MORPHISMS FOR RESISTIVE ELECTRICAL NETWORKS

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Dedicated to Dr. David Imagawa MD of UC Irvine's Chao Cancer Center, whose surgical skill, dedication to healing, and compassion made this work possible.

ABSTRACT. This paper presents a notion of morphism for resistive electrical networks. It is obvious that symmetries of a resistive network are (trivial) morphisms and easy to show that they induce a nontrivial quotient (their symmetrizer). We extend this result to arbitrary RESNET morphisms having a common domain (called the pushout in category theory). Our principle application of RESNET morphisms is to solve Kirchoff's problem for certain products of paths.

1. MacLane's Categorical Imperative

In his classic monograph, Categories for the Working Mathematician ([20], the last paragraph of Chapter 1) Saunders Mac Lane writes, "Category theory asks of every mathematical structure, 'What are its morphisms?'" This paper might be regarded as advancing the applications of category theory advocated by Mac Lane. However, for this author the search for morphisms did not begin with Mac Lane's dogmatic question, but with a combinatorial problem posed by G.-C. Rota [23]. Rota asked if there is an analog of Sperner's theorem (the largest antichain in the Boolean lattice of subsets of an n-set is the largest rank) for the lattice of partitions of an n-set. In collaborative work on Rota's question [2], Ron Graham and the author observed a nice notion of morphism (called *flowmorphism* in [9]) for the MaxFlow problem of Ford-Fulkerson. Eventually we realized that the weighted Sperner problem was dual to a MinFlow problem and therefore also preserved by flowmorphisms. The definition of flowmorphism then led to a strengthening of the Sperner property for posets: A poset is called *normal* if its Hasse diagram is the domain of a flowmorphism whose range is a path (total order). The Product Theorem of [9] gave conditions under which the product of normal posets is normal. At the time this generalized all known variants of Sperner's theorem and produced several new ones. Subsequently, [11] began the systematic investigation of flowmorphisms. Work on Rota's question culminated (after thirty years) in a remarkably precise answer (See [4] and [5]). Variants of Sperner's theorem have been surveyed in [3].

Around 1975, having made progress with Sperner problems using flowmorphisms and having discovered MacLane's book [20] the author decided to try the same approach, what we now call *morphismology*, with another combinatorial problem. The decision as to which problem or established theory to morphologize was somewhat

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arbitrary. The hope was that this approach would work for many (*all* seemed overly optimistic) combinatorial theories. The very first attempt, looking for a notion of morphism for the edge-isoperimetric problem on graphs (that just happened to have been the subject of the author's first paper [8]) proved successful [10]. Rota pointed out that the notion of morphism for the edge-isoperimetric problem, called *stabilization,* was a discrete analog of Steiner symmetrization, so its generalizations were dubbed *Steiner operations*. A monograph on Steiner operations, based on the author's graduate course notes, was published in 2004 [15]. In the monograph it is observed that Steiner operations are special cases of a more general notion of morphism for minimum path problems on a network. Those were called *pathmorphisms*. A companion volume on flowmorphisms is in preparation. Optimization of paths and flows are still the only combinatorial problems whose morphisms the author has studied in depth, but other papers do define morphisms for related problems: [12], [13], [14] & [16].

The reason for limiting major efforts to just flowmorphisms and Steiner operations was the feeling that morphismology needed depth more than breadth. Both combinatorics and category theory (the abstract study of morphisms) have suffered from reputations for superficiality. Using concrete morphisms and the categorical methods they generate to solve hard combinatorial problems would demonstrate their synergy and benefit both fields. Also, these two problems, MaxFlow and MinPath, are the premier problems of combinatorial optimization and algorithmic analysis.

Other authors have studied combinatorial morphisms and categories. The monograph by Hell & Nesetril on graph homomorphisms [7] cites over 350 papers. There is also Joyal's monograph on enumeration [17] and the work by Crapo and others on morphisms for matroids [6]. The text by Lawvere & Schanuel [19], in presenting their case for category theory being fundamental for all of mathematics, employs many combinatorial examples.

2. Morphisms for Resistive Networks

Why resistive networks? Again, the choice was somewhat arbitrary; this just seemed the best of many possibilities. The structures involved are simple and similar to those for flowmorphisms and pathmorphisms. There is a large body of literature, stretching back to Kirchoff and Maxwell, upon which to draw. But the primary reason was that the theory of resistive networks has the characteristics (simple (polynomial bounded) algorithms, infinite and even continuous analogs) that seem to go with a nice, effective notion of morphism. For years, when people asked about other possibilities for this kind of categorification, we have suggested electrical networks (positional games (see [1]) is another outstanding candidate), but are not aware that anyone has taken up the project.

2.1. **Basic Definitions & Classical Theory.**

The analysis of electrical circuits is the theoretical foundation of electrical engineering and has been for more than a hundred years. The basic ideas are summarized by Wikipedia and also in many textbooks such as Electric Circuits by Nilsson & Riedel [21]. The topological structure underlying electrical circuit theory is that of an "ordinary graph", *i.e.* a collection of "edges" (wires through which current may flow in either direction), meeting at "vertices" (electronic components or just junctions of wires). In the standard presentations an arbitrary direction is selected for each edge of the ordinary graph to facilitate calculation. Here we begin with the definition of a directed graph because it is simpler, avoids the extra step of selecting directions and facilitates constructions that come later.

A *directed graph (digraph),* $G = (V, E, \partial_+, \partial_-)$, consists of a set, V, of *vertices*, a set, E, of *edges* and functions $\partial_\pm : E \to V$ that give the head and tail ends, respectively, of each edge. A *circuit* (undirected) in a digraph is a (circular) sequence of edges, $e_1, e_2, ..., e_k$, such that each consecutive pair of edges, $e_i, e_{i+1(\bmod k)}$ have a common vertex (which can be any of the four possible combinations: $\partial_\pm (e_i) = \partial_{\pm'} (e_{i+1(\bmod k)})$).

