F$ is a functor $G: D \to C$ such that $FG$ and $GF$ are naturally isomorphic to the identity; if such a functor $G$ exists then $F$ is an equivalence. However, even if no such 'inverse' exists, the functor $F$ may have a kind of 'dual', namely an adjoint functor! A right adjoint $F^*: C \to D$, for example, would satisfy:

$$\hom(Fx, y) \cong \hom(x, F^*y)$$

for all $x \in C$, $y \in D$. Note that this is very similar to the definition of the adjoint of a linear map between Hilbert spaces, with 'hom' playing the role of the inner product.

The analogy between adjoint functors and adjoint linear operators relies upon a deeper analogy: just as in quantum theory the inner product $\langle \phi, \psi \rangle$ represents the *amplitude* to pass from $\phi$ to $\psi$, in category theory $\hom(x, y)$ represents the *set of ways* to go from $x$ to $y$. A precise working out of this analogy can be found in the author's paper [2] on '2-Hilbert spaces'. These are to Hilbert spaces as categories are to sets. The analogues of adjoint linear operators between Hilbert spaces are certain adjoint functors between 2-Hilbert spaces. Just as the primordial example of a category is Set, the primordial example of a 2-Hilbert space is Hilb. Also, just as the 2-category Cat is a 3-category, it appears that the 2-category 2Hilb is an example of a '3-Hilbert space' — a concept which has not yet been given a proper definition.

More generally, it appears that $n$Hilb is an $n$-category with duals, and that '$n$-Hilbert spaces' are needed for the proper treatment of $n$-dimensional topological quantum field theories [3, 19]. Thus, just as mathematics has been forced to ascend the ladder of $n$-categories, so may be physics!

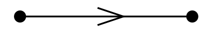
## 4 A Definition of Weak $n$-Category

As discussed in Section 2, any definition of $n$-categories involves a choice of the basic shapes of $j$-morphisms and a choice of allowed ways to glue them together. Any definition of weak $n$-categories also requires a careful treatment of coherence laws. In what follows we present a definition of weak $n$-categories in which all these issues are handled in a tightly linked way. In this definition, the basic shapes of $j$-morphisms are the $j$-dimensional 'opetopes'. The allowed ways of gluing together the $j$-dimensional opetopes correspond precisely to the $(j+1)$-dimensional opetopes. Moreover, the coherence laws satisfied by composition correspond to still higher-dimensional opetopes!

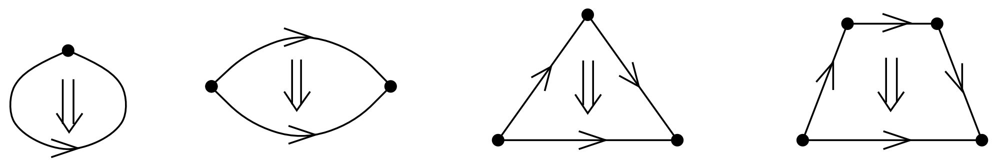
Before going into the details, let us give a rough sketch of this works. First consider some low-dimensional opetopes. The only 0-dimensional opetope is the point:

$$\bullet$$

There is no way to glue together 0-dimensional opetopes. The only 1-dimensional opetope is the interval, or more precisely the arrow:

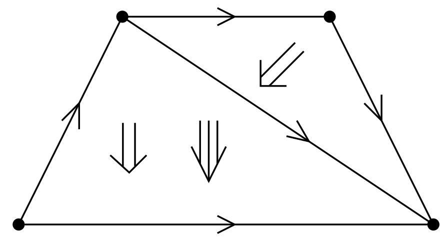


The allowed ways of gluing together 1-dimensional opetopes are given by the 2-dimensional opetopes. The first few 2-dimensional opetopes are as follows:



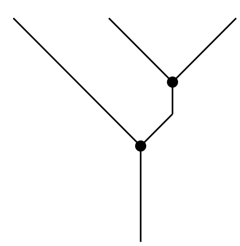
For any $k \geq 0$, there is a 2-dimensional opetope with $k$ 'infaces' and one 'outface'. (We are glossing over some subleties here; for reasons noted later, there are really $k!$ such opetopes.)

