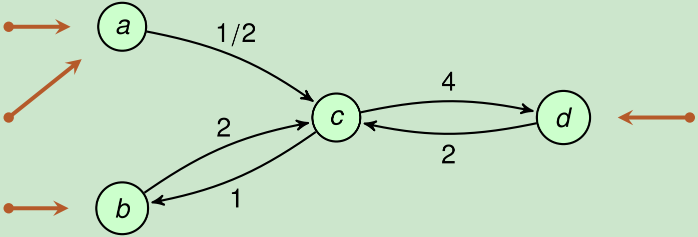
COARSE-GRAINING OPEN MARKOV PROCESSES



John Baez and Kenny Courser

ACT2020

NEW PARADIGMS

Category theory gives new ways to model open systems.

I'll illustrate this using continuous-time finite-state Markov chains — or "Markov processes" for short:

- JB, Brendan Fong and Blake Pollard, A compositional framework for Markov processes, arXiv:1508.06448.
- Kenny Courser, *Open Systems: A Double Categorical Perspective*, https://tinyurl.com/courser-thesis.
- JB and Kenny Courser, Coarse-graining open Markov processes, arXiv:1710.11343.

START WITH ANY KIND OF SYSTEM

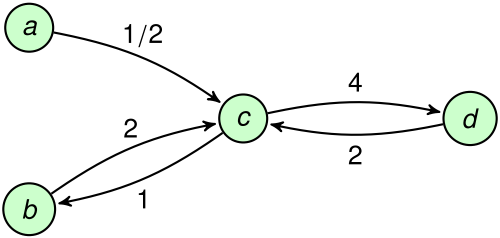
A **Markov process** describes how the probability of being in various states changes with time:

$$\frac{d}{dt}\begin{bmatrix} p_a(t) \\ p_b(t) \\ p_c(t) \\ p_d(t) \end{bmatrix} = \begin{bmatrix} -1/2 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 1/2 & 2 & -5 & 2 \\ 0 & 0 & 4 & -2 \end{bmatrix} \begin{bmatrix} p_a(t) \\ p_b(t) \\ p_c(t) \\ p_d(t) \end{bmatrix}$$

- Each column in the matrix sums to zero.
- The off-diagonal entries are nonnegative.

MAKE IT BETTER WITH DIAGRAMS

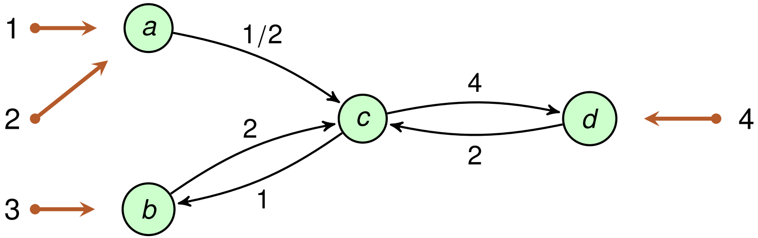
It's better to imagine a *picture* of a Markov process:



$$\frac{d}{dt}\begin{bmatrix} p_a(t) \\ p_b(t) \\ p_c(t) \\ p_d(t) \end{bmatrix} = \begin{bmatrix} -1/2 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 1/2 & 2 & -5 & 2 \\ 0 & 0 & 4 & -2 \end{bmatrix} \begin{bmatrix} p_a(t) \\ p_b(t) \\ p_c(t) \\ p_d(t) \end{bmatrix}$$

OPEN IT UP

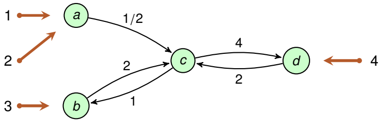
In a **open Markov process**, probability can flow in or out from outside:



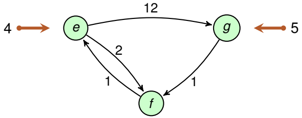
$$\frac{d}{dt}\begin{bmatrix}\pi_a(t)\\ \pi_b(t)\\ \pi_c(t)\\ \pi_d(t)\end{bmatrix} = \begin{bmatrix}-1/2 & 0 & 0 & 0\\ 0 & -2 & 1 & 0\\ 1/2 & 2 & -5 & 2\\ 0 & 0 & 4 & -2\end{bmatrix}\begin{bmatrix}\pi_a(t)\\ \pi_b(t)\\ \pi_c(t)\\ \pi_d(t)\end{bmatrix} + \begin{bmatrix}I_1(t)+I_2(t)\\ I_3(t)\\ 0\\ -O_4(t)\end{bmatrix}$$

TREAT OPEN SYSTEMS AS MORPHISMS

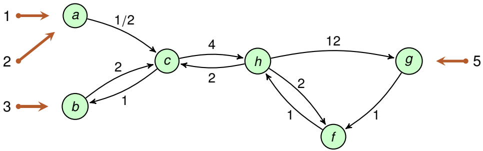
Composing this:



and this:

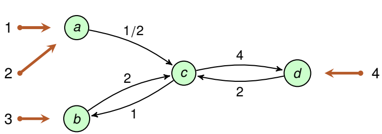


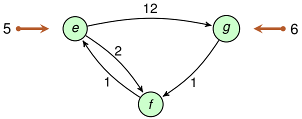
we get this:



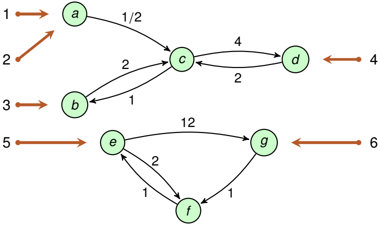
MAKE YOUR CATEGORY MONOIDAL

The tensor product of these open Markov processes:



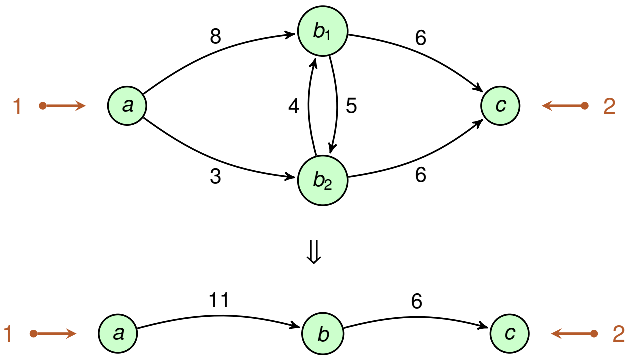


is this:



DON'T FORGET THE 2-MORPHISMS

We can use 2-morphisms between open Markov processes to describe **coarse-graining**:



PUT IT ALL TOGETHER

There's a symmetric monoidal double category $\mathbb{M}$**ark** where:

- An object is a finite set.
- A horizontal 1-cell is an **open Markov process**:

$$S \xrightarrow{i} (X, H) \xleftarrow{o} T$$

that is, a pair of injections $S \xrightarrow{i} X \xleftarrow{o} T$ together with a **Markov process** $H \colon \mathbb{R}^X \to \mathbb{R}^X$: an $X \times X$ matrix of reals with columns summing to 0 and off-diagonal entries ≥ 0.

- A vertical 1-morphism is a function $f\colon S \to S'$.
- A **2-morphism of open Markov processes**:

$$
\begin{array}{ccccc}
S & \xrightarrow{i_1} & (X,H) & \xleftarrow{o_1} & T \\
{\scriptstyle f}\downarrow & & {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\
S' & \xrightarrow[i'_1]{} & (X',H') & \xleftarrow[o'_1]{} & T'
\end{array}
$$

consists of pullback squares in the category of finite sets:

$$
\begin{array}{ccccc}
S & \xrightarrow{i_1} & X & \xleftarrow{o_1} & T \\
{\scriptstyle f}\downarrow & & {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\
S' & \xrightarrow[i'_1]{} & X' & \xleftarrow[o'_1]{} & T'
\end{array}
$$

such that $H'p_* = p_*H$, where $p_*\colon \mathbb{R}^X \to \mathbb{R}^{X'}$ is the linear map coming from $p\colon X \to X'$.

