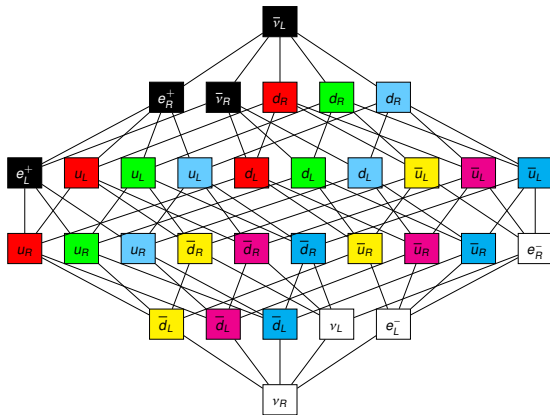


HIDDEN PATTERNS IN THE STANDARD MODEL



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One physics popularizer claimed that while the Standard Model is “the most successful physical theory ever proposed, it is also one of the ugliest”.

Maybe the Standard Model is more beautiful than we realize, and holds clues we do not yet understand.

There are many things about the Standard Model that would be great to understand better. In rough order of difficulty:

1. the Standard Model gauge group
2. its representation on one generation of fermions
3. the Lorentz group action on fermions, especially the chirality
4. the three generations
5. the Higgs and its couplings (22 dimensionless constants)
6. the gauge field coupling constants (3 dimensionless constants)

Here I will only consider the first two:

1. the Standard Model gauge group
2. its representation on one generation of fermions

We usually say the Standard Model gauge group is

$$U(1) \times SU(2) \times SU(3)$$

As a representation of this group, the fermions in one generation look random and ugly.

Let's write $\mathbb{C}_n$ for the representation of $U(1)$ on $\mathbb{C}$ where $\alpha \in U(1)$ acts as multiplication by α^n .

One Generation of Fermions

Particles		$U(1) \times SU(2) \times SU(3)$ rep
Left-handed leptons	$\begin{pmatrix} \nu_L \\ e_L^- \end{pmatrix}$	$\mathbb{C}_{-3} \otimes \mathbb{C}^2 \otimes \mathbb{C}$
Left-handed quarks	$\begin{pmatrix} u_L^r, u_L^g, u_L^b \\ d_L^r, d_L^g, d_L^b \end{pmatrix}$	$\mathbb{C}_1 \otimes \mathbb{C}^2 \otimes \mathbb{C}^3$
Right-handed neutrino	ν_R	$\mathbb{C}_0 \otimes \mathbb{C} \otimes \mathbb{C}$
Right-handed electron	e_R^-	$\mathbb{C}_{-6} \otimes \mathbb{C} \otimes \mathbb{C}$
Right-handed up quarks	(u_R^r, u_R^g, u_R^b)	$\mathbb{C}_4 \otimes \mathbb{C} \otimes \mathbb{C}^3$
Right-handed down quarks	(d_R^r, d_R^g, d_R^b)	$\mathbb{C}_{-2} \otimes \mathbb{C} \otimes \mathbb{C}^3$

However, some elements of $U(1) \times SU(2) \times SU(3)$ act trivially on all known particles! Thus, the “true” Standard Model gauge group is a quotient of $U(1) \times SU(2) \times SU(3)$ by some subgroup.

However, some elements of $U(1) \times SU(2) \times SU(3)$ act trivially on all known particles! Thus, the “true” Standard Model gauge group is a quotient of $U(1) \times SU(2) \times SU(3)$ by some subgroup.

This subgroup consists of elements

$$(\zeta, \zeta^3, \zeta^2) \in U(1) \times SU(2) \times SU(3)$$

where ζ is a sixth root of 1 times a suitable identity matrix.

These elements form a copy of $\mathbb{Z}_6$.

So, we can call

$$\frac{U(1) \times SU(2) \times SU(3)}{\mathbb{Z}_6}$$

the **true gauge group** of the Standard Model.

