N-CATEGORIES IN HOMOLOGY THEORY

A heuristic introduction...

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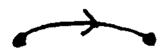
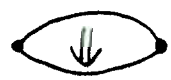
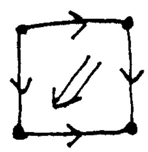
There are many competing approaches to n-category theory. In all, the <u>idea</u> is that an n-category has

objects :  (or "0-morphisms")

morphisms :  (or "1-morphisms")

2-morphisms : 

or 

or 

or ...

& so on up to n-morphisms, with various ways to compose these, satisfying various geometrically plausible laws: either <u>strictly</u> (as equations) or <u>weakly</u> (up to equivalence).

Batanin's definition of strict & weak n-categories begins with the concept of:

<u>Globular set</u> : a diagram of sets & functions

$$C_0 \underset{t}{\overset{s}{\leftleftarrows}} C_1 \underset{t}{\overset{s}{\leftleftarrows}} C_2 \underset{t}{\overset{s}{\leftleftarrows}} \cdots .$$

such that

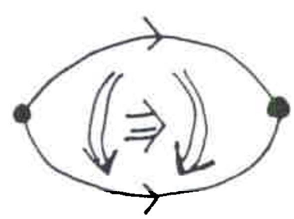
$$s(s(x)) = s(t(x))$$
$$t(s(x)) = t(t(x))$$

Elements of C_j are called <u>j-cells</u>, or in an n-category, <u>j-morphisms</u>:

$$\overset{x}{\bullet}$$ 0-cell

$$\begin{tikzcd} \underset{sx}{\bullet} \arrow[r, "x"'] & \underset{tx}{\bullet} \end{tikzcd}$$ 1-cell

$$\begin{tikzcd} s(s(x)) = s(t(x)) \arrow[r, bend left, "sx"] \arrow[r, bend right, "tx"'] \arrow[r, phantom, "\Downarrow x"] & t(s(x)) = t(t(x)) \end{tikzcd}$$ 2-cell