- The tensor product of 2-morphisms

$$
\begin{array}{ccccc}
S_1 & \xrightarrow{i_1} & (X_1, H_1) & \xleftarrow{o_1} & T_1 \\
{\scriptstyle f_1}\Big\downarrow & & {\scriptstyle p_1}\Big\downarrow & & \Big\downarrow{\scriptstyle g_1} \\
S_1' & \xrightarrow[i_1']{} & (X_1', H_1') & \xleftarrow[o_1']{} & T_1'
\end{array}
\qquad
\begin{array}{ccccc}
S_2 & \xrightarrow{i_2} & (X_2, H_2) & \xleftarrow{o_2} & T_2 \\
{\scriptstyle f_2}\Big\downarrow & & \Big\downarrow{\scriptstyle p_2} & & \Big\downarrow{\scriptstyle g_2} \\
S_2' & \xrightarrow[i_2']{} & (X_2', H_2') & \xleftarrow[o_2']{} & T_2'
\end{array}
$$

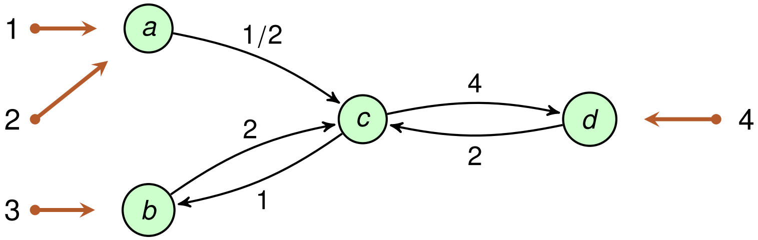
is:

$$
\begin{array}{ccccc}
S_1 + S_2 & \xrightarrow{i_1 + i_2} & (X_1 + X_2, H_1 \oplus H_2) & \xleftarrow{o_1 + o_2} & T_1 + T_2 \\
{\scriptstyle f_1 + f_2}\Big\downarrow & & {\scriptstyle p_1 + p_2}\Big\downarrow & & \Big\downarrow{\scriptstyle g_1 + g_2} \\
S_1' + S_2' & \xrightarrow[i_1' + i_2']{} & (X_1' + X_2', H_1' \oplus H_2') & \xleftarrow[o_1' + o_2']{} & T_1' + T_2'
\end{array}
$$

THEN USE FUNCTORIAL SEMANTICS

If open systems are *morphisms*, we can describe what they do using *functors*.

We can *completely* describe what an open Markov process does — or “black-box” it and describe only the relation between inputs and outputs that holds in steady states.



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There's a symmetric monoidal double functor called **black-boxing**:

$$\blacksquare \colon \mathbb{M}\mathbf{ark} \to \mathbb{L}\mathbf{inRel}$$

where $\mathbb{L}\mathbf{inRel}$ has:

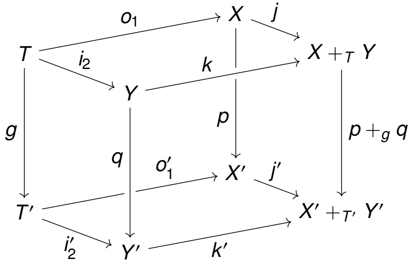
- finite-dimensional real vector spaces as objects,
- linear relations $R \subseteq V \oplus W$ as horizontal 1-cells,
- linear maps $f \colon V \to W$ as vertical 1-morphisms,
- squares

$$\begin{array}{ccc} V_1 & \xrightarrow{R \subseteq V_1 \oplus V_2} & V_2 \\ f\downarrow & & \downarrow g \\ W_1 & \xrightarrow[S \subseteq W_1 \oplus W_2]{} & W_2 \end{array}$$

obeying $(f \oplus g)R \subseteq S$ as 2-morphisms.

DON'T BE AFRAID TO SWEAT A BIT

To construct $\mathbb{M}$**ark** we use that FinSet is an "adhesive category", and thus:



If the top and bottom faces are pushouts, the left and back vertical faces are pullbacks, and the arrows o'_1 and i'_2 are monic, then the right and front vertical faces are pullbacks as well.