There is a homomorphism

$$\begin{aligned} \phi: \quad \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3) &\rightarrow \mathrm{SU}(5) \\ (\alpha, g, h) &\mapsto \begin{pmatrix} \alpha^3 g & 0 \\ 0 & \alpha^{-2} h \end{pmatrix} \end{aligned}$$

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Its kernel is the $\mathbb{Z}_6$ subgroup that acts trivially on all known particles:

$$\phi: (\zeta, \zeta^3, \zeta^2) \mapsto \begin{pmatrix} \zeta^3 \zeta^3 & 0 \\ 0 & \zeta^{-2} \zeta^2 \end{pmatrix} = 1$$

Its image is

$$\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3)) = \left\{ x \in \mathrm{SU}(5) : x = \begin{pmatrix} * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \end{pmatrix} \right\}$$

Thus, the true gauge group of the Standard Model is

$$\frac{U(1) \times SU(2) \times SU(3)}{\mathbb{Z}_6} = S(U(2) \times U(3))$$

All the fermions and antifermions in one generation, including a right-handed neutrino and its antiparticle, fit into the natural representation of $S(U(2) \times U(3))$ on

$$\Lambda \mathbb{C}^5 = \bigoplus_{n=0}^5 \Lambda^n \mathbb{C}^5$$

The left-handed fermions and antifermions live in

$$\Lambda^{\text{even}} \mathbb{C}^5 = \Lambda^0 \mathbb{C}^5 \oplus \Lambda^2 \mathbb{C}^5 \oplus \Lambda^4 \mathbb{C}^5$$

while the right-handed ones live in

$$\Lambda^{\text{odd}} \mathbb{C}^5 = \Lambda^1 \mathbb{C}^5 \oplus \Lambda^3 \mathbb{C}^5 \oplus \Lambda^5 \mathbb{C}^5$$

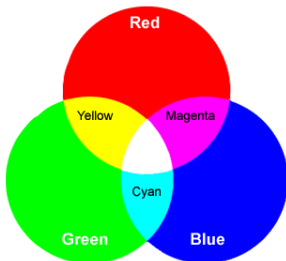
They are interchanged by the conjugate-linear Hodge star operator

$$\star: \Lambda \mathbb{C}^5 \rightarrow \Lambda \mathbb{C}^5$$

$\Lambda\mathbb{C}^5$ has a basis labelled by 5-bit strings, answering 5 yes-or-no questions:

- ▶ is the particle isospin up?
- ▶ is it isospin down?
- ▶ is it red?
- ▶ is it green?
- ▶ is it blue?

Here “up & down” means isospin 0, “red & blue” means “magenta” = anti-green, “red, green & blue” means colorless, etc.



$\Lambda^0 \mathbb{C}^5$	$\Lambda^1 \mathbb{C}^5$	$\Lambda^2 \mathbb{C}^5$	$\Lambda^3 \mathbb{C}^5$	$\Lambda^4 \mathbb{C}^5$	$\Lambda^5 \mathbb{C}^5$
$\bar{\nu}_L = 1$	$e_R^+ = u$	$e_L^+ = ud$	$e_R^- = rgb$	$e_L^- = drgb$	$\nu_R = udrgb$
	$\bar{\nu}_R = d$	$u_L = \begin{Bmatrix} \textcolor{red}{ur} \\ \textcolor{green}{ug} \\ \textcolor{blue}{ub} \end{Bmatrix}$	$\bar{u}_R = \begin{Bmatrix} \textcolor{violet}{drb} \\ \textcolor{yellow}{drg} \\ \textcolor{blue}{dgb} \end{Bmatrix}$	$\nu_L = urgb$	
	$d_R = \begin{Bmatrix} \textcolor{red}{r} \\ \textcolor{green}{g} \\ \textcolor{blue}{b} \end{Bmatrix}$	$d_L = \begin{Bmatrix} \textcolor{red}{dr} \\ \textcolor{green}{dg} \\ \textcolor{blue}{db} \end{Bmatrix}$	$\bar{d}_R = \begin{Bmatrix} \textcolor{yellow}{urg} \\ \textcolor{violet}{urb} \\ \textcolor{blue}{ugb} \end{Bmatrix}$	$\bar{d}_L = \begin{Bmatrix} \textcolor{yellow}{udrg} \\ \textcolor{violet}{udrb} \\ \textcolor{blue}{udgb} \end{Bmatrix}$	
		$\bar{u}_L = \begin{Bmatrix} \textcolor{yellow}{rg} \\ \textcolor{violet}{rb} \\ \textcolor{blue}{gb} \end{Bmatrix}$	$u_R^c = \begin{Bmatrix} \textcolor{red}{udr} \\ \textcolor{green}{udg} \\ \textcolor{blue}{udb} \end{Bmatrix}$		

