Black-boxing open reaction networks

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Dynamics, Thermodynamics, and
Information Processing in Chemical Networks

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Black-boxing

We saw that there is a functor

$$\color{gray}{\blacksquare}\color{black}\colon \mathtt{RxNet} \to \mathtt{Dynam},$$

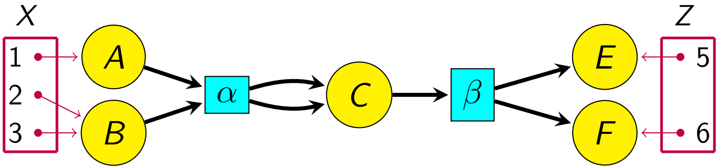
the gray-boxing functor, sending an open reaction network to the open dynamical system generated by mass-action kinetics.

Here,

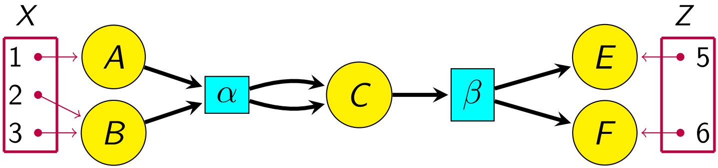
- $\mathtt{RxNet}$ is the category of open reaction networks and
- $\mathtt{Dynam}$ is the category of open dynamical systems.

I'll describe a functor $\blacksquare\colon \mathtt{Dynam} \to \mathtt{Rel}$ sending an open dynamical system to the space of possible *steady state* inflows and outflows.

Black boxing

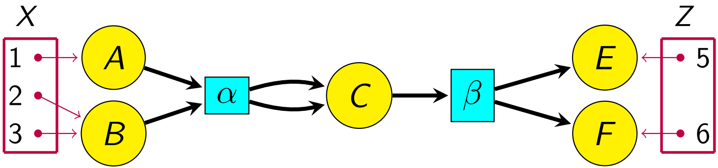


Black boxing



$$
\begin{aligned}
v_A &= -r(\alpha)AB \\
v_B &= -r(\alpha)AB \\
v_C &= 2r(\alpha)AB - r(\beta)C \\
v_E &= r(\beta)C \\
v_F &= r(\beta)C
\end{aligned}
$$

Black boxing



$$\frac{dA}{dt} = -r(\alpha)AB + I_1$$

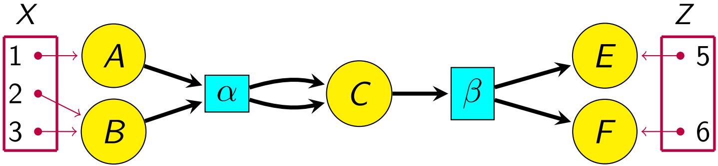
$$\frac{dB}{dt} = -r(\alpha)AB + I_2 + I_3$$

$$\frac{dC}{dt} = 2r(\alpha)AB - r(\beta)C$$

$$\frac{dE}{dt} = r(\beta)C - O_5$$

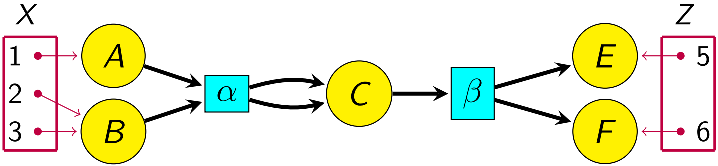
$$\frac{dF}{dt} = r(\beta)C - O_6$$

Black-boxing



$$
\begin{aligned}
0 &= -r(\alpha)AB + I_1 \\
0 &= -r(\alpha)AB + I_2 + I_3 \\
0 &= 2r(\alpha)AB - r(\beta)C \\
0 &= r(\beta)C - O_5 \\
0 &= r(\beta)C - O_6
\end{aligned}
$$

Black-boxing



$$I_1 = r(\alpha)AB$$

$$I_2 + I_3 = r(\alpha)AB$$

$$2r(\alpha)AB = r(\beta)C$$

$$O_5 = r(\beta)C$$

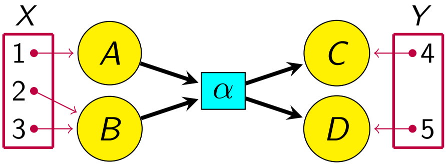
$$O_6 = r(\beta)C.$$

Open reaction networks

Theorem (Baez, BP.)

