The Octonions

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April 7, 2010

1 Introduction

Since grade school or even earlier, we have all been learning about numbers. We start with counting, followed by addition and subtraction, multiplication and—the hard part—division. We learn that numbers are good for quantifying things, not just by counting, but by measurement. The venerated number system we are learning about is called the **real numbers**. The real numbers form a line:

Figure 1: The real numbers.

Every point on this line is a number. There are infinitely many, but we have marked a few of them for reference. The real numbers are one-dimensional, because lines are one-dimensional. Though we could turn this on its head: the line is one-dimensional because each point on it corresponds to one number.

Yet there are other number systems in which we can add, multiply, subtract, and—still the hard part—*divide* in much the same way as we can with real numbers. In fact, there are precisely four of them. These four number systems are technically called "normed division algebras", but we shall cut corners and call them "division algebras". The real number system is but one example. It is the simplest, most familiar division algebra.

The the next most familiar is the system of **complex numbers**, which many of us learn about in college. Complex numbers play a crucial role in subjects ranging from electrical engineering to quantum mechanics. The complex numbers form a plane:

Figure 2: The complex numbers.

The complex numbers are two-dimensional, one more than the reals. This is because it takes two *real* numbers to specify a point on a plane, one more than it takes to specify a point on a line. The extra dimension comes from having another number: the number i, which acts as a square root of -1. This is something that the reals lack, because the square of a real number is always positive, or zero. The fact that the complex numbers have i allows one to solve more equations with complex numbers than real numbers. For example, the equation $x^2 = -1$ has the solution $x = i$ when x is allowed to be complex, but no solution at all in the real numbers. This makes the complex numbers incredibly nice for many applications, and that usefulness is part of their charm. Yet complex numbers have a deeply mathematical charm too, and this is most readily apparent in their relationship with geometry. As we shall see shortly, multiplication of complex numbers can be described using the geometric operations of stretching and rotating.

The signifcance of complex numbers for two-dimensional geometry led the nineteenth century mathematician and physicist William Rowan Hamilton to seek a similar system of numbers to play the same role in three-dimensions. This problem vexed him for many years, and Hamilton's breakthrough came only when he began to think of even higher dimensions. He discovered the **quaternions**, a four-dimensional number system in which you can add, subtract multiply and divide. Like the complex numbers which owe their two-dimensions to 1 and a single square root of -1, the quaternions owe their four dimensions to 1 and *three* square roots of -1, called i, j and k respectively.

Inspired by Hamilton's work, his friend John Graves went on to discover the fourth and most mysterious of the division algebras: the **octonions**. This is an 8-dimensional number system, with *seven* square roots of -1. Their significance for physics only became clear with the rise of string theory. In fact, the octonions are the reason superstrings live in 10-dimensional spacetime!

We want to explain the octonions and their role in string theory. But we should start at the beginning...

2 Real numbers and their complex cousins

In the 1500s, Italian mathematicians had contests where they solved equations. The square root of -1, now called i, was introduced as a trick for solving these equations by Gerolamo Cardano, a mathematician, physician and gambler. For Cardano, i would sometimes show up as an intermediate step in a longer calculation: solving a cubic equation. The solutions Cardano found were always real numbers, and Cardano's trick worked—he got the right answers. But this raised a question: does the square root of -1 "really exist?" Or is it only a trick, a mere "imaginary" number? Indeed, the word "imaginary" haunts i to this day, and was first used to describe i in a derogatory way, by no less a thinker than Descartes.

Today, despite the word "imaginary", mathematicians consider i to be as real as any other number, thanks in large part to the work of Leonhard Euler and Carl Frederick Gauss, who proved such beautiful theorems using i that no doubt remained about the merit of the concept, and to Robert-Jean Argand, who popularized the geometric interpretation of complex numbers. In the 1800s, Argand realized that complex numbers $a + bi$ can be thought of as points on the *plane*. This is done by using the two *real* numbers a and b as coordinates: a tells us how much to the left or right the point $a + bi$ is, and b tells us how far up or down. In this way, we can think of any pair of real numbers as a point in the plane, but Argand went a step further: he showed how to think of the operations one can do with complex numbers—addition, subtraction, multiplication and division—as geometric operations on the plane.