A *resistive network*, $N = (G, \{s, t\}, R)$ consists of

(1) A digraph G,
(2) A pair of distinct vertices, s, t, the *terminals* of the network.
(3) A function $R : E \to \mathbb{R}^+$, which specifies the *resistance* of each edge.

Example 1. *Let Q_3 be the network whose graph is the 3-cube (see Fig. 1, s, t are two antipodal vertices, all edges are directed upward and have resistance 1.*

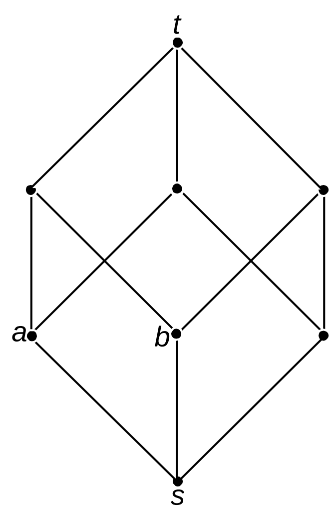


Fig. 1-The network Q_3

Kirchoff's Problem: If a current (of say 1 Amp) flows in at s and out at t, how will it flow through N?

A *flow* is a function $f : E \to \mathbb{R}$. $f(e)$ is the *rate of flow* in $e \in E$. If $f(e) > 0$ it is flowing from $\partial_-(e)$ to $\partial_+(e)$. If $f(e) < 0$ it is flowing from $\partial_+(e)$ to $\partial_-(e)$. Note that every flow can be made nonnegative by reversing the direction of those edges where it is negative. Empirically, a flow of electrical current in N determines a potential energy $V(v) \in \mathbb{R}$ at each vertex, v (Unfortunately, V is being used in two different ways here since V stands for vertex in graph theory and voltage in electrical theory. Hopefully, context will make the meaning clear). Then the potential function, V, and the electrical (current) flow function, i, are determined by the following three laws:

(1) Kirchoff's Current Law: $\forall v \in V, v \neq s, t$,

$$\sum_{\substack{e \in E \\ \partial_+(e)=v}} i(e) = \sum_{\substack{e \in E \\ \partial_-(e)=v}} i(e).$$

(2) Ohm's Law: $\forall e \in E$, *the voltage drop across* e, $V(\partial_-(e)) - V(\partial_+(e))$, is equal to

$$i(e) R(e).$$

(3) Kirchoff's Voltage Law: In any circuit, $e_1, e_2, ..., e_k$, the sum of the potential drops around the circuit is zero.

Kirchoff's Current Law (KCL) is essentially the Law of conservation of mass plus the statement that electrons do not accumulate anywhere. It follows from KCL that *the net flow out of* s,

$$Net(i) = \sum_{\substack{e \in E \\ \partial_-(e)=s}} i(e) - \sum_{\substack{e \in E \\ \partial_+(e)=s}} i(e),$$

is equal to *the net flow into* t, *i.e.*

$$Net(i) = \sum_{\substack{e \in E \\ \partial_+(e)=t}} i(e) - \sum_{\substack{e \in E \\ \partial_-(e)=t}} i(e).$$

Example 2. *In* Q_3 *above,* $|V| = 8$ *and* $|E| = 12$*, so there are* $8 + 12 = 20$ *variables to solve for. However, theory shows that symmetric edges must have the same values of* i *and symmetric vertices must have the same values of* V*. The three edges out of* s *are symmetric. Their flows must total* 1 *and so for each of them* $i(e) = 1/3$*. Proceeding in this way to the other edges we have the current flow in Fig. 2,*

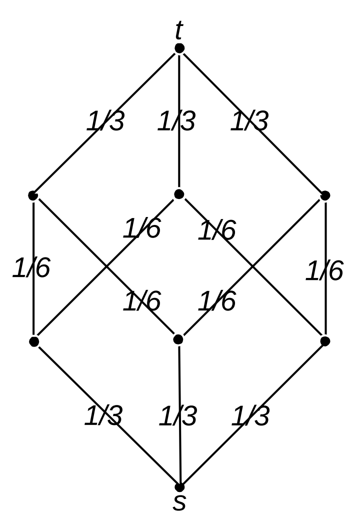


Fig. 2-Current flow in Q_3

and the voltages (with $V(t)$ taken to be 0) in Fig. 3.

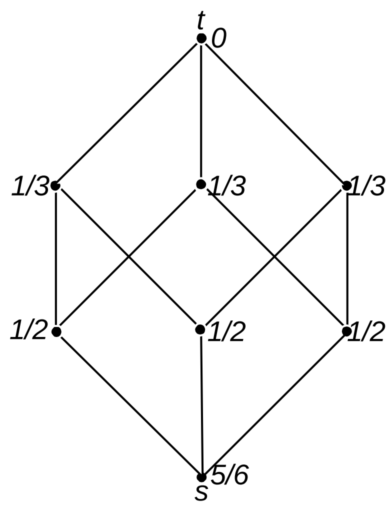


Fig. 3-Voltages in Q_3

Thevenin's Theorem (see [21]) says that any (2-terminal) subnetwork M (of a larger network N) may be replaced by a single resistor of resistance $\overline{R}(M)$ without altering the response of the network to inputs at the terminals, s & t, of N. The preceding calculation for Q_3 shows that

$$\begin{aligned} \overline{R}(Q_3) &= 1 \cdot \overline{R}(Q_3) \\ &= V(s) - V(t), \text{ by Ohm's Law,} \\ &= 5/6 - 0, \\ &= 5/6 \end{aligned}$$

Suppose in the preceding example we identify just two of the vertices that are symmetric (a, b in Fig. 1). They are of the same potential so we reduce the network to that of Fig. 4.