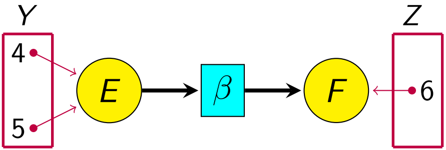
There is a category `RxNet` *whose objects are finite sets and whose morphisms correspond to open reaction networks.*

$$R\colon X \to Y$$



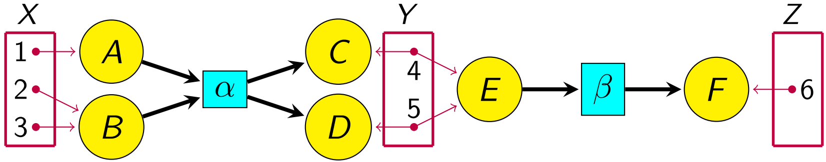
Composition of open reaction networks

Given another open reaction network $R' \colon Y \to Z$



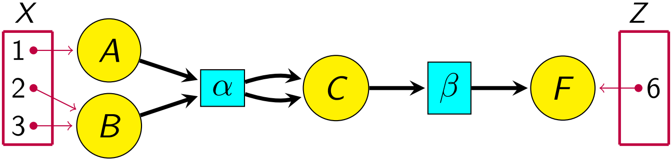
Composition of open reaction networks

To compose $R\colon X \to Y$ and $R'\colon Y \to Z$ we first combine them



Composition of open reaction networks

Then, we identify any species which are in the image of the same point in Y



This gives a new open reaction network $RR' \colon X \to Z$.

A category of open dynamical systems

Definition

An **open dynamical system** $D\colon X \to Y$ *on* S *consists of a cospan of finite sets*

$$X \xrightarrow{i} S \xleftarrow{o} Y$$

together with a polynomial vector field v *on* $\mathbb{R}^S$.

Theorem (Baez, P.)

There is a category `Dynam` *where objects are finite sets and morphisms are isomorphism classes of open dynamical systems.*

The gray-boxing functor

Theorem (Baez, P.)

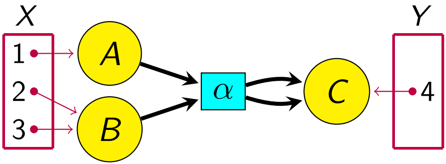
There is a functor $\blacksquare\colon \mathtt{RxNet} \to \mathtt{Dynam}$ *sending an open reaction network to its corresponding open dynamical system generated by the rate equation.*

For open reaction networks $R\colon X \to Y$ and $R'\colon Y \to Z$, the gray-boxing functor satisfies