As a warm up for this, let us think about the real numbers, and how operations with real numbers can be thought of as geometric operations on a line. Adding any real number slides the whole real line to the left or right, depending on whether the number is positive or negative. Subtracting is done by sliding in the opposite direction. Multiplying by any positive real number stretches or squashes the real line. For example, multiplying by 2 stretches the line by a factor of 2. Dividing by the same number squashes or stretches—dividing by 2 squashes everything to be half as big as before.

Likewise, adding any complex number slides the whole complex plane along the arrow between 0 and the given number. Subtracting corresponds to sliding in the opposite direction. So, for addition and subtraction, the story with complex numbers is the same as for real numbers, just in two dimensions rather than one.

It is multplication that really gives us something new: multiplying by a complex number stretches or squashes but also *rotates* the complex plane counter clockwise, opposite the direction of a clock. The number i corresponds to rotating a quarter turn.

(Explain using a couple of pictures.)

So, of course it is possible to divide by nonzero complex numbers: just shrink instead of stretching, or vice versa, and then rotate in the opposite sense—clockwise. Doing this will undo any multiplication, just as division should. It is the *inverse* to multiplication.

(Explain using a few more pictures.)

In fact, almost everything one can do with real numbers—raise to powers, extract roots, define logarithms, study calculus—can also be done with complex numbers, and it often works even better in that context, thanks to the fact that more equations can be solved with complex numbers than with real numbers. Their main deficit is merely that they are a litte more abstract than real numbers: whatever kind of number the square root of -1 is, it is not the right kind to describe many things in our familiar experience, such as the amount money we have in the bank or the volume of water we bought in a water bottle.

There is far more to say about the complex numbers than we have space for here. We have only had a taste of them and their world, the plane. But we have other worlds to explore, so let us set the complex numbers aside now, and turn to their stranger cousins, the quaternions and the octonions.

3 Hamilton's Alchemy: The Quaternions

In 1835, at the age of 30, the Irish mathematician and physicist William Rowan Hamilton discovered how to treat complex numbers as pairs of real numbers. Fascinated by the relation between complex numbers and 2-dimensional geometry, he tried for many years to invent a bigger algebra of "triplets" that would play a similar role in 3-dimensional geometry.

His quest built to its climax in October 1843. He later wrote to his son, "Every morning in the early part of the above-cited month, on my coming down to breakfast, your (then) little brother William Edwin, and yourself, used to ask me: 'Well, Papa, can you *multiply* triplets?' Whereto I was always obliged to reply, with a sad shake of the head: 'No, I can only *add* and *subtract* them'."

Finally, on the 16th of October, 1843, while walking with his wife along the Royal Canal to a meeting of the Royal Irish Academy in Dublin, he made his momentous discovery. "That is to say, I then and there felt the galvanic circuit of thought *close*; and the sparks which fell from it were the fundamental equations between i, j, k; exactly such as I have used them ever since." And in a famous act of mathematical vandalism, he carved these equations into the stone of the Brougham Bridge.

It turns out that his quest to multply triplets was hopeless. We now know this is because Hamilton was seeking a 3-dimensional division algebra, which *does not exist!* He really needed a system consisting of numbers like $a + bi + cj + dk$, where i, j, k are *three* square roots of -1, and this is what he discovered how to multiply on that fateful day. With numbers like this, you can describe rotations in 3d space.

He called this number system the quaternions, since it's 4-dimensional: Just as we think of the complex numbers

$$a + bi$$

as points in a two-dimensional plane, so too can we think of the quaternions

$$a + bi + cj + dk$$

as points in a four-dimensional space. It may seem odd that we need points in a 4d space to describe rotations and stretching in 3d space, but it's true! Three of the numbers come from describing rotations, which we can see most readily if we imagine trying to fly an airplane. To orient the plane however we want, we need to control the *pitch*, or angle with the horizontal:

Figure 3: Pitch of a plane.

We also may need to adjust the *yaw*, by turning left or right, like a car:

Figure 4: Yaw of a plane.

And finally, we may need to adjust the *roll*, the angle of the plane's wings:

Figure 5: Roll of a plane.