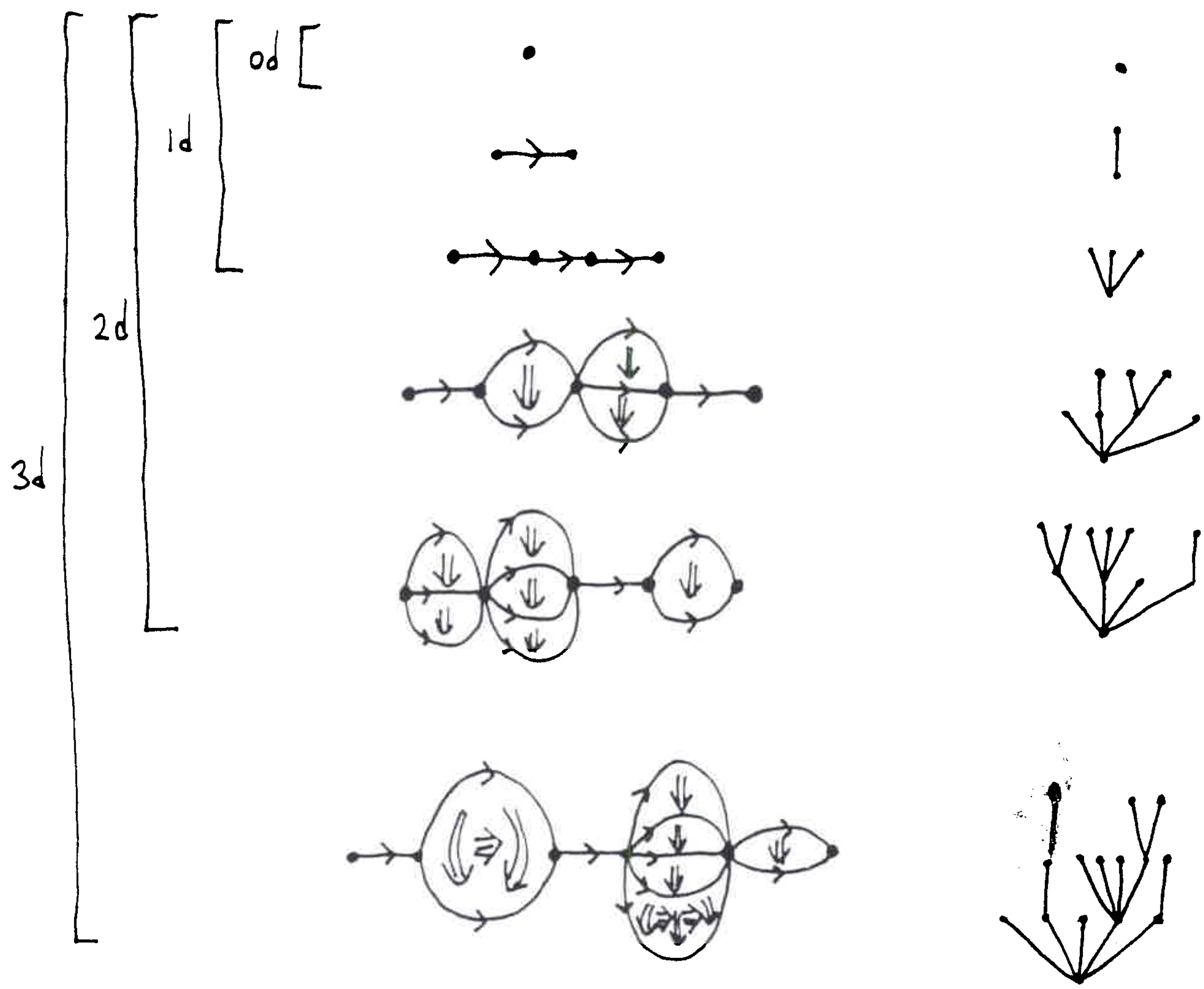


3-cell

$\vdots$

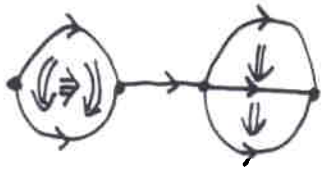
Composition is defined using the concept of :

<u>Cell colony</u> : a k-dimensional cell colony is a globular set of this sort:

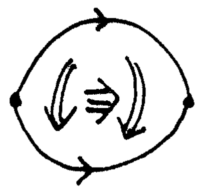


k-dimensional cell colonies correspond to $\leq k$-stage planar trees.

A strict or weak <u>ω-category</u> is a globular set C where we can <u>compose</u> cells arranged in any k-dimensional cell colony and get a k-cell:



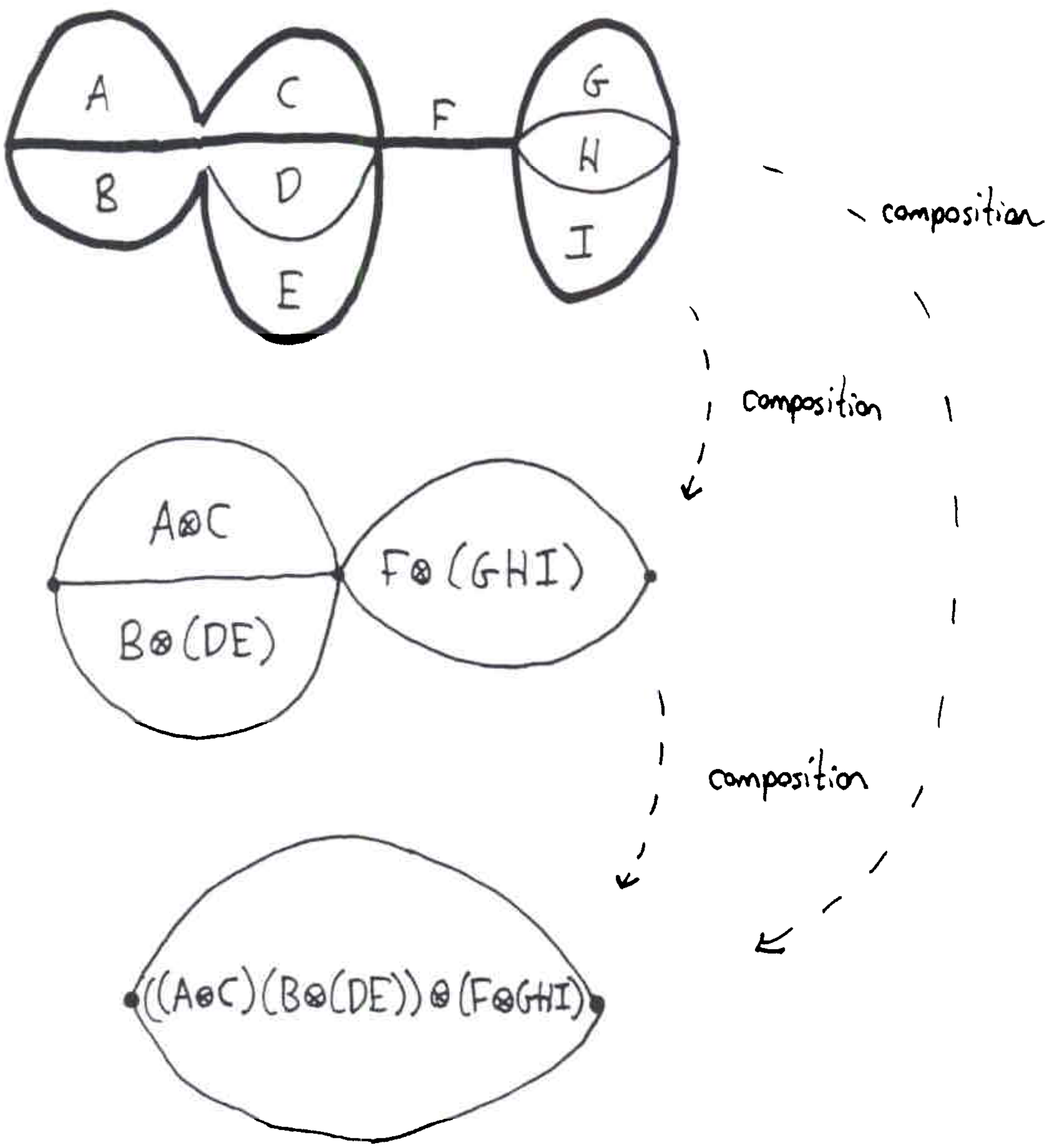
gives



whose source & target are themselves composites in an obvious way.

In a <u>strict</u> ω-category these composition operations satisfy "all possible laws"; in a <u>weak</u> one they satisfy "all possible laws up to equivalence", where these equivalences are constructed using extra operations which themselves satisfy "all possible laws up to equivalence", etc. ad infinitum!

More precisely, in a strict ω-category we can compose the cells arranged in some cell colony "all at once", or "a bit at a time", & get the same result:



Weak ω-categories are far more complicated!

Example: In a strict ω-category, composition of 1-morphisms is associative:

$$\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \xrightarrow{h} \bullet$$

$$(fg)h = fgh = f(gh)$$

In a weak ω-category we instead have operations that produce 2-morphisms:

$$\begin{tikzcd}
(fg)h \arrow[r, Rightarrow, shift left, "\alpha_{f,g,h}"] & fgh \arrow[l, Rightarrow, shift left, "\bar{\alpha}_{f,g,h}"] \arrow[r, Rightarrow, shift left, "\beta_{f,g,h}"] & f(gh) \arrow[l, Rightarrow, shift left, "\bar{\beta}_{f,g,h}"]
\end{tikzcd}$$

$\left.\begin{array}{l}\alpha, \bar{\alpha} \\ \beta, \bar{\beta}\end{array}\right\}$ not inverses, but there are operations

$$A_{f,g,h} : 1 \Rrightarrow \alpha_{f,g,h} \bar{\alpha}_{f,g,h}$$

$$A'_{f,g,h} : 1 \Rrightarrow \bar{\alpha}_{f,g,h} \alpha_{f,g,h}$$

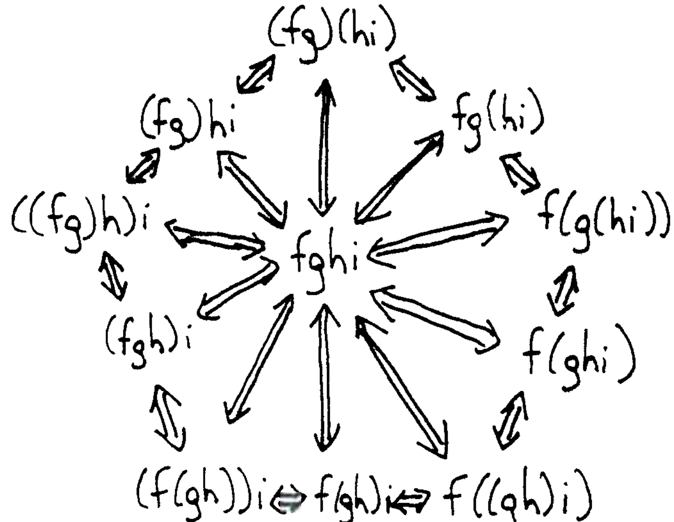
$$B_{f,g,h} : 1 \Rrightarrow \beta_{f,g,h} \bar{\beta}_{f,g,h} \qquad \text{etc.!}$$