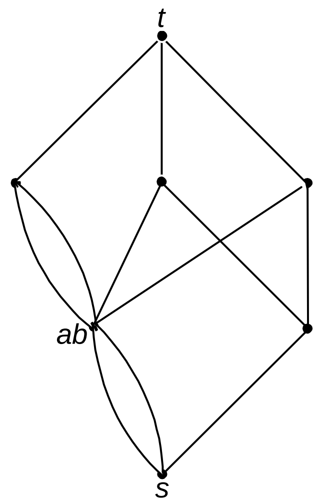
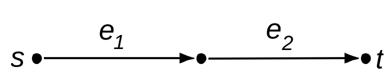


Fig.4-Vertices a, b coalesced into ab

Question: All the symmetry of Q_3 has been destroyed and we can no longer use symmetry to justify further collapsing. Obviously, from its construction, vertices on the same rank must still be of the same potential. But how could we know that without knowledge of the construction of the network? Can we verify, just from the diagram itself (without going to the extreme of calculating the voltage at every vertex and defeating the purpose) that further collapsing will lead to the correct value of $\overline{R}$? This question is answered affirmatively in Section 2.

Furthermore the textbooks tell us that:

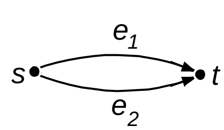
(1) For resistors in series



$$\overline{R} = R(e_1) + R(e_2),$$

the sum of the resistances.

(2) For resistors in parallel



$$\overline{R} = \frac{1}{\frac{1}{R(e_1)} + \frac{1}{R(e_2)}},$$

the *parallel sum* of the resistances.

When parallel edges in the previous diagram are replaced by a single equivalent edge, we have, by Thevenin, the equivalent diagram

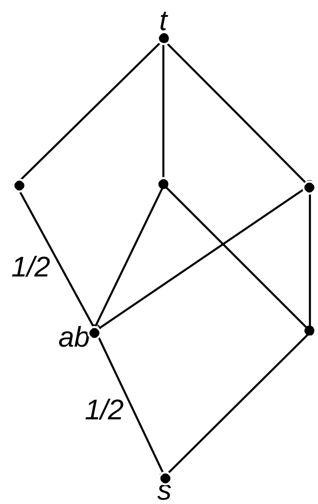


Fig. 5-Q_3 with parallel edges coalesced

3. The Main Question

So what are morphisms for resistive networks? Generally speaking, useful morphisms are "nice" maps between the structures involved (resistive networks in this case) that "preserve" significant parameters such as the equivalent resistance, $\overline{R}(N)$. Our answer to Mac Lane's question for resistive networks involves 3 steps:

Step 1: Recast the problem: An *s-t flow on* N is a function $f : E \to \mathbb{R}$, satisfying Kirchoff's Current Law. The *power dissipated by* f is

$$P(f) = \sum_{e \in E} f^2(e) R(e).$$

Lemma 1. *(Rayleigh, c. 1870) The s-t flow, f, with $Net(f) = 1$ (or any fixed value), that minimizes $P(f)$ over all such flows, is the current flow, i, determined by Ohm's Law and Kirchoff's Voltage Law.*

Proof. By the Lagrange Multiplier Theorem: Differentiating the Lagrangian

$$\sum_{e \in E} f^2(e) R(e) + \sum_{v \in V} \lambda(v) \left(\sum_{\partial_+(e)=v} f(e) - \sum_{\partial_-(e)=v} f(e) \right)$$

with respect to the value $f(e)$ and setting to 0 gives

$$2f(e) R(e) = \lambda(v) - \lambda(w)$$

where $\partial_+(e) = v$ and $\partial_-(e) = w$. Taking $V(v)$ to be $\lambda(v)/2$ gives Ohm's Law. Kirchoffs Voltage Law follows from the existence of Lagrange multipliers, $\lambda : V \to \mathbb{R}$ and the definition of a circuit. □

So the current flow, i, is the solution to an optimization problem on s-t flows very similar to the Ford-Fulkerson MaxFlow Problem.

Step 2: A morphism $\varphi : M \to N$, from a resistive electrical network $M = (G, \{s_M, t_M\}, R_M)$ to a resistive electrical network $N = (H, \{s_N, t_N\}, R_N)$ should preserve that structure in some sense, so we take it to consist of

(1) A digraph homomorphism $(\varphi_V, \varphi_E) : G \to H$, which in turn consists of functions $\varphi_V : V_G \to V_H$ and $\varphi_E : E_G \to E_H$ such that $\forall e \in E_G$, $\varphi_V(\partial_\pm(e)) = \partial_\pm(\varphi_E(e))$.

(2) $\varphi_V^{-1}(s_N) = \{s_M\}$ and $\varphi_V^{-1}(t_N) = \{t_M\}$.

(3) Some function F, such that $F(R_M) = R_N$. F should be locally defined in the sense that $F(R_M)(e)$ should only depend on the values of R_M restricted to $\varphi_E^{-1}(e)$. Also, $F(R_M)(e)$ should be easy to calculate. A candidate satisfying these preconditions is parallel sum,

$$F(R_M)(e) = \frac{1}{\sum_{e' \in \varphi_E^{-1}(e)} 1/R_M(e')} = R_N(e).$$

The intuition behind this choice is that φ collapses the vertices of $\varphi_V^{-1}(\partial_+(e))$ to a single vertex, which is going to preserve voltage only if all members of $\varphi_V^{-1}(\partial_+(e))$ have the same voltage. If that is true at $\varphi_V^{-1}(\partial_+(e))$ and $\varphi_V^{-1}(\partial_-(e))$, then all the edges of $\varphi_E^{-1}(e)$ are effectively in parallel.