We have

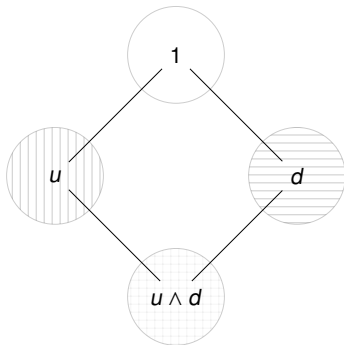
$$\Lambda\mathbb{C}^5 \cong \Lambda\mathbb{C}^2 \otimes \Lambda\mathbb{C}^3$$

It's good to study these two factors separately.

The first is a representation of isospin $SU(2)$, while the second is a representation of color $SU(3)$.

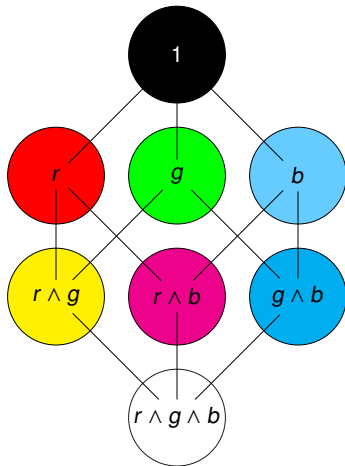
Hypercharge $U(1)$ acts on both.

$$\Lambda \mathbb{C}^2 = \Lambda^0 \mathbb{C}^2 \oplus \Lambda^1 \mathbb{C}^2 \oplus \Lambda^2 \mathbb{C}^2 \cong \mathbf{1} \oplus \mathbf{2} \oplus \mathbf{1}$$



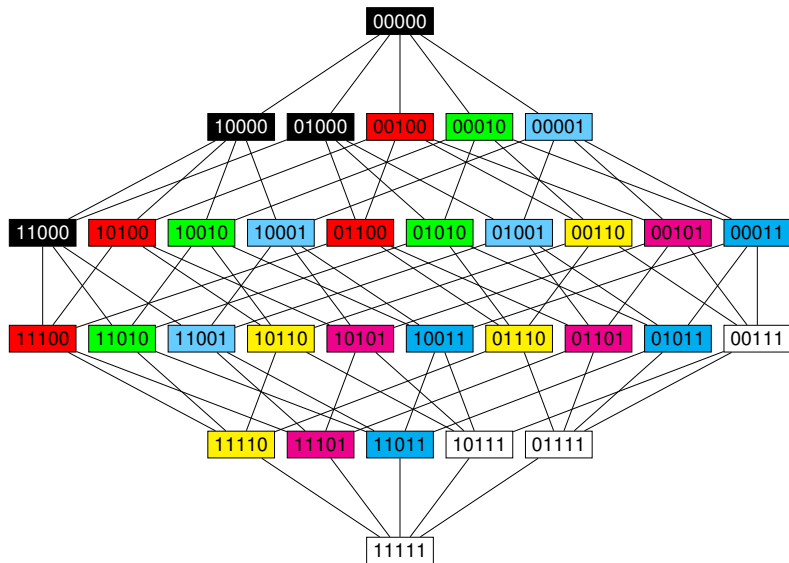
Half the fermions come in isospin doublets, while the other half come in isospin singlets.

