§5.14 COMBINATORIAL QUANTUM THEORY AND QUANTIZED DIRECTIONS *by R. Penrose*

According to conventional quantum theory, angular momentum can take only integral values (measured in units of $1/2\,\hbar$) and the (probabilistic) rules for combining angular moments are of a combinatorial nature. Also, according to quantum theory, a system with zero total angular momentum must be spherically symmetrical. A system must, thus, involve a relatively large angular momentum in order to determine a well-defined direction in space, and we may picture the axis of the angular momentum as giving such a direction. In the limit of large angular momenta, we may therefore expect that the quantum rules for angular momentum will determine the geometry of directions in space. We may imagine these directions to be determined by a number of spinning bodies. The angles between their axes can then be defined in terms of the probabilities that their total angular momenta (i.e. their "spins") will be increased or decreased when, say, an electron is thrown from one body to another. In this way the geometry of directions may be built up, and the problem is then to see whether the geometry so obtained agrees with what we know of the geometry of space and time.

Although there is a standard procedure for the treatment of such a problem (in the non-relativistic case: namely the use of 3-j and 6-j symbols, etc.), it is convenient to make use of an alternative (but equivalent) formalism which can be described very briefly as follows. Consider, first, an n-dimensional Kronecker delta δ_{ab}. Then a set of "isotropic" Cartesian tensors can be built up from this one symbol and scalars by means of the operations of addition, outer multiplication, transposition of indices, and contraction. The only identities satisfied by δ_{ab} which hold independently of n are $\delta_{ab} = \delta_{ba}$ and $\delta_{ab}\,\delta_{bc} = \delta_{ac}$ (summation convention assumed). Also, we have $\delta_{aa} = n$ (> 0) and a certain identity, which depends on n, holds. Now consider an abstract structure which behaves formally exactly like the above set of tensors except that the formal equation $\delta_{aa} = -2$ is imposed, together with the one identity: $\delta_{ab}\,\delta_{cd} + \delta_{ac}\,\delta_{db} + \delta_{ad}\,\delta_{bc} = 0$. The "scalars" of this abstract structure will be taken to be the *rational* numbers. The tensor-like quantities which can be constructed in this way I call *binors*. It can then be shown that a necessary and sufficient condition for a binor $A_{ab\ldots d}$ to vanish is that its *norm* $\|A \ldots\| = A_{ab..d}A_{ab..d}$ (a rational number) should vanish.

To obtain the physical interpretation of the binors, I envisage the following type of situation. Imagine the universe to be represented as a

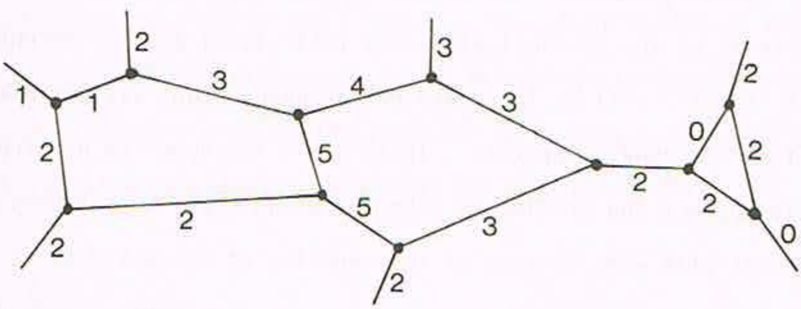

Figure A

network of segments (see Fig. A) where, for simplicity, it will be assumed that each internal segment connects two vertices and each vertex joins just three segment ends. Associated with each segment is a non-negative integer. Each segment is to be thought of as representing the world line of a particle, nucleus, atom, etc. Or, generally, some structure which may be momentarily considered as stationary and isolated from the rest of the universe. The network thus gives a kind of combinatorial space-time picture of the universe. The integer associated with each segment is twice the total intrinsic angular momentum quantum number of the particle (or structure). Hence, for a pi-meson or ground state helium atom this integer is zero; for an electron or proton it is one; for a deuteron it is two; and so on. In order for the proposed calculus to give an accurate description of a part of the universe, it must be supposed that it is possible to neglect effects due to relative velocities of the particles (or structures). In this sense the theory is a non-relativistic one.