$$\blacksquare(RR') = \blacksquare(R)\ \blacksquare(R').$$

The gray-boxing functor

$\blacksquare(R\colon X \to Y)$



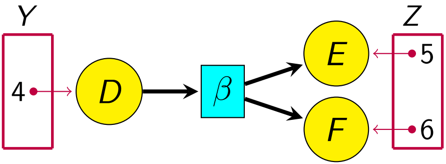
$$v_A = -r(\alpha)A(t)B(t)$$

$$v_B = -r(\alpha)A(t)B(t)$$

$$v_C = 2r(\alpha)A(t)B(t)$$

The gray-boxing functor

■ $(R' \colon Y \to Z)$



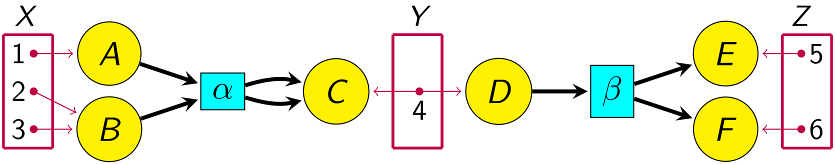
$$v_D = -r(\beta)D(t)$$

$$v_E = r(\beta)D(t)$$

$$v_F = r(\beta)D(t)$$

Composition in Dynam

■$(R\colon X \to Y)$■$(R'\colon Y \to Z)$



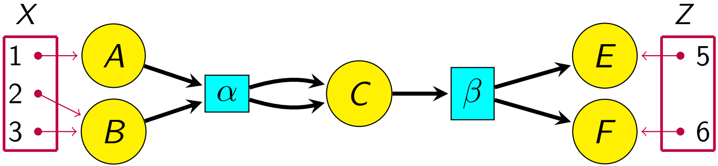
$$v_A = -r(\alpha)AB \qquad v_D = -r(\beta)D$$

$$v_B = -r(\alpha)AB \qquad v_E = r(\beta)D$$

$$v_C = 2r(\alpha)AB \qquad v_F = r(\beta)D$$

Composition in Dynam

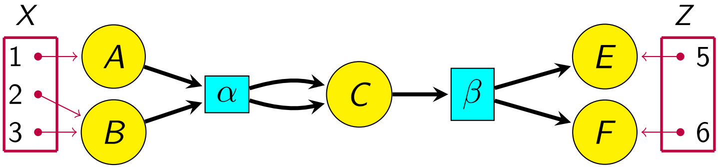
■$(R\colon X \to Y)$■$(R'\colon Y \to Z)$



$$
\begin{aligned}
v_A &= -r(\alpha)AB \\
v_B &= -r(\alpha)AB \\
v_C + v_D &= 2r(\alpha)AB - r(\beta)D \text{ and } C = D \\
v_E &= r(\beta)D \\
v_F &= r(\beta)D
\end{aligned}
$$

The gray-boxing functor

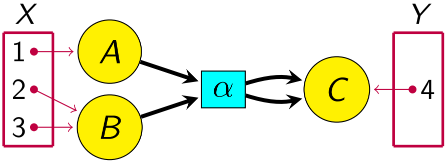
■$(RR' \colon X \to Z)$



$$
\begin{aligned}
v_A &= -r(\alpha)AB \\
v_B &= -r(\alpha)AB \\
v_C &= 2r(\alpha)AB - r(\beta)C \\
v_E &= r(\beta)C \\
v_F &= r(\beta)C
\end{aligned}
$$

The open rate equation

Let $I\colon \mathbb{R} \to \mathbb{R}^X$ and $O\colon \mathbb{R} \to \mathbb{R}^Y$ be arbitrary smooth functions of time specifying the **inflows** and **outflows**.



$$\frac{dA(t)}{dt} = -r(\alpha)A(t)B(t) + I_1(t)$$

$$\frac{dB(t)}{dt} = -r(\alpha)A(t)B(t) + I_2(t) + I_3(t)$$

$$\frac{dC(t)}{dt} = 2r(\alpha)A(t)B(t) - O_4(t)$$

The open rate equation

Given an open dynamical system together with specified inflows $I \in \mathbb{R}^X$ and outflows $O \in \mathbb{R}^X$, we define the pushforward $i_* \colon \mathbb{R}^X \to \mathbb{R}^S$ by

$$i_*(I)_\sigma = \sum_{\{x : i(x) = \sigma\}} I_x$$

and define $o_* \colon \mathbb{R}^Y \to \mathbb{R}^S$ by

$$o_*(O)_\sigma = \sum_{\{y : o(y) = \sigma\}} O_y .$$

We can then write down the **open rate equation** as

$$\frac{dc(t)}{dt} = v(c(t)) + i_*(I(t)) - o_*(O(t)).$$

Steady states

A **steady state** solution of the open rate equation is a concentration vector $c \in \mathbb{R}^S$ such that

$$\frac{dc}{dt} = 0.$$

From the open rate equation

$$\frac{dc}{dt} = v(c) + i_*(I) - o_*(O)$$

we see that this implies

$$v(c) = o_*(O) - i_*(I).$$

This imposes relations among the steady state concentrations and flows along the boundary.