$$\bar{B}_{f,g,h} : 1 \Rrightarrow \bar{\beta}_{f,g,h} \beta_{f,g,h}$$

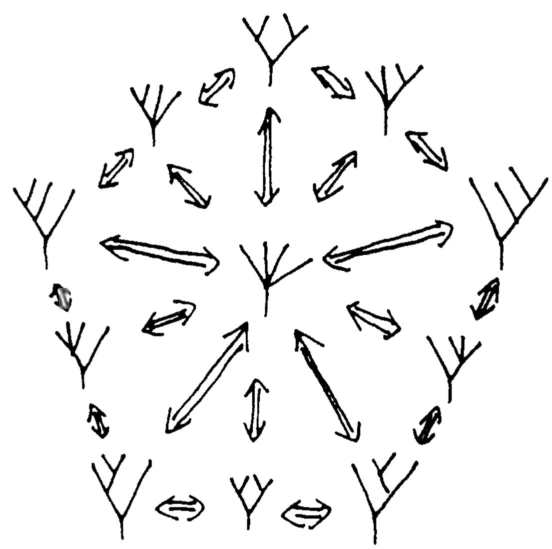
In this situation:

$$\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \xrightarrow{h} \bullet \xrightarrow{i} \bullet$$

we get:



Barycentric subdivision of pentagon!
(& higher-dimensional associahedra)



In either the strict or weak worlds...

- An <u>ω-groupoid</u> is an ω-category where all j-morphisms $(j>0)$ are invertible (strictly or weakly, as the case may be).
- An <u>n-category</u> is an ω-category where all j-morphisms for $j>n$ are identities.
- An <u>n-groupoid</u> is an n-category that is an ω-groupoid.
- A <u>k-tuply</u> <u>monoidal</u> <u>n-category</u> is an $(n+k)$-category with only one j-morphism for $j<k$.
- A <u>k-tuply</u> <u>groupal</u> <u>n-groupoid</u> is an $(n+k)$-groupoid with only one j-morphism for $j<k$.

includes $n=\omega$!

WEAK N-CATEGORIES — THE DREAM

Let a <u>k-tuply</u> <u>monoidal</u> <u>n-category</u> be a weak $(n+k)$-category with only one j-morphism for $j<k$. We expect:

	$n=0$	$n=1$	$n=2$	(etc.)
$k=0$	sets	categories	2-categories	
$k=1$	monoids	monoidal categories	monoidal 2-categories	
$k=2$	commutative monoids	braided monoidal categories	braided monoidal 2-categories	
$k=3$	" "	symmetric monoidal categories	sylleptic monoidal 2-categories	
$k=4$	" "	" "	symmetric monoidal 2-categories	
(etc.)				

In the weak world, it is conjectured that:

ω-groupoids	$\approx$	homotopy types
n-groupoids	$\approx$	homotopy types of spaces with $\pi_i = 0$ for $i > n$.
k-tuply groupal ω-groupoids	$\approx$	homotopy types of spaces with $\pi_i = 0$ for $i < k$.
k-tuply groupal n-groupoids	$\approx$	homotopy types of spaces with $\pi_i = 0$ for $i > n+k$ or $i < k$.
	$\approx$	homotopy n-types of k-fold loop spaces

equivalence of homotopy categories, or more.

Famous operations in homotopy theory have extensions to the world of weak n-categories – extensions from $n\mathrm{Gpd}_k$ to $n\mathrm{Cat}_k$:

$$
\begin{tikzcd}
n\mathrm{Cat}_k \arrow[rr, bend left, "\text{Discrete}"'] \arrow[d, bend right, "\text{Suspension}"'] & & (n+1)\,\mathrm{Cat}_k \arrow[ll, bend right, "\text{Decategorification}"'] \arrow[dll, bend left, "\text{Looping}"] \\
n\mathrm{Cat}_{k+1} \arrow[u, bend right, "\text{Forgetful}"'] \arrow[urr, bend left, "\text{Delooping}"] & &
\end{tikzcd}
$$

Decategorification identifies isomorphic n-morphisms & discards (n+1)-morphisms: it's like "killing π_{n+1}".

Stabilization hypothesis : suspension / forgetting form an equivalence for $k \geq n+2$.

In the strict world, it is known that:

ω-groupoids $\approx$ homotopy types of spaces with trivial Postnikov k-invariants

groupal ω-groupoids $\approx$ homotopy types of connected spaces with trivial Postnikov k-invariants

$\approx \prod_{n \geq 2} K(\pi_n, n)$-bundles over a $K(\pi_1, 1)$

$\approx$ crossed complexes

doubly groupal ω-groupoids $\approx$ homotopy types of simply connected spaces with trivial Postnikov k-invariants

$\approx \prod_{n \geq 1} K(\pi_n, n)$'s

$\approx$ chain complexes

& then it stabilizes!