The allowed ways of gluing together 2-dimensional opetopes are given by the 3-dimensional opetopes. There are many of these; a simple example is as follows:



This may be a bit hard to visualize, but it depicts a 3-dimensional shape whose front consists of two 3-sided 'infaces', and whose back consists of a single 4-sided 'outface'. We have drawn double arrows on the infaces but not on the outface. Note that while this shape is topologically a ball, it cannot be realized as a polyhedron with planar faces. This is typical of opetopes.

In general, a $(n+1)$-dimensional opetope has any number of infaces and exactly one outface: the infaces are $n$-dimensional opetopes glued together in a tree-like pattern, while the outface is a single $n$-dimensional opetope. For example, the 3-dimensional opetope above corresponds to the following tree:

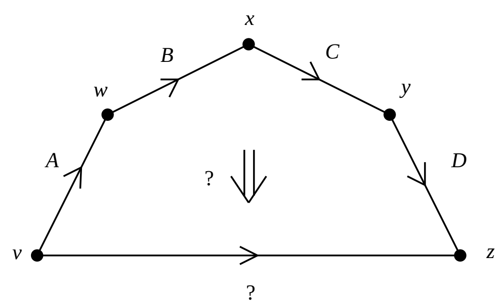


The two triangular infaces of the opetope correspond to the two nodes in this tree. This is a rather special tree; in general, we allow nonplanar trees with any number of nodes and any number of edges coming into each node.

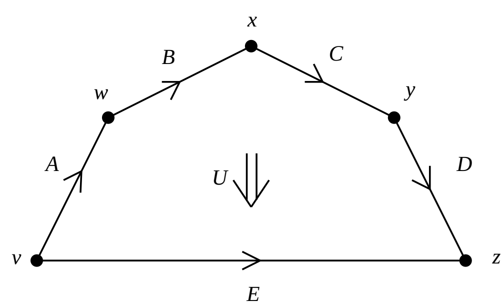
In our approach, a weak $n$-category is a special sort of 'opetopic set'. Basically, an opetopic set is a set of 'cells' shaped like opetopes, such that any face of any cell is again a cell. In a weak $n$-category, the $j$-dimensional cells play the role of

$j$-morphisms. An opetopic set $C$ is an $n$-category if it satisfies the following two properties:

1) *"Any niche has a universal occupant."* A 'niche' is a configuration where all the infaces of an opetope have been filled in by cells of $C$, but not the outface or the opetope itself:

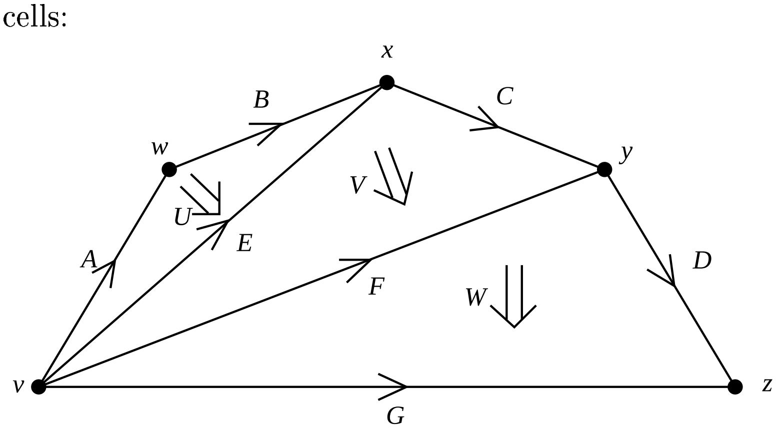



An 'occupant' of the niche is a way of extending this configuration by filling in the opetope (and thus its ouface) with a cell:




The 'universality' of an occupant means roughly that every other occupant factors through the given one *up to equivalence*. To make this precise we need to define universality in a rather subtle recursive way. We may think of a universal occupant of a niche as 'a process of composing' the infaces, and its outface as 'a composite' of the infaces.