Together, giving all three numbers tells us how to *rotate* a plane from any one orientation to any other. So, in short it takes three numbers to describe a rotation, and this accounts for three of the four dimensions of the quaternions. What about the fourth?

The fourth dimension arises because quaternions can *also* describe stretching or shrinking, which takes one more number:

Figure 6: Stretching or shrinking in 3d.

Loosely, these four numbers represent the four dimensions of the quaternions, though it takes a complicated formula to relate them to the four numbers a, b, c, and d.

The utility of the quaternions doesn't stop with rotations in three-dimensions. We can use them for something even stranger—rotations in 4-dimensions, which we can think of as the space of quaternions itself! Now, however, it takes more than three numbers to describe a rotation. In fact, it takes six. So, we can't expect to pack all that information into a single quaternion, with only four numbers available for us to tweak. Instead, it takes *two* quaternions to describe a rotation in 4-dimensions.

Hamilton spent the rest of his life obsessed with the quaternions, and found many applications for them. Today, in many of these appications the quaternions have been replaced by their simpler cousins, called *vectors*, in 3d. These are basically the triplets which Hamilton struggled for years to multiply, without success. Yet quaternions still have their niche: they provide an efficient way to represent rotations on a computer, and show up wherever this is needed, from the attitude-control system of a spacecraft to the graphics engine of a video game. Despite these applications, however, we might still wonder: do these square roots of -1 really exist? If we accept i, we might as well accept j and k. But does this mean we can just keep throwing in square roots of -1 to our heart's content?

4 The octonions

This question was asked by Hamilton's college friend, the lawyer John T. Graves. It was Graves' interest in algebra that got Hamilton thinking about complex numbers and triplets in the first place. The very day after his fateful walk, Hamilton sent an 8-page letter describing the quaternions to Graves. Graves replied on October 26th, complimenting Hamilton on the boldness of the idea, but adding "There is still something in the system which gravels me. I have not yet any clear views as to the extent to which we are at liberty arbitrarily to create imaginaries, and to endow them with supernatural properties." And he asked: "If with your alchemy you can make three pounds of gold, why should you stop there?"

Graves then set to work on some gold of his own! On December 26th, he wrote to Hamilton describing a new 8-dimensional algebra, which he called the 'octaves'. These are now called 'octonions'. He later tried a 16-dimensional algebra, but "met with an unexpected hitch". It was later realized that the octonions are the last algebra where you can add, subtract, multiply and *divide*: this is also possible in dimensions 1 (real numbers), 2 (complex numbers), 4 (quaternions) and 8 (octonions). Such algebras, as we have already said, are called "division algebras".

Graves was unable to in get Hamilton interested in the octonions, who focused on his own quaternions instead. Hamilton promised to speak about Graves' octaves at the Irish Royal Society, which is one way mathematical results were published at the time. But he kept putting it off, until the young genius Arthur Cayley rediscovered the octonions and beat Graves to publication. For this reason, the octonions are also sometimes known as "Cayley numbers".

5 Nonassociativity

Why didn't Hamilton like the octonions? Let's look at how to multiply quaternions, and octonions.

(A picture explaining how to multiply quaternions, with a couple of examples.) The quaternions are a bit strange: they're "noncommutative":

$$xy \neq yx$$

But this is okay, because rotations in 3 dimensions are noncommutative. Take a piece of paper, and turn it a little bit clockwise. Then flip it over, and note that it looks like it was rotated a little bit counter clockwise. Now, do these two operations in reverse order: flip first, and then rotate a little bit clockwise. Now the paper looks like it's been rotated clockwise. Since the result depends on the order, rotations don't commute! (picture)

(Picture to explain multiplication of octonions, with a few examples.) The octonions are much stranger: they're "nonassociative":

$$a(bc) \neq (ab)c$$

We've all seen a nonassociative operation in our study of mathematics: subtraction!

$$3 - (2 - 1) \neq (3 - 2) - 1$$

But we're used to multiplication being associative, and most mathematicians still feel this way, even though they have gotten used to noncommutative operations. Rotations are associative, for example, even though they don't commute.

Despite this weirdness, octonions are still closely related to the geometry of 7 and 8 dimensions, and we can still describe rotations in those dimensions using the multiplication of octonions! It's just that, because rotations are associative and octonions are not, the relationship is more subtle than it is for the other division algebras.