`Rel`

A relation $L\colon U \rightsquigarrow V$ is a subspace $L \subseteq U \oplus V$.

There is a category `Rel` where an object is real vector space and a morphism is a relation between real vector spaces.

Given relations $L\colon U \rightsquigarrow V$ and $L'\colon V \rightsquigarrow W$, their composite $LL'\colon U \rightsquigarrow W$ is given by

$$LL' = \{(u, w)\colon \exists v \in V \text{ with } (u, v) \in L \text{ and } (v, w) \in L'\}.$$

Composition in `Rel` requires that the subspaces agree on their overlap.

Composition in $\mathtt{Rel}$

Given the relation $L\colon \mathbb{R} \rightsquigarrow \mathbb{R}^2$

$$L = \{\ (w, x, y) \mid w = y^2\ \}$$

and the relation $L'\colon \mathbb{R}^2 \rightsquigarrow \mathbb{R}$

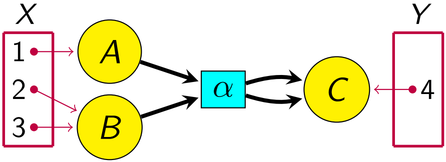
$$L' = \{\ (x', y', z) \mid y' = z\ \},$$

their composite $LL'\colon \mathbb{R} \rightsquigarrow \mathbb{R}$ is

$$LL' = \{\ (w, z) \mid w = z^2\ \}.$$

Steady state behavior

We characterize the steady state behavior of an open reaction network in terms of the relation imposed between inputs and outputs.



$$\frac{dA(t)}{dt} = -r(\alpha)A(t)B(t) + I_1(t)$$

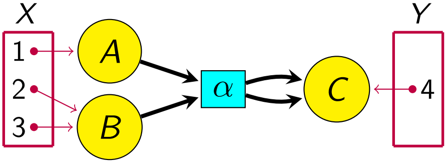
$$\frac{dB(t)}{dt} = -r(\alpha)A(t)B(t) + I_2(t) + I_3(t)$$

$$\frac{dC(t)}{dt} = 2r(\alpha)A(t)B(t) - O_4(t)$$

Steady state behavior

$$c_X = (c_1, c_2, c_3) \in \mathbb{R}^X, \quad I_X = (I_1, I_2, I_3) \in \mathbb{R}^X$$

$$c_Y = c_4 \in \mathbb{R}^Y, \quad O_Y = O_4 \in \mathbb{R}^Y$$



$$(c_X, I_X, c_Y, O_Y) \subseteq \mathbb{R}^X \oplus \mathbb{R}^X \oplus \mathbb{R}^Y \oplus \mathbb{R}^Y$$

such that

$$I_1 = r(\alpha)AB$$

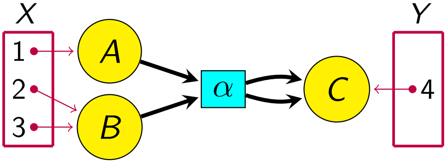
$$I_2 + I_3 = r(\alpha)AB$$

$$O_4 = 2r(\alpha)AB$$

Steady state behavior

$$c_X = (c_1, c_2, c_3) \in \mathbb{R}^X, \quad I_X = (I_1, I_2, I_3) \in \mathbb{R}^X$$

$$c_Y = c_4 \in \mathbb{R}^Y, \quad O_Y = O_4 \in \mathbb{R}^Y$$



$$(c_1, c_2, c_3, I_1, I_2, I_3, c_4, O_4) \subseteq \mathbb{R}^X \oplus \mathbb{R}^X \oplus \mathbb{R}^Y \oplus \mathbb{R}^Y$$