Note: A flow, f, on M induces a flow $\varphi(f)$ on N by additivity,

$$\varphi(f)(e) = \sum_{e' \in \varphi_E^{-1}(e)} f(e'),$$

and then $Net(\varphi(f)) = Net(f)$.

Lemma 2. *$\forall$s-tflows, f, on M, we have $P_N(\varphi(f)) \leq P_M(f)$.*

Proof. Replace resistance by conductance (see [21]), $C(e) = 1/R(e)$. Then

$$\begin{aligned} C_N(e) &= \frac{1}{R_N(e)} \\ &= \sum_{e' \in \varphi_E^{-1}(e)} 1/R_M(e') \\ &= \sum_{e' \in \varphi_E^{-1}(e)} C_M(e'), \end{aligned}$$

so conductance is also transformed additively by φ. The theorem then follows from

$$\sum_{e' \in \varphi_E^{-1}(e)} f^2(e')\, C_M^{-1}(e') \geq \left(\sum_{e' \in \varphi_E^{-1}(e)} f(e') \right)^2 \left(\sum_{e' \in \varphi_E^{-1}(e)} C_M(e') \right)^{-1},$$

a special case of the Reverse Hölder Inequality (See Wikipedia). It may also be proved directly by induction on the size of $\varphi_E^{-1}(e)$. $\square$

Step 3: By Steps 1 & 2 above, if $\exists$ a map $\varphi : M \to N$ with properties 1, 2 & 3 (of Step 2), then

$$P_M(i) = \min_{Net(f)=1} P_M(f) \geq \min_{Net(f)=1} P_N(f) = P_N(i').$$

Technically, φ satisfies MacLane's requirements for a morphism and justifies the definition of $R_N(e)$. However, in the light of results obtained for flowmorphisms and pathmorphisms, we ask for more:

Question: What additional conditions on φ will give equality,

$$\min_{Net(f)=1} P_M(f) = \min_{Net(f)=1} P_N(f)?$$

If such conditions exist, the problem of computing electrical flows and their significant parameters is completely preserved by such a map.

Answer: The existence of a right inverse, $\rho : N \to M$ for φ. We do not need a right inverse for the set functions, φ_V & φ_E, just for the linear transformations they generate. Given that $\overline{E} = \{f : E \to \mathbb{R}\}$, the vector space generated by the set E, we require that the linear map $\rho : \overline{E}_N \to \overline{E}_M$ take flows to flows in such a way that $Net(\rho(f)) = Net(f)$ & $P(\rho(f)) = P(f)$. This means that:

(1) For each $e \in E_N$ the Reverse Hölder Inequality above must give equality. It then follows (See Wikipedia on the Reverse Hölder Inequality) that $f(e')/C_M(e') = \alpha$ is constant (the same for all $e' \in \varphi_E^{-1}(e)$).

(2) Suppose that $f \in \overline{E}_N$ is a flow and $\rho(f) = f' \in \overline{E}_M$ is a flow such that $Net(f') = Net(f)$ and $P_M(f') = P_N(f)$. By the above, $f'(e') = \alpha(e) C_M(e')$. Then

$$\begin{aligned} f(e) &= \sum_{e' \in \varphi_E^{-1}(e)} f'(e') \\ &= \sum_{e' \in \varphi_E^{-1}(e)} \alpha(e) C_M(e') \\ &= \alpha(e) \sum_{e' \in \varphi_E^{-1}(e)} C_M(e') \\ &= \alpha(e) C_N(e). \end{aligned}$$

Therefore $\alpha(e) = f(e)/C_N(e)$ and

$$f'(e') = f(e) \frac{C_M(e')}{C_N(e)}.$$

(3) The function $C_M/C_N(e)$ on the bipartite network $\varphi_E^{-1}(e)$ is what was called a "normalized flow" in [2]. Its sources are $\varphi_V^{-1}(\partial_-(e))$ and its sinks are $\varphi_V^{-1}(\partial_+(e))$ .The flow out of $v' \in \varphi_V^{-1}(\partial_-(e))$ is

$$\widetilde{C}_{M,e,-}(v') = \sum_{\substack{e' \in \varphi_E^{-1}(e) \\ \partial_-(e') = v'}} \frac{C_M(e')}{C_N(e)},$$

and the flow into $w' \in \varphi_V^{-1}(\partial_+(e))$ is

$$\widetilde{C}_{M,e,+}(w') = \sum_{\substack{e' \in \varphi_E^{-1}(e) \\ \partial_+(e') = w'}} \frac{C_M(e')}{C_N(e)}.$$

Note that $\widetilde{C}_{M,e,\pm}$ are probability measures.

Lemma 3. *If* $\forall v \in V_N$ *and* $\forall e \in E_N$ *such that* $v = \partial_{\pm}(e)$*, the probability measure* $\widetilde{C}_{M,e,\pm}$ *induced on* $\varphi_V^{-1}(v)$ *by the normalized flow* $C_M/C_N(e)$ *is the same (independent of* e*), then every s-t flow,* f*, in* N *gives a flow* f' *on* M *by*

$$f'(e') = f(e)\frac{C_M(e')}{C_N(e)}.$$

Also $Net(f') = Net(f)$ *and* $P_M(f') = P_N(f)$*.*

We now summarize the results of Steps 1, 2 & 3 with a definition of morphism for the flow of electricity in a resistive network.