If division algebras were people, they would have very different personalities. The real numbers are the dependable breadwinner of the family, the one we all learn about in grade school. The complex numbers are a slightly flashier but still respectable younger brother. The quaternions, being noncommutative, are the eccentric cousin who is shunned at important family gatherings. But the octonions are the crazy old uncle nobody lets out of the attic! They've suffered from long neglect, but now this may be changing.

6 Supersymmetry

What are the octonions good for? Can a mathematical object exist without a "purpose"? In the late 20th century, it became clear that the octonions may indeed have a role to play in physics. This role is related to supersymmetry—a symmetry between “matter” particles like electrons and “force” particles like photons. So, to say how it works, first we must say what these particles are.

The idea that everything, all the stuff we see around us, is made of particles of matter should be familiar to most people. We know that it is all made of atoms, and that atoms are made of electrons, protons, and neutrons. Of these matter particles, electrons are truly fundamental: they are not composed of anything smaller, while protons and neutrons are made of fundmental particles called *quarks*. It is particles like these, fundamental particles of matter, that we talk about when we discuss supersymmetry.

Besides these particles of matter, there are also particles of “force”, which are less familiar, but no less a part of modern particle physics. The fundamental forces in physics arise to the emission and absorption of force particles. For instance, electrons repel each other via the electromagnetic force, and the mechanism by which this occurs is the emission and absorption of photons.

Supersymmetry is the idea that the laws of physics will remain unchanged if we exchanged all the matter and force particles. Imagine viewing the universe in a strange mirror that, rather than interchanging left and right, traded every force particle for a matter particle, and vice versa. If supersymmetry is true, if it really describes our universe, this mirror universe would act the same as ours. Unfortunately, we do not know if supersymmetry *does* describe our universe. No evidence for it is seen in nature, but it’s so seductively beautiful that many physicists believe in it.

According to quantum mechanics, particles are also waves. The simplest sort of wave only wiggles up and down, so we need one ordinary real number to describe its height at each time and place. But there are also waves that wiggle in more directions! These are described by fancier kinds of numbers. Usually we need different kinds of numbers to describe matter particles and force particles. Physicists call the numbers needed to describe matter particles *spinors*, and the kind used to describe force particles *vectors*. The properties of vectors and spinors depends on the dimension of spacetime we’re talking about.

But imagine a strange universe with no time, only space. Then if this universe had dimension 1, 2, 4, or 8, both “matter” and “force” particles would be waves described by a number in a division algebra! In other words, the vectors and spinors *coincide*, and simplify: they are each just real numbers, complex numbers, quaternions or octonions. Indeed, one could turn this fact around: division algebras only exist when vectors and spinors coincide, and this only happens in dimensions 1, 2, 4 and 8. This is in fact one way that mathematicians can *prove* the only division algebras are the real numbers, complex numbers, quaternions and octonions.

Even better, in this strange universe of dimension 1, 2, 4 or 8, the interaction of matter and force particles would be described by multiplication in these number systems! In physics, such interactions are usually drawn using *Feynman diagrams*, named for physicist Richard Feynman:

Figure 7: Feynman diagram depicting an electron absorbing a photon.

So in dimensions 1, 2, 4 and 8, we can use the same diagram to depict multiplication in a division algebra. The two edges coming in represent the two numbers we start with, and the edge coming out represents their product.

So, in these strange universes with no time and special dimensions, nature would have ”supersymmetry”: a unified description of matter and forces.

7 String theory

But we really need to take time into account, since we live in a universe with time. In string theory this has a curious effect. At any moment in time a string is a 1-dimensional thing, like a curve or a line. But this string traces out a 2d surface as time passes. This changes the dimensions in which supersymmetry naturally arises, by adding two.

Namely: the simplest supersymmetric string theories arise in spacetimes of dimension 3, 4, 6, and 10. These numbers are 2 more than 1, 2, 4, and 8, the dimensions of the division algebras. Though the details are formidable, the basic idea is simple. The string traces out a 2-dimensional surface in spacetime, but it wiggles in 1, 2, 4, or 8 *extra* spatial dimensions. This is analogous to the way a drum vibrates: strike a drum, and it wiggles mostly up and down, with little wiggling along the drum itself, which is pulled taut.