such that

$$I_1 = r(\alpha)AB$$

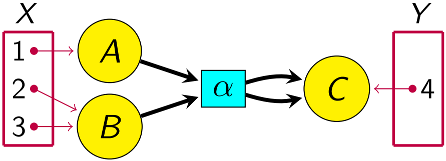
$$I_2 + I_3 = r(\alpha)AB$$

$$O_4 = 2r(\alpha)AB$$

Steady state behavior

$$c_X = (c_1, c_2, c_3) \in \mathbb{R}^X, \quad I_X = (I_1, I_2, I_3) \in \mathbb{R}^X$$

$$c_Y = c_4 \in \mathbb{R}^Y, \quad O_Y = O_4 \in \mathbb{R}^Y$$



$$\begin{gathered}(\; c_1, \; c_2, \; c_3, \; I_1, \; I_2, \; I_3, \; c_4, \; O_4) \\ = \\ (A, B, B, r(\alpha)AB, I_2, r(\alpha)AB - I_2, C, 2r(\alpha)AB)\end{gathered} .$$

The black-box functor

Theorem (Baez, P.)

There is a functor

$$\blacksquare \colon \mathtt{Dynam} \to \mathtt{Rel}$$

sending an open dynamical system to the relation characterizing its steady state boundary concentrations and flows.

The black-box functor

Theorem (Baez, P.)

There is a functor

$$\blacksquare : \texttt{Dynam} \to \texttt{Rel}$$

sending an open dynamical system to the relation characterizing its steady state boundary concentrations and flows.

Theorem (Baez, P.)

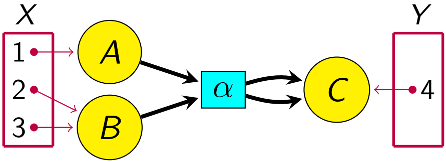
Composing the gray-boxing and black-boxing functors gives a functor

$$\texttt{RxNet} \xrightarrow{\color{gray}\blacksquare} \texttt{Dynam} \xrightarrow{\blacksquare} \texttt{Rel}$$

sending an open reaction network to the subspace of possible steady state boundary concentrations and flows.

Black-boxing

$$\blacksquare(R\colon X \to Y)$$



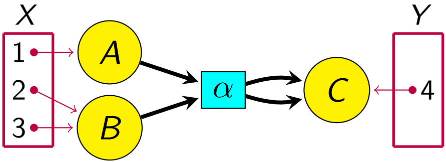
$$\frac{dA(t)}{dt} = -r(\alpha)A(t)B(t) + I_1(t)$$

$$\frac{dB(t)}{dt} = -r(\alpha)A(t)B(t) + I_2(t) + I_3(t)$$

$$\frac{dC(t)}{dt} = 2r(\alpha)A(t)B(t) - O_4(t)$$

Black-boxing

$$\blacksquare(\square(R)) \colon \mathbb{R}^X \oplus \mathbb{R}^X \rightsquigarrow \mathbb{R}^Y \oplus \mathbb{R}^Y$$



$$(c_X, I_X, c_Y, O_Y)$$

such that

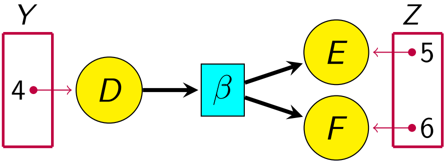
$$I_1 = r(\alpha)AB$$

$$I_2 + I_3 = r(\alpha)AB$$

$$O_4 = 2r(\alpha)AB$$

The ‘gray-boxing’ functor

$$\blacksquare(R' \colon Y \to Z)$$



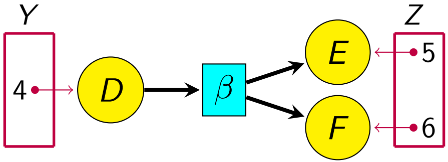
$$\frac{dD(t)}{dt} = -r(\beta)D(t) + I_4(t)$$

$$\frac{dE(t)}{dt} = r(\beta)D(t) - O_5(t)$$

$$\frac{dF(t)}{dt} = r(\beta)D(t) - O_6(t)$$

Black-boxing

$$\blacksquare(\square(R')) \colon \mathbb{R}^Y \oplus \mathbb{R}^Y \rightsquigarrow \mathbb{R}^Z \oplus \mathbb{R}^Z$$