2) *"Composites of universal cells are universal."* Suppose that $U, V$, and $W$ below are universal cells:



Then we can compose them, and we are guaranteed that their composite is again universal, and thus that the outface $G$ is a composite of the cells $A, B, C, D$. Note that a process of composing $U, V, W$ is described by a universal occupant of a niche of one higher dimension.

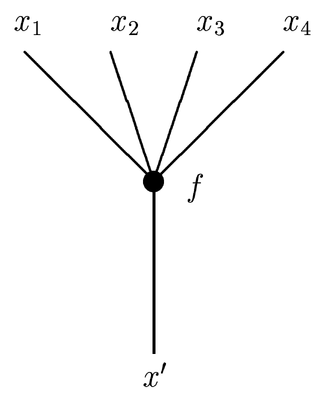
Note that in this approach to weak $n$-categories, composition of cells is not an operation in the traditional sense: the composite is defined by a universal property, and is thus unique only *up to equivalence*. Only at the top level, for the $n$-cells in an $n$-category, is the composite truly unique. This may seem odd at first glance, but in fact it closely reflects actual mathematical practice. For example, it is unnatural to think of the Cartesian product as an operation on Set. We can do it, but there are as many ways to do so as there are ways to define ordered pairs in set theory; there is certainly nothing sacred about the von Neumann definition. If we arbitrarily choose a way, we can think of Set as a weak monoidal category, i.e., a bicategory with one object. However, we can avoid this arbitrariness if we define the Cartesian product of sets by a universal property, using the category-theoretic concept of 'product'. Then the product of sets is only defined up to a natural isomorphism, and Set becomes our sort of weak 2-category with one object. In this approach, all the necessary coherence laws *follow automatically* from the universal property defining the product.

In the following sections we first review the theory of operads, and then explain how this theory can be used to define the opetopes. After a brief discussion of some notions related to opetopic sets, we give the definition of 'universal occupant' of a niche, and then the definition of weak $n$-category. At various points we skim over technical details; these can all be found in our paper [5].

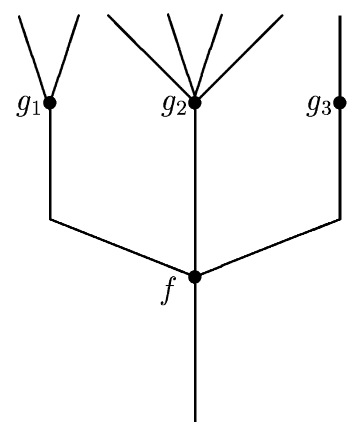
## 4.1 Operads

To describe the opetopes we need to specify exactly which tree-like patterns we can use to glue together the opetopes of a given dimension; these are the opetopes of the next higher dimension. For this we use the theory of 'operads'. An operad is a gadget consisting of abstract $k$-ary 'operations' and various ways of composing them, and the $n$-dimensional opetopes will be the operations of a certain operad. This is another example of how $n$-category theory is indebted to homotopy theory, since operads were first developed for the purposes of homotopy theory [1, 12, 31].

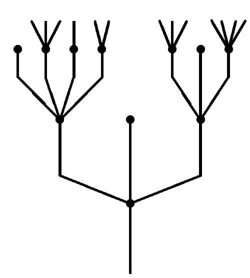
In what follows we work with typed operads having a set $S$ of types, or '$S$-operads' for short. The basic idea of an $S$-operad $O$ is that given types $x_1, \ldots, x_k, x' \in S$, there is a set $O(x_1, \ldots, x_k; x')$ of abstract $k$-ary 'operations' with inputs of type $x_1, \ldots, x_k$ and output of type $x'$. We can visualize such an operation as a tree with only one node:




In an operad, we can get new operations from old ones by composing them, which we can visualize in terms of trees as follows:




We can also obtain new operations from old by permuting arguments, and there is a unary 'identity' operation of each type. Finally, we demand a few plausible axioms: the identity operations act as identities for composition, permuting arguments is compatible with composition, and composition is 'associative', making composites of the following sort well-defined:



Formally, we have:

**Definition 1.** *For any set $S$, an '$S$-operad' $O$ consists of*

1. *for any* $x_1, \ldots, x_k, x' \in S$*, a set* $O(x_1, \ldots, x_k; x')$

2. *for any* $f \in O(x_1, \ldots, x_k; x')$ *and any* $g_1 \in O(x_{11}, \ldots, x_{1i_1}; x_1), \ldots,$ $g_k \in O(x_{k1}, \ldots, x_{ii_k}; x_k)$*, an element*

$$f \cdot (g_1, \ldots, g_k) \in O(x_{11}, \ldots, x_{1i_1}, \ldots\ldots, x_{k1}, \ldots, x_{ii_k}; x')$$

3. *for any* $x \in S$*, an element* $1_x \in O(x; x)$

4. *for any permutation* $\sigma \in S_k$*, a map*

$$\begin{array}{rcl} \sigma\colon O(x_1, \ldots, x_k; x') & \to & O(x_{\sigma(1)}, \ldots, x_{\sigma(k)}; x') \\ f & \mapsto & f\sigma \end{array}$$

*such that:*

(a) *whenever both sides make sense,*

$$f \cdot (g_1 \cdot (h_{11}, \ldots, h_{1i_1}), \ldots, g_k \cdot (h_{k1}, \ldots, h_{ki_k})) =$$

$$(f \cdot (g_1, \ldots g_k)) \ \cdot \ (h_{11}, \ldots, h_{1i_1}, \ldots\ldots, h_{k1}, \ldots, h_{ki_k})$$

(b) *for any* $f \in O(x_1, \ldots, x_k; x')$,

$$f = 1_{x'} \cdot f = f \cdot (1_{x_1}, \ldots, 1_{x_k})$$

(c) *for any* $f \in O(x_1, \ldots, x_k; x')$ *and* $\sigma, \sigma' \in S_k$,

$$f(\sigma\sigma') = (f\sigma)\sigma'$$

(d) *for any* $f \in O(x_1, \ldots, x_k; x')$, $\sigma \in S_k$*, and* $g_1 \in O(x_{11}, \ldots, x_{1i_1}; x_1), \ldots,$ $g_k \in O(x_{k1}, \ldots, x_{ki_k}; x_k)$,

$$(f\sigma) \cdot (g_{\sigma(1)}, \ldots, g_{\sigma(k)}) = (f \cdot (g_1, \ldots, g_k))\, \rho(\sigma),$$

*where* $\rho\colon S_k \to S_{i_1+\cdots+i_k}$ *is the obvious homomorphism.*

(e) *for any* $f \in O(x_1, \ldots, x_k; x')$, $g_1 \in O(x_{11}, \ldots, x_{1i_1}; x_1), \ldots,$ $g_k \in O(x_{k1}, \ldots, x_{ki_k}; x_k)$*, and* $\sigma_1 \in S_{i_1}, \ldots, \sigma_k \in S_{i_k}$,

$$(f \cdot (g_1\sigma_1, \ldots, g_k\sigma_k)) = (f \cdot (g_1, \ldots, g_k))\, \rho'(\sigma_1, \ldots, \sigma_k),$$

*where* $\rho'\colon S_{i_1} \times \cdots \times S_{i_k} \to S_{i_1+\cdots+i_k}$ *is the obvious homomorphism.*

There is an obvious notion of a morphism from an $S$-operad $O$ to an $S$-operad $O'$: a function mapping each set $O(x_1, \ldots, x_k, x')$ to the corresponding set $O'(x_1, \ldots, x_k, x')$, preserving composition, identities, and the symmetric group actions. An important example is an 'algebra' of an operad, in which its abstract operations are represented as actual functions:

**Definition 2.** *For any $S$-operad $O$, an '$O$-algebra' $A$ consists of:*

1. *for any $x \in S$, a set $A(x)$.*
2. *for any $f \in O(x_1, \ldots, x_k; x')$, a function*

$$\alpha(f)\colon A(x_1) \times \cdots \times A(x_k) \to A(x')$$

*such that:*

(a) *whenever both sides make sense,*

$$\alpha(f \cdot (g_1, \ldots, g_k)) = \alpha(f)(\alpha(g_1) \times \cdots \times \alpha(g_k))$$