Definition 1. *A flowmorphism* $\varphi : M \to N$*, from a resistive electrical network* $M = (G, \{s_M, t_M\}, 1/C_M)$ *to a resistive electrical network* $N = (H, \{s_N, t_N\}, 1/C_N)$ *consists of*

(1) A digraph homomorphism $(\varphi_V, \varphi_E) : G \to H$, such that
(2) $\varphi_V^{-1}(s_N) = \{s_M\}$ and $\varphi_V^{-1}(t_N) = \{t_M\}$,
(3) Conductance is additive, *i.e.* $\forall e \in E_N$,

$$C_N(e) = \sum_{e' \in \varphi_E^{-1}(e)} C_M(e'),$$

(4) φ has a right inverse, ρ. This means that $\forall v \in V_N$ and $\forall e \in E_N$ such that $v \in \partial_{\pm}(e)$, the probability measure $\widetilde{C}_{M,e,\pm}$ on $\varphi_V^{-1}(v)$ is the same (independent of e).

Then for a flow, f, on N

$$\rho(f)(e') = f(e)\frac{C_M(e')}{C_N(e)}$$

for $e' \in \varphi_E^{-1}(e)$.

Theorem 1. *If there is a flowmorphism* $\varphi : M \to N$*, then* $\forall$*s-t flows,* f*, on* M*,* $Net(\varphi(f)) = Net(f)$ *and* $P_M(\varphi(f)) \geq P_N(f)$*. Also,* $\forall$*s-t flows,* f'*, on* N*,* $Net(\rho(f')) = Net(f')$ *and* $P_M(\rho(f')) = P_N(f')$*.*

Flowmorphisms are morphisms for a category, RESNET, whose objects are resistive networks. We have chosen to reuse the term "flowmorphism", introduced in [8] for Ford-Fulkerson flows on networks because they represent essentially the same concept. In [8] and [11] the networks are vertex-weighted and here they are edge-weighted but Definition 1 answers the question, posed in [11] as to how flowmorphisms might be extended to edge-weighted networks. Another difference is that the functions $f : E \to V$ representing Ford-Fulkerson flows are required to take non-negative values, whereas those representing electrical flows are allowed to be negative. Yet the overall structure: Being generated by a digraph homomorphism that preserves s & t, transforms flows and weights (whether capacity or conductance) by additivity and having a right inverse determined by normalized flows makes it clear that they have a common generalization (to networks that are vertex- **and** edge-weighted).

Another difference between FLOW (the category of vertex-weighted Ford-Fulkerson networks) and RESNET, which makes the fact that they have essentially the same morphisms seem even more shocking, is that the Ford-Fulkerson objective functions, Net(f), are linear while the electrical objective function, $P(f) = \sum_{e \in E} f^2(e) R(e)$, is quadratic.

3.1. **Some Examples.**

3.1.1. *The Network of Fig. 5.* There is a flowmorphism, φ, illustrated in Fig. 6. Its domain is the network of Fig. 5 (a quotient of Q_3) and its range is $[4] = \{0 < 1 < 2 < 3\}$. Vertices and edges are mapped to the right. Note that $ab \in \varphi^{-1}(1)$. The normalized flow into ab from $\varphi^{-1}(01)$ is $2/(2+1) = \frac{2}{3}$ and out of ab by $\varphi^{-1}(12)$ is $(2+1+1)/(2+1+1+1+1) = 2/3$. Since they are equal and there are only those two edges incident to 1 in [4], we have verified the condition for a flowmorphism at ab. The condition is similar at the other vertices of the domain. This answers the question asked about Fig. 4 since the conditions to be verified are determined by the diagram, purely local and easily computed.

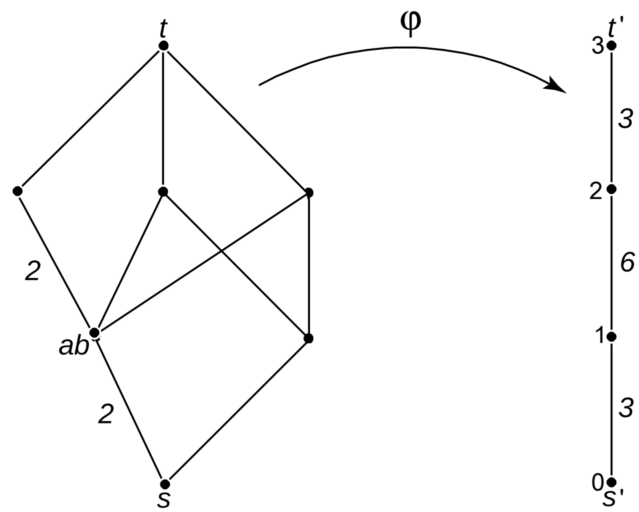


Fig. 6-A flowmorphism

3.1.2. *The Graph of the n-cube.* Q_n is the graph whose vertices are n-tuples of 0s and 1s. Edges, $e = (v, w)$, are pairs of vertices that differ in exactly one coordinate. The vertex, v, with 0 in that coordinate is $\partial_-(e)$ and w, with 1 in that coordinate is $\partial_+(e)$. $s = 0^n$, $t = 1^n$ and $R(e) = 1 \; \forall e \in E$. The map which takes $Q_{n,k} = \{v \in V : |v| = k\}$ to k

(1) Gives a digraph homomorphism $\varphi : Q_n \to [n+1] = \{0 \to 1 \to ... \to n\}$
(2) $\varphi^{-1}(0) = \{0^n\}$ and $\varphi^{-1}(n) = \{1^n\}$,
(3) We define conductance on the edges of $[n+1]$ so that φ is additive:

$$\begin{aligned} C(k, k+1) &= (n-k)\binom{n}{k} = n\binom{n-1}{k-1} \\ &= (k+1)\binom{n}{k+1}, \end{aligned}$$

and
(4) The two weights induced on $Q_{n,k}$ by the normalized conductance weights on $\varphi^{-1}(k-1, k)$ and $\varphi^{-1}(k, k+1)$ are both uniform probability measures (and thus the same).