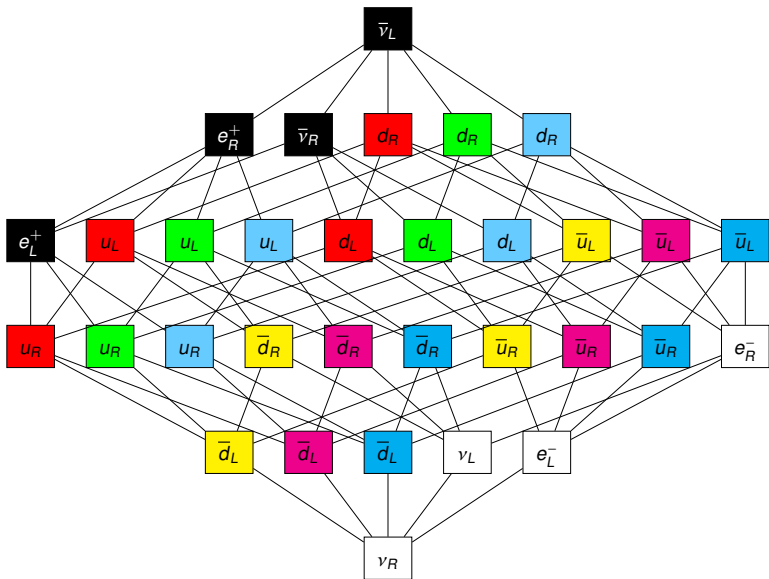
$$\Lambda \mathbb{C}^3 = \Lambda^0 \mathbb{C}^3 \oplus \Lambda^1 \mathbb{C}^3 \oplus \Lambda^2 \mathbb{C}^3 \oplus \Lambda^3 \mathbb{C}^3 \cong \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{3}^* \oplus \mathbf{1}$$



Each quark comes in three colors and has a partner: a colorless lepton.

Putting them together, we get $\Lambda\mathbb{C}^5 \cong \Lambda\mathbb{C}^2 \otimes \Lambda\mathbb{C}^3$:

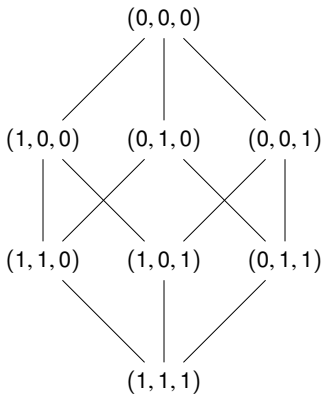




It's all very beautiful. But what's so special about $\Lambda\mathbb{C}^2$ and $\Lambda\mathbb{C}^3$?
Why not $\Lambda\mathbb{C}^4$ and $\Lambda\mathbb{C}^7$?

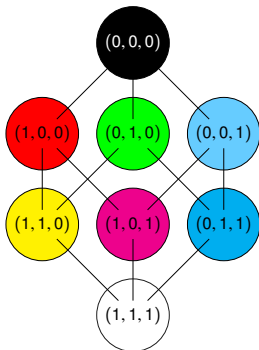
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$\Lambda\mathbb{C}^3$ resembles the algebra of octonions.



The octonions $\mathbb{O}$ are an 8-dimensional *real* algebra with basis given by 3-bit strings $(i, j, k) \in \mathbb{Z}_2^3$, with multiplication

$$(i, j, k) (i', j', k') = \pm(i + i', j + j', k + k')$$



The automorphisms of $\mathbb{O}$ fixing any square root of -1 , e.g. $(1, 1, 1)$, form a copy of $SU(3)$.