(b) *for any $x \in C$, $\alpha(1_x)$ acts as the identity on $A(x)$*

(c) *for any $f \in O(x_1, \ldots, x_k, x')$ and $\sigma \in S_k$,*

$$\alpha(f\sigma) = \alpha(f)\sigma,$$

*where $\sigma \in S_k$ acts on the function $\alpha(f)$ on the right by permuting its arguments.*

We can think of an operad as a simple sort of theory, and its algebras as models of this theory. Thus we can study operads either 'syntactically' or 'semantically'. To describe an operad syntactically, we list:

1. the set $S$ of *types*,
2. the sets $O(x_1, \ldots, x_k; x')$ of *operations*,
3. the set of all *reduction laws* saying that some composite of operations (possibly with arguments permuted) equals some other operation.

This is like a presentation in terms of generators and relations, with the reduction laws playing the role of relations. On the other hand, to describe an operad semantically, we describe its algebras.

## 4.2 Opetopes

The following fact is the key to defining the opetopes. Let $O$ be an $S$-operad, and let $\mathrm{elt}(O)$ be the set of all operations of $O$.

**Theorem 3.** *There is an* $\mathrm{elt}(O)$*-operad* $O^+$ *whose algebras are* $S$*-operads over* $O$*, i.e.,* $S$*-operads equipped with a homomorphism to* $O$*.*

We call $O^+$ the 'slice operad' of $O$. One can describe $O^+$ syntactically as follows:

1. The types of $O^+$ are the operations of $O$.
2. The operations of $O^+$ are the reduction laws of $O$.
3. The reduction laws of $O^+$ are the ways of combining reduction laws of $O$ to give other reduction laws.

The 'level-shifting' going on here as we pass from $O$ to $O^+$ captures the process by which equational laws are promoted to equivalences and these equivalences satisfy new coherence laws of their own. In this context, the new laws are just *the ways of combining the old laws.*

Note that we can iterate the slice operad construction. Let $O^{n+}$ denote the result of applying the slice operad construction $n$ times to the operad $O$ if $n \geq 1$, or just $O$ itself if $n = 0$.

**Definition 4.** *An* $n$*-dimensional '*$O$*-opetope' is a type of* $O^{n+}$*, or equivalently, if* $n \geq 1$*, an operation of* $O^{(n-1)+}$*.*

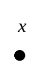
In particular, we define an $n$-dimensional 'opetope' to be an $n$-dimensional $O$-opetope when $O$ is the simplest operad of all:

**Definition 5.** *The 'initial untyped operad'* $I$ *is the* $S$*-operad with:*

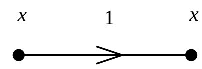
1. *only one type:* $S = \{x\}$
2. *only one operation, the identity operation* $1 \in O(x; x)$
3. *all possible reduction laws*

Semantically, $I$ is the operad whose algebras are just sets.

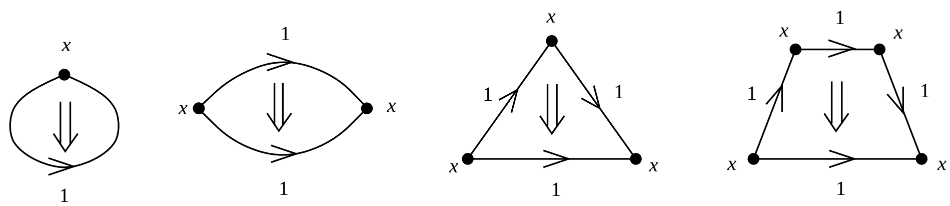
The opetopes emerge from $I$ as follows. The 0-dimensional opetopes are the types of $I$, but there is only one type, so there is only one 0-dimensional opetope:



The 1-dimensional opetopes are the types of $I^+$, or in other words, the operations of $I$. There is only one operation of $I$, the identity operation, so there is only one 1-dimensional opetope:

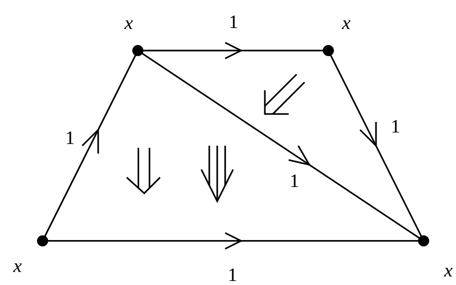



The 2-dimensional opetopes are the types of $I^{++}$, or in other words, the operations of $I^+$, which are the reduction laws of $I$. These reduction laws all state that the identity operation composed with itself $k$ times equals itself. This leads to 2-dimensional opetopes with $k$ infaces and one outface, as follows:



Actually there are $k!$ different 2-dimensional opetopes with $k$ infaces, since the permutation group $S_k$ acts freely on the set of $k$-ary operations of $I^+$. We could keep track of these by labelling the infaces with some permutation of $k$ distinct symbols.

The 3-dimensional opetopes are the types of $I^{+++}$, or in other words, the operations of $I^{++}$, which are the reduction laws of $I^+$. These state that some composite of 2-dimensional opetopes equals some other 2-dimensional opetope. This leads to 3-dimensional opetopes like the following:



In general, the $(n+1)$-dimensional opetopes describe all possible ways of composing $n$-dimensional opetopes. Since the $(n+1)$-dimensional opetopes are the operations of the operad $I^{n+}$, all allowed ways of composing them can be described by trees. One can use this to describe the opetopes of all dimensions using 'metatree notation'. In this notation, an $n$-dimensional opetope is represented as a list of $n$ labelled trees. This notation nicely handles the nuances of how the permutation group $S_k$ acts on the set of opetopes with $k$ infaces.

## 4.3 Opetopic sets

A weak $n$-category will be an 'opetopic set' with certain properties. An opetopic set consists of collections of 'cells' of different shapes, one collection for each opetope.

The face of any cell is again a cell, and one can keep track of this using 'face maps' going between the collections of cells. These maps also satisfy certain relations. We omit the details here, but it is worth noting that all this can be handled nicely using a trick widely used in algebraic topology [31]. The idea is that there is a category Op whose objects are opetopes. The morphisms in this category describe how one opetope is included as a specified face of another. An opetopic set may then be defined as a contravariant functor $S\colon \mathrm{Op} \to \mathrm{Set}$. Such a functor assigns to each opetope $t$ a set $S(t)$ of cells of that shape, or '$t$-cells'. Moreover, if $f\colon s \to t$ is a morphism in Op describing how $s$ is a particular face of $t$, the face map $S(f)\colon S(t) \to S(s)$ describes how each $t$-cell of $S$ has a given $s$-cell as this particular face.

For the definition of a weak $n$-category we need some terminology concerning opetopic sets. If $j \geq 1$, we may schematically represent a $j$-dimensional cell $x$ in an opetopic set as follows:

$$(a_1, \ldots, a_k) \xrightarrow{\quad x \quad} a'$$

Here $a_1, \ldots, a_k$ are the infaces of $x$ and $a'$ is the outface of $x$; all these are cells of one lower dimension. A configuration just like this, satisfying all the incidence relations satisfied by the boundary of a cell, but with $x$ itself missing:

$$(a_1, \ldots, a_k) \xrightarrow{\quad ? \quad} a'$$

is called a 'frame'. A 'niche' is like a frame with the outface missing:

$$(a_1, \ldots, a_k) \xrightarrow{\quad ? \quad} ?$$

Similarly, a 'punctured niche' is like a frame with the outface and one inface missing:

$$(a_1, \ldots, a_{i-1}, ?, a_{i+1}, \ldots, a_k) \xrightarrow{\quad ? \quad} ?$$

If one of these configurations (frame, niche, or punctured niche) can be extended to an actual cell, the cell is called an 'occupant' of the configuration. Occupants of the same frame are called 'frame-competitors', while occupants of the same niche are called 'niche-competitors'.

## 4.4 Universality

The only thing we need now to define the notion of weak $n$-category is the concept of a 'universal' occupant of a niche. This is also the subtlest aspect of the whole theory. We explain it briefly here, but it seems that the only way to really understand it is to carefully work through examples.