Therefore φ is a flowmorphism and

$$\overline{R}\left(Q_n\right) = \overline{R}\left([n+1]\right) = \frac{1}{n}\sum_{k=1}^{n}\frac{1}{\binom{n-1}{k-1}}.$$

In particular

$$\overline{R}\left(Q_3\right) = \frac{1}{3}\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{1}\right) = \frac{5}{6},$$

as calculated in Section 1. Note the difference between this analysis and that of Section 1: There we observed that the values of the function $i : E \to \mathbb{R}$ are the same on symmetric edges (so if there is a symmetry such that $\varphi(e) = e'$, then $i(e) = i(e')$. Since symmetries are transitive on the edges from $Q_{n,k}$ to $Q_{n,k+1}$, this means that i is constant on those edges. In this example the same simplification takes the form of a quotient, $\varphi : Q_n \to [n+1]$. Symmetries always induce such quotients. In Section 3.2 we extend that construction to all flowmorphisms having a common domain.

Whenever a network, N, has a morphism $\varphi : N \to [n]$, a path, $\overline{R}(N) = \overline{R}([n]) = \sum_{k=1}^{n-1} 1/C(k-1,k)$ is immediate. Such a network is called *normal* (not to denote that it is the norm but as a shortened version of the original term "has the normalized flow property"). Note that $Q_n \simeq [2]\times[2]\times ...\times[2]$ is normal.

Exercise 1. *Verify that* $[3]\times[3]$ *is normal, but* $[2]\times[3]\times[3]$ *and* $[4]\times[4]$ *are not (all vertex and edge weights are 1).*

3.1.3. $[n_1]\times[n_2]\times...\times[n_k]$ *with Vertex Weights.* In the preceding exercise we showed that the normality of $Q_n \simeq [2]\times[2]\times...\times[2]$ does not generally extend to products of longer total orders. However, if we regard the edges of $[n]$ as superconducting wires (infinite conductance) and the vertices as components with unit conductance, then the Product Theorem of [9] tells us that their product is normal. In fact the sequence of vertex-weights on any of the factors, $[n_i]$, $i = 1,2,...,k$ may be quite general as long as they are 2-positive, and the product will still be normal. In [9] the Product Theorem was designed to solve Sperner problems, but since flowmorphisms also preserve dissipated power, it solves Kirchoff problems!

3.2. **Pushouts in RESNET.** Symmetries are morphisms in every category. They are trivial from the point of view of computational complexity since the range is the same size as the domain. However, in many categories symmetries induce nontrivial morphisms whose range is the quotient of the domain (the symmetry having been modded out). In category theory these are colimits called *symmetrizers* (See [20]). Another colimit, more general and fundamental than the symmetrizer is the pushout: Given morphisms, $\varphi_1 : M \to N_1$ and $\varphi_2 : M \to N_2$ their *pushout* consists of morphisms $\sigma_1 : N_1 \to P$ and $\sigma_2 : N_2 \to P$ that makes the diagram (of

Fig. 7)

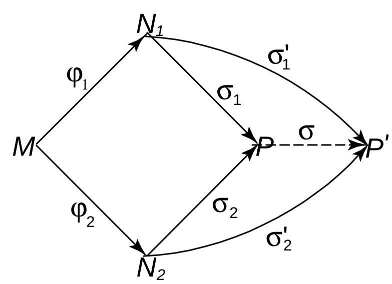


Fig. 7-Pushout diagram

commute and is universal in the sense that if $\sigma_1' : N_1 \to P'$ and $\sigma_2' : N_2 \to P'$ are any morphisms satisfying the same condition, then there is a unique morphism, $\sigma : P \to P'$ such that $\sigma_1' = \sigma \circ \sigma_1$ and $\sigma_2' = \sigma \circ \sigma_2$ (See [20] for further details).

Theorem 2. *RESNET has pushouts.*

Proof. To construct σ_1, σ_2 and show that they are flowmorphisms, we proceed through the four parts of Definition 1:

(1) DIGRAPH, the category of directed graphs, is a functor category, DIGRAPH $\simeq$ Funct(D, SET), where D is the category whose diagram is (Fig. 8)

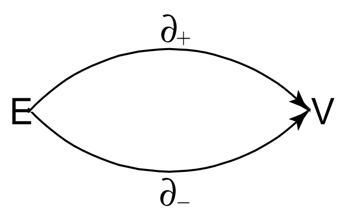


Fig. 8-Diagram category

By the Corollary of Theorem V.3.1 of [20] (p. 116), DIGRAPH inherits limits (and by duality, colimits) from SET. Also, they can be constructed componentwise. Thus the vertex set of P is the disjoint union of V_{N_1} and V_{N_2} modulo the equivalence relation $v_1 \sim v_2$ if $\exists v' \in V_M$ such that $\varphi_1(v') = v_1$ and $\varphi_2(v') = v_2$. Similarly, E_P is the disjoint union of E_{N_1} and E_{N_2} modulo the equivalence relation $e_1 \sim e_2$ if $\exists e' \in E_M$ such that $\varphi_1(e) = e_1$ and $\varphi_2(e) = e_2$. The edge and vertex maps of σ_1 and σ_2 are defined in the obvious way, $\sigma_1(v_1) = \overline{v_1}$, the equivalence class of v_1.

(2) $s_P = \{s_{N_1} \sim s_{N_2}\}$ and $t_P = \{t_{N_1} \sim t_{N_2}\}$, so $\sigma_1^{-1}(s_P) = \{s_{N_1}\}$, $\sigma_2^{-1}(s_P) = \{s_{N_2}\}$, $\sigma_1^{-1}(t_P) = \{t_{N_1}\}$ and $\sigma_2^{-1}(t_P) = \{t_{N_2}\}$.