But $\Lambda\mathbb{C}^3$ is an 8-dimensional *complex* space. To bridge the gap, form the **bioctonions**

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{O}$$

As vector spaces

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{O} \cong \Lambda\mathbb{C}^3$$

Their products are different, but $SU(3)$ acts on both as algebra automorphisms.

Thus, the color degrees of freedom in one generation of the Standard Model correspond to bioctonions with a chosen square root of -1 .

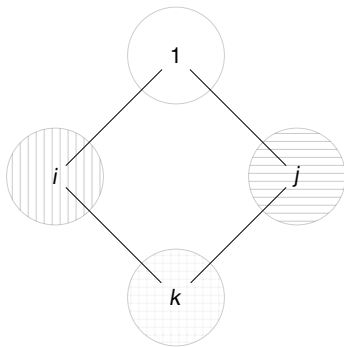
What about the isospin degrees of freedom? We can try to describe these using **biquaternions**:

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \cong \Lambda \mathbb{C}^2$$

as vector spaces, where

$$\mathbb{H} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\}$$

is Hamilton's algebra of quaternions.



The internal degrees of freedom in one generation of fermions and antifermions are thus described by the **Dixon algebra**

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \otimes_{\mathbb{R}} \mathbb{O} \cong \underbrace{(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H})}_{\text{biquaternions}} \otimes_{\mathbb{C}} \underbrace{(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{O})}_{\text{bioctonions}}$$

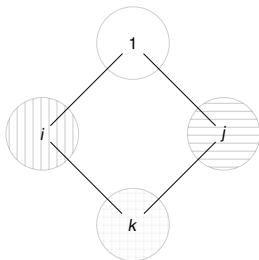
which is isomorphic to $\Lambda \mathbb{C}^5$ as a vector space.

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which is isomorphic to $\Lambda \mathbb{C}^5$ as a vector space.

An esthetic problem: while $SU(3)$ acts as automorphisms of the bioctonions fixing a square root of -1 , $SU(2)$ does *not* act as automorphisms of the biquaternions fixing a square root of -1 .



An alternative approach is to consider “positive hermitian Jordan pairs”. Any such thing gives a semisimple real Lie algebra $\mathfrak{k}$ with a $\mathbb{Z}_2$ -grading $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1$:

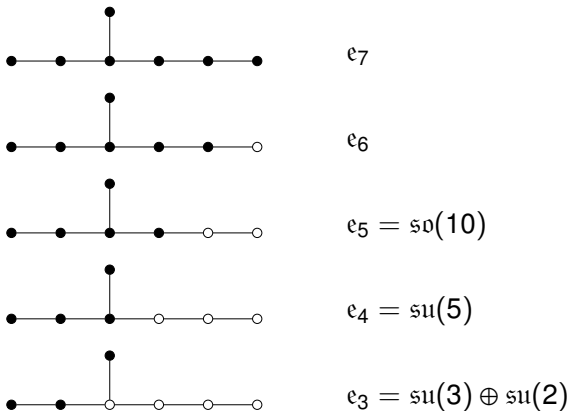
$$[\mathfrak{k}_i, \mathfrak{k}_j] \subseteq \mathfrak{k}_{i+j} \quad (\text{addition mod } 2)$$

such that the **bosonic part** $\mathfrak{k}_0$ is the Lie algebra of a compact Lie group and the **fermionic part** $\mathfrak{k}_1$ is a complex representation of $\mathfrak{k}_0$.

They have deep connections to differential geometry and the foundations of quantum physics:

- ▶ John Baez, Endre Bokor and Latham Boyle, Jordan pair quantum theory and the Standard Model, [arXiv:2607.10833](#).

There is an exceptional series of positive hermitian Jordan pairs:



Each Lie algebra e_n here has $e_{n-1} \oplus \mathfrak{u}(1)$ as its zeroth grade.

$\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ is the Standard Model Lie algebra!