$$(c_Y, I_Y, c_Z, O_Z)$$

$$I_4 = r(\beta)D$$

$$O_5 = r(\beta)D$$

$$O_6 = r(\beta)D$$

Composing relations

$$\blacksquare({\color{gray}\blacksquare}(R))\blacksquare({\color{gray}\blacksquare}(R'))$$

$$(c_X, I_X, c_Y, O_Y)(c_Y, I_Y, c_Z, O_Z)$$

$$\begin{aligned} I_1 &= r(\alpha)AB & I_4 &= r(\beta)D \\ I_2 + I_3 &= r(\alpha)AB & O_5 &= r(\beta)D \\ O_4 &= 2r(\alpha)AB & O_6 &= r(\beta)D \end{aligned}$$

Composing relations

$$\blacksquare({\color{gray}\blacksquare}(R))\blacksquare({\color{gray}\blacksquare}(R'))$$

$$(c_X, I_X, c_Y, O_Y)(c_Y, I_Y, c_Z, O_Z)$$

$$\begin{array}{rr} I_1 = r(\alpha)AB & I_4 = r(\beta)D \\ I_2 + I_3 = r(\alpha)AB & O_5 = r(\beta)D \\ O_4 = 2r(\alpha)AB & O_6 = r(\beta)D \end{array}$$

$$C = D$$

$$O_4 = I_4$$

Composing relations

$$\blacksquare(\textcolor{gray}{\blacksquare}(R))\blacksquare(\textcolor{gray}{\blacksquare}(R'))$$

$$(c_X, I_X, c_Y, O_Y)(c_Y, I_Y, c_Z, O_Z)$$

$$\begin{array}{rr} I_1 = r(\alpha)AB & I_4 = r(\beta)D \\ I_2 + I_3 = r(\alpha)AB & O_5 = r(\beta)D \\ O_4 = 2r(\alpha)AB & O_6 = r(\beta)D \end{array}$$

$$C = D$$

$$2r(\alpha)AB = r(\beta)D$$

Composing relations

$$\blacksquare({\color{gray}\blacksquare}(R))\blacksquare({\color{gray}\blacksquare}(R')) \colon \mathbb{R}^X \oplus \mathbb{R}^X \rightsquigarrow \mathbb{R}^Z \oplus \mathbb{R}^Z$$

$$(c_X, I_X, c_Z, O_Z)$$

$$I_1 = r(\alpha)AB$$

$$I_2 + I_3 = r(\alpha)AB$$

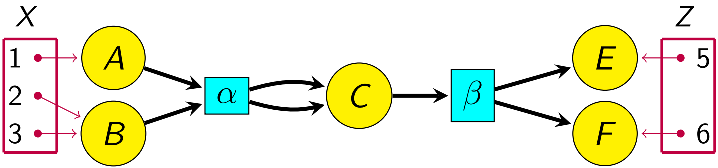
$$2r(\alpha)AB = r(\beta)C$$

$$O_5 = r(\beta)C$$

$$O_6 = r(\beta)C.$$

Black-boxing

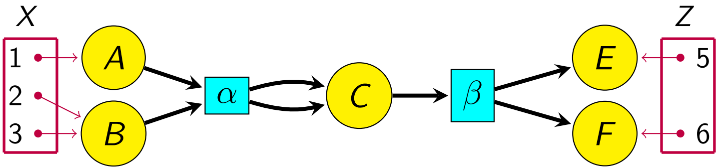
$$\blacksquare(RR' \colon X \to Y)$$



$$
\begin{aligned}
\frac{dA}{dt} &= -r(\alpha)AB + I_1 \\
\frac{dB}{dt} &= -r(\alpha)AB + I_2 + I_3 \\
\frac{dC}{dt} &= 2r(\alpha)AB - r(\beta)C \\
\frac{dE}{dt} &= r(\beta)C - O_5 \\
\frac{dF}{dt} &= r(\beta)C - O_6
\end{aligned}
$$