Corresponding to each possible such (open) network is associated a binor which is a contracted product of binors described as follows. Each segment numbered 0 is represented by the scalar 1. Each segment numbered 1 is represented by a δ_{ab}. Segments marked 2 by $\delta_{ac,bd} = 1/2! \, (\delta_{ab}\,\delta_{cd} - \delta_{ad}\,\delta_{cb})$, marked 3 by

$$\delta_{abc,\,def} = 1/3! \begin{vmatrix} \delta_{ad} & \delta_{ae} & \delta_{af} \\ \delta_{bd} & \delta_{be} & \delta_{bf} \\ \delta_{cd} & \delta_{ce} & \delta_{cf} \end{vmatrix}$$

and so on. The first group of indices of $\delta_{ab..d,ef..h}$ is to be associated with one end of the segment and the second group with the other end. (This is symmetrical since $\delta_{ab} = \delta_{ba}$ implies $\delta_{ab..d,ef..h} = \delta_{ef..h,ab..d}$.) At each vertex the three relevant groups of indices involved must all be paired off and contracted so that none of these indices remains uncontracted, and no two belonging to the same $\delta_{...,...}$ are contracted together. This implies that the sum of the three integers involved is even, and is at least twice the greatest of them. Then, the result of these contractions will be unique up to sign. The free indices of the resultant binor are then just those corresponding to the free ends of the network.

Consider, now, a situation in which a portion of the universe is represented by a known network X (see Fig. B) having, among its free ends, one numbered m and one numbered n. Suppose the particles or structures represented by these two ends combine together to form a new structure. (See Fig. C). We wish to know, for any allowable p, what is the probability

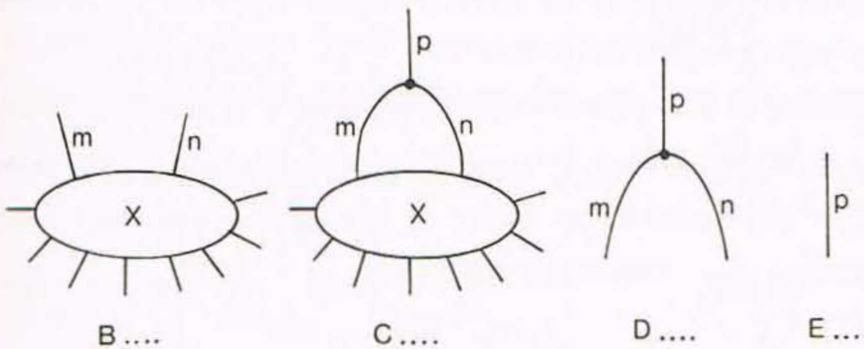

Figure B Figure C Figure D Figure E

that the angular momentum number of this new structure be p. Fig. D is the network representing the combining of these two structures to form the third, but ignoring the rest of the universe; Fig. E is the "network" for the final structure alone. Let B..., C..., D..., E..., be the binors representing the networks of Figs. B, C, D and E, respectively. Then,

$$\textit{required probability} = \frac{\|C...\|}{\|B...\|} \times \frac{\|E...\|}{\|D...\|}$$

(in fact $\| E\ldots \| = (-1)^p(p+1)$ and $\| D\ldots \| = (-1)^{(m+n+p)/2}(\frac{m+n-p}{2})!$ $(\frac{n+p-m}{2})!\ (\frac{p+m-n}{2})!\ (\frac{m+n+p+2}{2})!/m!n!p!)$. As a particular case of this we can deduce the result that the binor corresponding to a network vanishes if and only if the situation is "forbidden" according to the rules of quantum theory.