More precisely:

$$\mathfrak{k} \cong \mathfrak{k}_0 \oplus \mathfrak{k}_1$$

$$\mathfrak{e}_7 \cong \mathfrak{e}_6 \oplus \mathfrak{u}(1) \oplus \mathfrak{h}_3(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{O})$$

$$\mathfrak{e}_6 \cong \mathfrak{so}(10) \oplus \mathfrak{u}(1) \oplus (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{O})^2$$

$$\mathfrak{e}_5 \cong \mathfrak{so}(10) \cong \mathfrak{su}(5) \oplus \mathfrak{u}(1) \oplus \Lambda^2(\mathbb{C}^5)$$

$$\mathfrak{e}_4 \cong \mathfrak{su}(5) \cong \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1) \oplus \mathbb{C}^2 \otimes \mathbb{C}^3$$

$\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ is the Standard Model Lie algebra. It acts on the 16-dimensional space $(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{O})^2$ just as it does on one generation of fermions (not antifermions).

Other clues:

- ▶ The Standard Model gauge group is the group of automorphisms of the Jordan algebra $\mathfrak{h}_3(\mathbb{O})$ that preserve a copy of $\mathfrak{h}_3(\mathbb{C})$ sitting inside it and a copy of $\mathfrak{h}_2(\mathbb{C})$ sitting in that:
- John Baez and Paul Schwahn, **The Standard Model gauge group from the exceptional Jordan algebra.**

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 - John Baez and Paul Schwahn, **The Standard Model gauge group from the exceptional Jordan algebra.**
- ▶ The Lie algebra $\mathfrak{e}_7$ contains a copy of the Standard Model Lie algebra acting via the Lie bracket on 3 generations of fermions and their antiparticles:
 - John Baez, **Three generations in E_7 .**

Can we put some of these jigsaw pieces together and make solid progress?

Details: Jordan pairs as $\mathbb{Z}_2$ -graded Lie algebras

Theorem. Positive hermitian Jordan pairs are equivalent to pairs $(\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1, J)$, where $\mathfrak{k}$ is a real semisimple Lie algebra with a $\mathbb{Z}_2$ -grading

$$[\mathfrak{k}_i, \mathfrak{k}_j] \subseteq \mathfrak{k}_{i+j} \quad (\text{addition mod } 2)$$

such that three conditions hold:

1. $[\mathfrak{k}_1, \mathfrak{k}_1] = \mathfrak{k}_0$;
2. the Killing form of $\mathfrak{k}$ is negative definite;
3. J is a $\mathfrak{k}_0$ -invariant complex structure on $\mathfrak{k}_1$.

Here “equivalent” means an equivalence of groupoids: on the Jordan side the morphisms are isomorphisms of hermitian Jordan pairs, and on the Lie side they are grading-preserving Lie algebra isomorphisms $\phi: \mathfrak{k} \rightarrow \mathfrak{k}'$ with $\phi \circ J = J' \circ \phi$ on $\mathfrak{k}_1$.

Here Jordan pairs are finite-dimensional over $\mathbb{C}$. A hermitian Jordan pair is the same as a hermitian Jordan triple system V : a complex vector space with a product $\{x, y, z\}$ that is complex-linear in x, z , antilinear in y , and satisfies the Jordan identity [4]. It is *positive* if the hermitian form $(x | y) = \text{tr } D(x, y)$ is positive definite, where $D(x, y)z = \{x, y, z\}$. We write B for Killing forms.