Before confronting the precise definition of universality, it is important to note that the main role of universality is to define the notion of 'composite':

**Definition 6.** *Given a universal occupant $u$ of a $j$-dimensional niche:*

$$(a_1, \ldots, a_k) \xrightarrow{\quad u \quad} b$$

*we call $b$ a 'composite' of $(a_1, \ldots, a_k)$.*

It is also important to keep in mind the role played by cells of different dimensions. In our framework an $n$-category usually has cells of arbitrarily high dimension. For $j \leq n$ the $j$-dimensional cells play the role of $j$-morphisms, while for $j > n$ they play the role of 'equations', 'equations between equations', and so on. The definition of universality depends on $n$ in a way that has the following effects. For $j \leq n$ there may be many universal occupants of a given $j$-dimensional niche, which is why we speak of 'a' composite rather than 'the' composite. There is at most one occupant of any given $(n+1)$-dimensional niche, which is automatically universal. Thus composites of $n$-cells are unique, and we may think of the universal occupant of an $(n+1)$-dimensional niche as an equation saying that the composite of the infaces equals the outface. For $j > n+1$ there is exactly one occupant of each $j$-dimensional frame, indicating that the composite of the equations corresponding to the infaces equals the equation corresponding to the outface.

The basic idea of universality is that a $j$-dimensional niche-occupant is universal if all of its niche-competitors factor through it uniquely, *up to equivalence.* For $j \geq n+1$ this simply amounts to saying that each niche has a unique occupant, while for $j = n$ it means that each niche has an occupant through which all of its niche-competitors factor uniquely. In general, we require that composition with a universal niche-occupant set up a 'balanced punctured niche' of one higher dimension. Heuristically, one should think of a balanced punctured niche as defining an equivalence between occupants of its outface and occupants of its missing outface.

**Definition 7.** *A $j$-dimensional niche-occupant:*

$$(c_1, \ldots, c_k) \xrightarrow{\quad u \quad} d$$

*is said to be 'universal' if and only if $j > n$ and $u$ is the only occupant of its niche, or $j \leq n$ and for any frame-competitor $d'$ of $d$, the $(j+1)$-dimensional punctured niche*

$$\begin{array}{c} ((c_1, \ldots, c_k) \xrightarrow{u} d,\ d \xrightarrow{?} d') \\ \Big\downarrow {\scriptstyle ?} \\ (c_1, \ldots, c_k) \xrightarrow{?} d' \end{array}$$

*and its mirror-image version*

$$\begin{array}{c}(d \xrightarrow{?} d',\ (c_1, \ldots, c_k) \xrightarrow{u} d) \\ \Big\downarrow ? \\ (c_1, \ldots, c_k) \xrightarrow{?} d'\end{array}$$

*are balanced.*

Finally we must define the concept of 'balanced punctured niche'. The reader may note that the first numbered condition in the following definition generalizes the concept of an 'essentially surjective' functor, while the second generalizes the concept of a 'fully faithful' functor.

**Definition 8.** *An $m$-dimensional punctured niche:*

$$(a_1, \ldots, a_{i-1}, ?, a_{i+1}, \ldots, a_k) \xrightarrow{\quad ? \quad} ?$$

*is said to be 'balanced' if and only if $m > n+1$ or:*

1. *any extension*

$$(a_1, \ldots, a_{i-1}, ?, a_{i+1}, \ldots, a_k) \xrightarrow{\quad ? \quad} b$$

*extends further to:*

$$(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_k) \xrightarrow{\quad u \quad} b$$

*with $u$ universal in its niche, and*

2. *for any occupant*

$$(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_k) \xrightarrow{\quad u \quad} b$$

*universal in its niche, and frame-competitor $a'_i$ of $a_i$, the $(m+1)$-dimensional punctured niche*

$$\begin{array}{c}(a'_i \xrightarrow{?} a_i,\ (a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_k) \xrightarrow{u} b) \\ \Big\downarrow ? \\ (a_1, \ldots, a_{i-1}, a'_i, a_{i+1}, \ldots, a_k) \xrightarrow{?} b\end{array}$$