So for strings in dimensions 3, 4, 6 or 10, wiggling in 1, 2, 4, or 8 spatial directions, the vibrations are described using a division algebra!

Figure 8: A string traces out a surface in 3-dimensional spacetime, wiggling in one direction, like a drum.

This allows for supersymmetry. Different vibrations of the string describe different kinds of particles. When we describe these vibrations using a division algebra, this allows us to unify the "matter" and "force" aspect of the string's vibrations.

Curiously, when we fully take quantum mechanics into account, only the 10d theory is consistent! This is the theory that uses octonions. So, if superstring theory is right, the 10-dimensionality of spacetime arises from the octonions.

8 M-theory

Recently physicists have become interested in going beyond strings to consider "membranes". For example, a 2-dimensional membrane or "2-brane" looks like a soap bubble at any moment in time. As time passes, it traces out a 3d surface in spacetime - one more dimension than we had in string theory.

And, it turns out that "supersymmetric 2-branes" only work in dimensions 4, 5, 7 and 11. The reason is very much the same as in string theory, but now these numbers are 3 more than 1, 2, 4 and 8. Again, it's the octonionic case—the 11 dimensional theory—that seems to work best. This is a *part* of what physicists call "M-theory". Alas, no one understands M-theory well enough to even write down its basic equations! So, it is difficult to know if 2-branes will continue to be a part of the theory, because we just don't know what shape it may take in the future.

Indeed, we should emphasize that string theory and M-theory have made *no experimentally testable predictions so far.* They are beautiful dreams... but so far only dreams. The spacetime we see does not look 10- or 11-dimensional, and we do not see supersymmetry—though there's a chance that evidence for it will turn up at the new particle experiments at CERN.

9 Open questions

Let us turn from speculative physics to speculative mathematics. In the mathematics of the division algebras, we have met an example of a *classification theorem*. We can think of this as a mathematics problem of the following sort: "Given a list of properties, describe *every* mathematical object with those properties."

For instance, given the list of properties describing a "division algebra", describe every such object. In this article, we've done it! There are exactly four: the real numbers, the complex numbers, the quaternions and the octonions.

This kind of simple classification is the exception to the rule in mathematics: usually, the list of objects in a classification theorem is *infinite*. In that case, we resort to giving a recipe to build each one. And the recipes tend to be quite few: a few recipes that give infinite families, and a short list of recipes that can each only be used *once*. The gadgets we cook up with these one-time recipes are called *exceptional objects*. They are weirdos that we find it hard to fit into a grand scheme.

The octonions are an example: our recipe for building them works only in dimension 8. But they are just one of many "exceptional objects" in mathematics. They have exotic names like "the exceptional Jordan algebra", the "exceptional group E_8", and the "Monster group".

In an eerie way, the exceptional objects of one sort tend to be related to exceptional gadgets of other sorts. This suggests that someday there will be a "unified theory of exceptions", paradoxical as this might seem. And the simplest way to start seeking such a theory is to start with the octonions, which play a crucial role in constructing many exceptional objects in mathematics.

For example, there are several known ways to use the octonions to build the exceptional group E_8 - a huge 248-dimensional symmetry group which plays an important role in some versions of superstring theory. But the relevance to physics remains unclear, since the group E_8 showed up in string theory for reasons that seem unrelated to the octonions!

So, there are many mysteries to puzzle over... and these force us back to basic questions such as: "what does supersymmetry really mean?", "why should it be related to division algebras?", and "if it's really important in physics, why don't we see it in nature?"

IT WOULD BE GOOD TO SAY MORE ABOUT THE HUMAN SIDE OF HOW PEOPLE DISCOVERED THE RELATION BETWEEN DIVISION ALGEBRAS AND SUPERSYMMETRY. THIS WOULD MAKE THE TONE OF THE ARTICLE MORE CONSISTENT.

About the authors

John Baez is a mathematician at the University of California, Riverside. John Huerta is a graduate student there, working on algebra inspired by particle physics and string theory.

A link to our webpages
http://math.ucr.edu/home/baez/
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would be nice.