Step 1: from V to $(\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1, J)$. Let $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be the Tits–Kantor–Koecher (TKK) algebra of V , with $\mathfrak{g}_1 = V$, $\mathfrak{g}_{-1} = \overline{V}$ and $\mathfrak{g}_0 = \text{Inn}(V)$ spanned by the operators $D(x, y)$ [3]. The hermitian structure gives a grade-reversing antilinear involution θ of $\mathfrak{g}$. Let ϵ be the automorphism acting as $(-1)^j$ on $\mathfrak{g}_j$, and set $\theta_c = \theta\epsilon$, again a grade-reversing antilinear involution. Let $\mathfrak{k} = \mathfrak{g}^{\theta_c}$, so that

$$\mathfrak{k}_0 = (\mathfrak{g}_0)^\theta, \quad \mathfrak{k}_1 = \{x - \theta x : x \in \mathfrak{g}_1\} \cong V.$$

The bracket relations $[\mathfrak{k}_i, \mathfrak{k}_j] \subseteq \mathfrak{k}_{i+j}$ hold because $\mathfrak{g}_{\pm 1}$ are abelian.

Semisimplicity. A positive V has nondegenerate trace form, so V is nondegenerate, hence semisimple, and its TKK algebra $\mathfrak{g}$ is semisimple [3]. Then k , a real form of $\mathfrak{g}$, is semisimple.

Condition 1. The complexification of $[k_1, k_1]$ is $[\mathfrak{g}_1, \mathfrak{g}_{-1}]$, which is $\text{Inn}(V) = \mathfrak{g}_0$ by construction.

Condition 2. For $x, y \in \mathfrak{g}_1$, the form $B(x, \theta y)$ is a nonzero constant multiple of $\text{tr } D(x, y)$; with the normalizations of [4, 5] the constant is positive. Since $\mathfrak{g}_1$ is isotropic for B , for $u = x - \theta x \in k_1$ we get $B(u, u) = -2B(x, \theta x)$. So positivity of V is exactly negative definiteness of B on k_1 . Negative definiteness on k_0 , hence on all of k , then follows from Lemma 1. Equivalently, k is a compact real form of $\mathfrak{g}$: the Lie algebra of the isometry group of the compact dual of the bounded symmetric domain of V [4].

Condition 3. Let $H \in \mathfrak{g}_0$ be the grading element, so $\text{ad } H$ is j on $\mathfrak{g}_j$. Since $\theta_c H = -H$, the element $Z = iH$ lies in k_0 , it is central there, and $J = \text{ad}(Z)|_{k_1}$ satisfies $J^2 = -1$. Under $k_1 \cong V$, J is multiplication by i .

Lemma

Suppose $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1$ is a $\mathbb{Z}_2$ -graded semisimple real Lie algebra with $[\mathfrak{k}_1, \mathfrak{k}_1] = \mathfrak{k}_0$ and B negative definite on $\mathfrak{k}_1$. Then $\mathfrak{k}_0$ acts faithfully on $\mathfrak{k}_1$, and B is negative definite on $\mathfrak{k}_0$.

Proof.

First, $\mathfrak{k}_0$ acts faithfully on $\mathfrak{k}_1$. If $X \in \mathfrak{k}_0$ and $[X, \mathfrak{k}_1] = 0$, then by the Jacobi identity X also commutes with $[\mathfrak{k}_1, \mathfrak{k}_1] = \mathfrak{k}_0$, so X is central in $\mathfrak{k}$, hence $X = 0$.

Next, each $\text{ad } X|_{\mathfrak{k}_1}$ is skew for the inner product $-B|_{\mathfrak{k}_1}$. So $\mathfrak{k}_0$ embeds in $\mathfrak{so}(\mathfrak{k}_1, -B)$ and is a compact Lie algebra. In particular $\text{ad } X|_{\mathfrak{k}_0}$ is skew for some inner product on $\mathfrak{k}_0$. Therefore

$$B(X, X) = \text{tr}((\text{ad } X|_{\mathfrak{k}_0})^2) + \text{tr}((\text{ad } X|_{\mathfrak{k}_1})^2),$$

where the first term is ≤ 0 and the second is < 0 for $X \neq 0$, by faithfulness.



Step 2: from $(\mathfrak{k}, \mathfrak{k}_0 \oplus \mathfrak{k}_1, J)$ to V .