*and its mirror-image version*

$$\begin{array}{c} ((a_1,\ldots,a_{i-1},a_i,a_{i+1},\ldots,a_k)\xrightarrow{u}b,\ a_i'\xrightarrow{?}a_i) \\ \Big\downarrow ? \\ (a_1,\ldots,a_{i-1},a_i',a_{i+1},\ldots,a_k)\xrightarrow{?}b \end{array}$$

*are balanced.*

Note that while the definitions of 'balanced' and 'universal' call upon each other recursively, there is no bad circularity. Using these definitions, it is easy to define a weak $n$-category. While the definition below does not explicitly depend on $n$, it depends on $n$ through the definition of 'universal' niche-occupant.

**Definition 9.** *A 'weak $n$-category' is an opetopic set such that 1) every niche has a universal occupant, and 2) composites of universal cells are universal.*

# 5 Conclusions

The above definition of weak $n$-category is really a beginning, rather than an end. We turn the reader to the papers by Dolan and the author [5] and the forthcoming work of Hermida, Makkai and Power for more. The weak $(n+1)$-category of weak $n$-categories is beginning to be understood; the generalizations of $n$-categories where we replace $I$ by an arbitrary operad have also turned out to be very interesting. However, before the really interesting applications of $n$-category theory can be worked out, there is still much basic work to be done.

In particular, it is important to compare various different definitions of weak $n$-category, so that the subject does not fragment. As one might expect, the question of when two definitions of weak $n$-category are 'equivalent' is rather subtle. This question seems to have first been seriously pondered by Grothendieck [24], who proposed the following solution. Suppose that for all $n$ we have two different definitions of weak $n$-category, say '$n$-category$_1$' and '$n$-category$_2$'. Then we should try to construct the $(n+1)$-category$_1$ of all $n$-categories$_1$ and the $(n+1)$-category$_1$ of all $n$-categories$_2$ and see if these are equivalent as objects of the $(n+2)$-category$_1$ of all $(n+1)$-categories$_1$. If so, we may say the two definitions are equivalent as seen from the viewpoint of the first definition.

There are some touchy points here worth mentioning. First, there is considerable freedom of choice involved in constructing the two $(n+1)$-categories$_1$ in question; one should do it in a 'reasonable' way, but this is not necessarily easy. Secondly, there is no guarantee that we might not get a different answer for the question if we reversed the roles of the two definitions. Nonetheless, it should be interesting to compare different definitions of weak $n$-category in this way.

A second solution is suggested by homotopy theory, which again comes to the rescue. Many different approaches to homotopy theory are in use, and though superficially very different, there is a well-understood sense in which they are fundamentally the same. Different approaches use objects from different categories to represent topological spaces, or more precisely, the homotopy-invariant information in topological spaces, called their 'homotopy types'. These categories are not equivalent, but each one is equipped with a class of morphisms playing the role of homotopy equivalences. Given a category $C$ equipped with a specified class of morphisms called 'equivalences', under mild assumptions one can adjoin inverses for these morphisms, and obtain a category called the 'homotopy category' of $C$. Two categories with specified equivalences may be considered the same for the purposes of homotopy theory if their homotopy categories are equivalent in the usual sense of category theory. Homotopy theorists have proved that all the popular approaches to homotopy theory are the same in this sense [10].

The same strategy should be useful in $n$-category theory. Any definition of weak $n$-category should come along with a definition of an '$n$-functor' for which there is a category with weak $n$-categories as objects and $n$-functors as morphisms, and there should be a specified class of $n$-functors called 'equivalences'. This allows the construction a homotopy category of $n$-categories. Then, for two definitions of weak $n$-category to be considered equivalent, we require that their homotopy categories be equivalent.

Dolan and the author have constructed the homotopy category of their $n$-categories, and Simpson [37] has constructed the homotopy category of Tamsamani's $n$-categories. Now we need machinery to check whether these homotopy categories, and those corresponding to other definitions, are equivalent. Once these preliminary chores are completed, there should be many exciting things we can do with $n$-categories.

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