Black-boxing

$$\blacksquare(\square(RR')) \colon \mathbb{R}^X \oplus \mathbb{R}^X \rightsquigarrow \mathbb{R}^Z \oplus \mathbb{R}^Z$$



$$I_1 = r(\alpha)AB$$

$$I_2 + I_3 = r(\alpha)AB$$

$$2r(\alpha)AB = r(\beta)C$$

$$O_5 = r(\beta)C$$

$$O_6 = r(\beta)C.$$

Summary

The fact that black-boxing is accomplished via a functor means that one can compute the steady state behavior of a composite open reaction network by composing the semialgebraic relations characterizing the steady state behaviors of its constituent systems:

$$\blacksquare(\square(R))\ \blacksquare(\square(R')) = \blacksquare(\square(R)\square(R')) = \blacksquare(\square(RR'))$$

This provides a compositional approach to studying both the dynamical and steady state behaviors of open reaction networks.

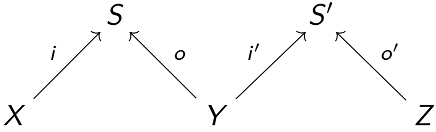
Thank you!

For more:

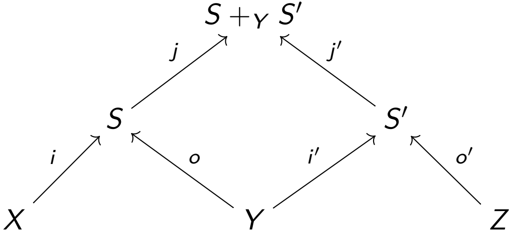
- John C. Baez and Blake S. Pollard, A compositional framework for reaction networks, submitted.
- John C. Baez, Brendan Fong and Blake S. Pollard, A compositional framework for Markov processes, *Journal of Mathematical Physics*.
- Blake S. Pollard, Open Markov processes: A compositional perspective on non-equilibrium steady states in biology, *Entropy*.
- Blake S. Pollard, A Second Law for open Markov processes, *Open Systems and Information Dynamics*.

Composition in Dynam

Given open dynamical systems $D\colon X \to Y$ on S and $D'\colon Y \to Z$ on S'



with vector fields $v\colon \mathbb{R}^S \to \mathbb{R}^S$ and $v'\colon \mathbb{R}^{S'} \to \mathbb{R}^{S'}$ to get an open dynamical system $DD'\colon X \to Z$ on $S +_Y S'$



we need to cook up a vector field $v''\colon \mathbb{R}^{S+_Y S'} \to \mathbb{R}^{S+_Y S'}$.

Composition in $\mathtt{Dynam}$

To get a vector field $v'' \colon \mathbb{R}^{S+_Y S'} \to \mathbb{R}^{S+_Y S'}$, first take the inclusion map

$$[j, j'] \colon S + S' \to S +_Y S'$$

and define two maps, $[j, j']_* \colon \mathbb{R}^{S+S'} \to \mathbb{R}^{S+_Y S'}$ as

$$[j, j']_*(v + v')_\sigma = \sum_{\{\sigma' | [j,j'](\sigma') = \sigma\}} (v + v')_{\sigma'},$$

and $[j, j']^* \colon \mathbb{R}^{S+_Y S'} \to \mathbb{R}^{S+S'}$ as

$$[j, j']^*(c'') = c'' \circ [j, j']$$

with $c'' \in \mathbb{R}^{S+_Y S'}$. We can then define our vector field via the expression

$$v''(c'') = [j, j']_*(v + v')[j, j']^*(c'').$$

Semialgebraic relations

A **semialgebraic subspace** of a vector space is a one defined in terms of polynomials and inequalities.

Semialgebraic relations

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Given semialgebraic relations $A\colon U \rightsquigarrow V$ and $B\colon V \rightsquigarrow W$, their composite $AB\colon U \rightsquigarrow W$ is given by

$$AB = \{(u, w)\colon \exists v \in V \text{ with } (u, v) \in A \;\text{ and }\; (v, w) \in B\}.$$