Consider now the situation of Fig. F involving two bodies with *large* angular momenta M and N. We might define the *angle* between the axes of the

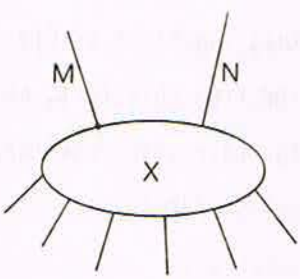

Figure F

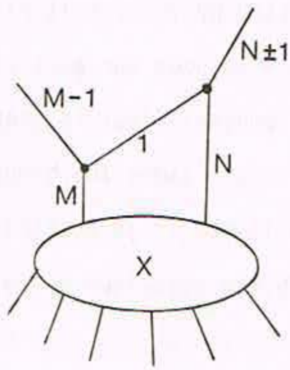

Figure G

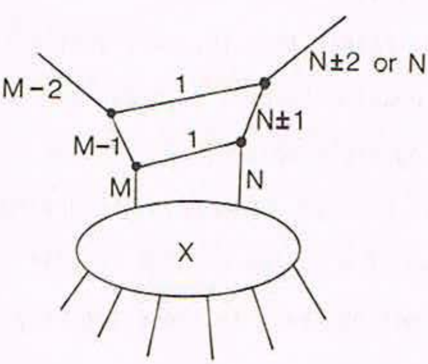

Figure H

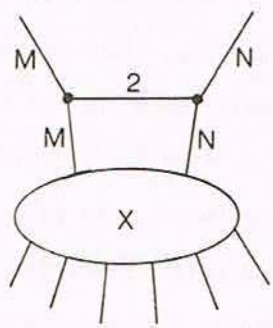

Figure I

bodies in terms of the relative probability (defined in terms of an ensemble of systems with the same Fig. F network) of occurrence of $N+1$ and $N-1$, respectively, in the "experiment" given by Fig. G. However, part of this probability may be due to *ignorance* of the relationship between the bodies. (This may manifest itself in the absence of sufficient connecting links in the known network X.) To eliminate the possibility that part of the probability be due to ignorance we envisage a repetition of the "experiment" as given in Fig. H. If the probability given in the second experiment is essentially unaffected by the result of the first experiment then we may say that the angle θ between the axes of the bodies is well-defined and is determined by this probability. It then turns out that the binor of Fig. I is essentially $1/2 \cos \theta$ times the binor of Fig. F and from this fact, and certain binor identities, it is possible to show that the angles obtained in this way satisfy the same laws as do angles in a three-dimensional Euclidean space.

This is very satisfactory and is perhaps a little surprising in at least two respects. In the first instance, since no complex numbers were used in the binor calculus it is somewhat remarkable that a full array of directions in three-dimensional space has been built up, rather than in, say, just a two-dimensional subspace, since in ordinary quantum theory, in order to build up all the wave functions for all the possible spin directions for an electron, *complex* linear combinations must be used. Secondly, according to standard quantum theory, the wave function of a system of high angular momentum will *not* normally determine a well-defined axis in space contrary to what has apparently been assumed here.

The answer to both these points seems to lie in the fact that the "directions" that emerge in the theory described here are things which are defined by the systems in relation to one another and they will not generally agree with the directions in a *previously* given (and unnecessary!) background space. The space that is obtained here is to be thought of (indeed *must* be thought of) as being the one *determined* by the systems themselves.

It is to be hoped that some modification to the above scheme might enable the effects due to relative velocities of systems to be taken into account so that perhaps a four-dimensional space-time might be constructed. (Time is absent from the above theory even to the extent that the time ordering of events is irrelevant!) Two additional features would have to be involved. The first is that the relativistic addition of angular momenta includes the possibility of multiple pair creation and many of the additional complications implied by relativistic field theory. The second is that relative velocity implies the possibility of a mixing of spin with orbital angular momentum so that the idea of "distance" between the world lines or particles is involved. Particularly because of this second feature, the fully relativistic theory would seem to be of a different order of difficulty from the one treated above.

REFERENCES

Penrose, R. 1971a. In *Combinatorial Mathematics and its Applications*, ed. D.J.A. Welsh. Academic Press.

Penrose, R. 1971b. In *Quantum Theory and Beyond*, ed. T. Bastin. Cambridge University Press.