(3) The conductances on P are defined additively

$$\begin{aligned} C_P(e) &= \sum_{\substack{e_1 \in E_{N_1} \\ \sigma_1(e_1)=e}} C_{N_1}(e_1) \\ &= \sum_{\substack{e_1 \in E_{N_1} \\ \sigma_1(e_1)=e}} \sum_{\substack{e' \in E_M \\ \varphi_1(e')=e_1}} C_M(e') \\ &= \sum_{\substack{e_2 \in E_{N_2} \\ \sigma_2(e_2)=e}} \sum_{\substack{e' \in E_M \\ \varphi_2(e')=e_2}} C_M(e') \\ &= \sum_{\substack{e_2 \in E_{N_2} \\ \sigma_2(e_2)=e}} C_{N_2}(e_2). \end{aligned}$$

(4) Given $v \in V_P$ and $e, f \in E_P$ with $\partial_{\pm}(e) = v = \partial_{\pm'}(f)$, we must show that $\widetilde{C}_{N_i,e,\pm} = \widetilde{C}_{N_i,f,\pm'}$for $i = 1, 2$. thus verifying that σ_i is a flowmorphism. Note that in the definition of flowmorphism, the equality $\widetilde{C}_{M,e,\pm} = \widetilde{C}_{M,f,\pm'}$is equivalent to the ratio

$$\frac{\sum_{\substack{e' \in \varphi^{-1}(e) \\ \partial_{\pm}(e')=v'}} C_M(e')}{\sum_{\substack{f' \in \varphi^{-1}(f) \\ \partial_{\pm'}(f')=v'}} C_M(f')}$$

being constant over all $v' \in \varphi^{-1}(v)$. In this case we look at the ratio

$$\frac{\sum_{\substack{e' \in \sigma_i^{-1}(e) \\ \partial_{\pm}(e')=v'}} C_{N_i}(e')}{\sum_{\substack{f' \in \sigma_i^{-1}(f) \\ \partial_{\pm'}(f'')=v'}} C_{N_i}(f')}.$$

Since φ_i is a flowmorphism and we have $v' \in V_{N_i}$ and $e', f' \in E_{N_i}$ with $\partial_{\pm}(e') = v' = \partial_{\pm}(f')$, $\widetilde{C}_{M,e',\pm}(v'') = \widetilde{C}_{M,f',\pm'}(v'')$ for all $v'' \in \varphi_i^{-1}(v')$.

Therefore

$$\begin{aligned}
\frac{\sum_{\substack{e'\in\sigma_i^{-1}(e)\\ \partial_\pm(e')=v'}} C_{N_i}(e')}{\sum_{\substack{f'\in\sigma_i^{-1}(f)\\ \partial_{\pm'}(f')=v'}} C_{N_i}(f')} &= \frac{\widetilde{C}_{M,e',\pm}(v'')\sum_{\substack{e'\in\sigma_i^{-1}(e)\\ \partial_\pm(e')=v'}} C_{N_i}(e')}{\widetilde{C}_{M,f',\pm'}(v'')\sum_{\substack{f'\in\sigma_i^{-1}(f)\\ \partial_{\pm'}(f')=v'}} C_{N_i}(f')} \\
&= \frac{\sum_{\substack{e'\in\sigma_i^{-1}(e)\\ \partial_\pm(e')=v'}} \widetilde{C}_{M,e',\pm}(v'')\, C_{N_i}(e')}{\sum_{\substack{f'\in\sigma_i^{-1}(f)\\ \partial_{\pm'}(f')=v'}} \widetilde{C}_{M,f',\pm'}(v'')\, C_{N_i}(f')} \\
&= \frac{\sum_{\substack{e'\in\sigma_i^{-1}(e)\\ \partial_\pm(e')=v'}} \sum_{\substack{e''\in\varphi_i^{-1}(e')\\ \partial_\pm(e'')=v"}} C_M(e'')}{\sum_{\substack{f'\in\sigma_i^{-1}(f)\\ \partial_{\pm'}(f')=v'}} \sum_{\substack{f''\in\varphi_i^{-1}(f')\\ \partial_\pm(f'')=v''}} C_M(f'')} \\
&= \frac{\sum_{\substack{e''\in(\sigma_i\circ\varphi_i)^{-1}(e)\\ \partial_\pm(e'')=v''}} C_M(e'')}{\sum_{\substack{f''\in(\sigma_i\circ\varphi_i)^{-1}(f)\\ \partial_\pm(f'')=v''}} C_M(f'')}.
\end{aligned}$$

So this ratio has the same value for all $v'' \in \varphi_i^{-1}\left(v'\right)$. However, the partition, $\left\{\varphi^{-1}(v) : v \in V_N\right\}$, of V_M induced by the pushout $\varphi = \sigma_1 \circ \varphi_1 = \sigma_2 \circ \varphi_2$ is the infimum (in the lattice of partitions of V_M) of the partitions, $\left\{\varphi_1^{-1}(v') : v' \in V_{N_1}\right\}$ and $\left\{\varphi_2^{-1}(v') : v' \in V_{N_2}\right\}$ induced by φ_1 and φ_2. That is to say, given $v'', w'' \in \varphi^{-1}(v)$, $\exists$a sequence, $v''_0, v''_1, ..., v''_k \in \varphi^{-1}(v)$ such that $v''_0 = v'', v''_k = w''$ and for $j = 0, 1, ..., k-1$ $\exists v' \in \sigma_i^{-1}(v)$ for $i = 1$ or 2, such that $\varphi_i\left(v''_j\right) = v' = \varphi_i\left(v''_{j+1}\right)$. Thus the ratio for v''_j and v''_{j+1} are the same and by the transitivity of equality, the ratio

$$\frac{\sum_{\substack{e'\in\sigma_i^{-1}(e)\\ \partial_\pm(e')=v'}} C_{N_i}(e')}{\sum_{\substack{f'\in\sigma_i^{-1}(f)\\ \partial_{\pm'}(f')=v'}} C_{N_i}(f')}$$

is constant over all $v' \in \sigma_i^{-1}(v)$.