Lemma

Under conditions 1–3 there is a unique $Z \in \mathfrak{z}(\mathfrak{k}_0)$ with $J = \text{ad}(Z)|_{\mathfrak{k}_1}$.

Sketch.

Let K be the simply connected group with Lie algebra $\mathfrak{k}$; by Condition 2 it is compact. The grading involution lifts to K , and its fixed-point group K_0 is connected with Lie algebra $\mathfrak{k}_0$ [1, Ch. VII]. So $M = K/K_0$ is a simply connected Riemannian symmetric space of compact type. The $\text{Ad}(K_0)$ -invariant J defines a K -invariant almost complex structure on M . Invariant tensor fields on a symmetric space are parallel [2, Ch. XI], so J is parallel and M is a hermitian symmetric space. Hence $J_o = \text{ad}(Z)|_{\mathfrak{k}_1}$ for some Z in the center of $\mathfrak{k}_0$ [1, Ch. VIII, Thm. 4.5].

Z is unique because $\text{ad } Z$ is then determined: it is J on $\mathfrak{k}_1$ and 0 on $\mathfrak{k}_0$. Since $\mathfrak{k}$ is semisimple, ad is injective. □

Now complexify: let $\mathfrak{g} = \mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C}$, graded by the eigenvalues $0, \pm 1$ of $\text{ad}(-iZ)$. Then

- ▶ $\mathfrak{g}_0 = \mathfrak{k}_0 \otimes \mathbb{C}$;
- ▶ $\mathfrak{g}_{\pm 1}$ are the $(\pm i)$ -eigenspaces of J on $\mathfrak{k}_1 \otimes \mathbb{C}$;
- ▶ $\mathfrak{g}_{\pm 1}$ are automatically abelian, since $[\mathfrak{g}_1, \mathfrak{g}_1]$ would have to lie in the $\text{ad}(-iZ)$ -eigenspace of eigenvalue 2, which is zero;
- ▶ complex conjugation θ_c with respect to $\mathfrak{k}$ is a grade-reversing antilinear involution, and so is $\theta = \theta_c \epsilon$.

This gives a hermitian Jordan pair $V = \mathfrak{g}_1 \cong (\mathfrak{k}_1, J)$, with triple product given by $\{x, y, z\} = [[x, \theta y], z] = -[[x, \theta_c y], z]$ up to a positive normalizing constant. Positivity of V is Condition 2, read through the same identity between $B(x, \theta y)$ and $\text{tr } D(x, y)$ used in Step 1.

Step 3: the constructions are mutually inverse

- ▶ *Starting from V .* Applying Step 2 to the pair produced in Step 1 returns $\mathfrak{g}_1 = V$ with its original product. This is the basic property of the TKK construction [3].
- ▶ *Starting from $(\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1, J)$.* The complexified grading satisfies $\mathfrak{g}_0 = [\mathfrak{g}_1, \mathfrak{g}_{-1}]$ (condition 2), and $\mathfrak{g}_0$ acts faithfully on $\mathfrak{g}_1 \oplus \mathfrak{g}_{-1}$ (Lemma 1). Under these two conditions the TKK algebra of the Jordan pair $(\mathfrak{g}_1, \mathfrak{g}_{-1})$ is canonically isomorphic to $\mathfrak{g}$ [3]. The isomorphism is compatible with θ_c , so it restricts to $\mathfrak{k}$, and it sends Z to Z , hence J to J .
- ▶ *Morphisms.* An isomorphism of hermitian Jordan pairs extends uniquely to an isomorphism of TKK algebras commuting with θ and ϵ , hence with θ_c , via $D(x, y) \mapsto D(fx, fy)$. Conversely, a graded isomorphism commuting with J fixes the grading element, by the uniqueness in Lemma 2, so it preserves each $\mathfrak{g}_j$ and θ_c . These assignments are inverse to each other. □

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