□

Note that this proof is not catagorical (diagram chasing) but set theoretic.

Corollary 1. *The symmetries of a network induce a quotient (their pushout, aka their symmetrizer).*

4. Conclusions and Comments

4.1. **Edge-weighted and/or Vertex-weighted Networks?** Is there any real difference? It is elementary that an edge-weighted network can be replaced by an "equivalent" vertex-weighted one (by replacing each edge by two edges with a vertex in between). Similarly, every vertex-weighted network can be replaced by an "equivalent" edge-weighted one. However, there are also qualitative differences which suggest that there is more to it:

(1) Current flow in edge-weighted networks is uniquely determined by Kirchoff's Laws, but not in vertex-weighted ones.
(2) The proof of pushouts for RESNET is set-theoretic, but the corresponding proof for FLOW [11] is analytic (involving a limit process). Logically, this is because the definition of flowmorphism for vertex-weighted networks entails an additional (existential) quantifier.

The question merits further study.

4.2. **Morphisms and Algorithms.** An intriguing possibility is that flowmorphisms might provide insights to improve the algorithm for Kirchoff's problem. It has long been an article of faith for the author that for combinatorial optimization problems there exists a close, but hidden, connection between morphisms and algorithms. Empirically, problems with nice morphisms, whose categories have nice structural properties (such as pushouts), also have nice (polynomial bounded) algorithms and *vice versa.* The details of this hypothetical connection remain a mystery but it has led to proveable conjectures. The algorithm given for Kirchoff's problem in modern texts (*cf.* Section 2.4 of [21]) is the one introduced by Kirchoff in 1845: For a network with n vertices, the flow in the $n-1$ edges of a tree form a basis for the flow and potential spaces. Ohm's Law and Kirchoff's Current Law at all vertices (except t, which is the same as that out of s) give $n-1$ independent linear equations that uniquely determine the current flow. Solving the equations by Gaussian elimination takes $O\left(n^3\right)$ operations. This is onerous if n is large. There must have been many attempts to improve on Kirchoff's algorithm. One can sense some frustration at the end of Rockafellar's treatise [22] at not being able to do this. The first complete paragraph on p. 600 concludes, "Surely some marriage of these two approaches (producing an algorithm for quadratic flow programs that utilizes graphical subroutines and enjoys finite termination) will be possible". The fact that the Ford-Fulkerson optimal flow problems and Kirchoff's problem are both preserved by flowmorphisms suggests that the Edmonds-Karp algorithm for MaxFlow might be carried over to Kirchoff's problem, giving what Rockafellar is hoping for.

The Alexeyeev-Engel problem (to minimize the variance of all "representations" (unit-increase functions) on a poset) is another optimization problem on networks having linear constraints and a quadratic objective function (See Section 4.4 of [3]). It is also preserved by flowmorphisms (Theorem 4.5.6 of [3]). In Section 4.4 Engel presents a combinatorial algorithm for minimizing variance which terminates in a finite number of steps (Theorem 4.4.2) but not proven strongly polynomial (See the last paragraph of Section 4.4).

4.3. **The Implications of Series-Parallel Duality.** We started out to find a notion of morphism, $\varphi : M \to N$, for resistive networks. In the process we replaced resistance by conductance. We could have stayed with resistance by defining the resistance of an edge, $e \in E_N$, as the parallel sum of the resistances on the edges $e' \in \varphi^{-1}(e)$. But in terms of conductance, the conductance of an edge, $e \in E_N$, is just the sum of the conductances on the edges $e' \in \varphi^{-1}(e)$. This makes φ additive and brings out the surprising similarity with the morphisms previously defined for the MaxFlow and MinFlow problems of Ford-Fulkerson. Furthermore, when a conductive network is normal (has a flowmorphism to a path), its effective conductance is the parallel sum of the conductances on the edges e' in the path. This suggests that, just as flowmorphisms are extensions of parallel reductions, there should also be a notion of morphism that extends series reductions. It is relatively straightforward then to define a notion of morphism for conductive networks that extends both flowmorphisms and series reductions. Since the series extensions for resistive networks differ from those for Ford-Fulkerson problems, they define different categories, RESNET, MAXFLOW and MINFLOW. We hope to report on these extensions in a sequel.

4.4. **Infinite Flowmorphisms?** Another inviting prospect, from the vantage point of this paper, is to extend flowmorphisms to infinite and even continuous structures. There are already instances of such things in the combinatorial literature: The celebrated Brooks-Smith-Stone-Tutte procedure for squaring the square involved a notion of quotient (RESNET morphism) of the unit square which they called a "Smith diagram" (See http://en.wikipedia.org/wiki/Squaring_the_square). The book by Klain-Rota [18] includes an analog of Sperner's theorem (Theorem 6.4.1) for the Grassmanians of real projective spaces. It essentially says that the poset of projective subspaces is normal.

Zemanian [24] has written extensively about infinite electrical networks; the special challenges and opportunities they present. Historically, Kirchoff's Laws (describing the static flow of electricity on a small scale) were extended by Maxwell's equations to describe the dynamic interactions between electric and magnetic fields on a large scale. Can flowmorphisms be extended to preserve Maxwell's equations? If so, can they cast light on problems in predicting space weather? (See John Baez's discussions of electromagnetic theory and category theory in his online column, This Week's Finds in Mathematical Physics (weeks 288 to 297) http://math.ucr.edu/home/baez/week288.html). Albert Einstein observed that uniform translations are symmetries of Maxwell's equations, which led him to Lorentz transforms and $\mathrm{E} = \mathrm{mc}^2$. Are flowmorphisms related to those epoch-making insights?

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