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On the Nature of Quantum Geometry

As a way of honoring Professor Wheeler on his sixtieth birthday, I propose to take this opportunity to elaborate upon certain somewhat speculative ideas which I have tried to hint at on occasion, concerning the possible nature of a quantized space-time. The reader will not need to be too discerning to recognize some substantial differences between the ideas I am proposing here and those which Professor Wheeler has on many occasions so eloquently and forcefully put forward. Nonetheless, there is little doubt in my own mind as to the very great inspirational influence that Professor Wheeler's own views have had in the development of several of the thoughts which I am expressing here.

To begin with, let me make clear that I do not necessarily mean by "quantized space-time" something which could be obtained by applying standard (or even non-standard) techniques of quantization to Einstein's general theory of relativity. What I wish to say has its roots in something which is really more primitive than either quantum theory or relativity as such. This is the question of the fundamental role played by the mathematical concept of *continuum* in virtually the whole of accepted present-day physical theory. Not only does the continuum occupy a basic position in our mathematical models of *space* and *time* (with the concomitant implication of a continuous nature for many related physical concepts such as velocity, energy, momentum, temperature, etc.), but so also does present-day *quantum theory* rest crucially on a continuum concept, namely on the two-dimensional complex continuum of probability amplitudes, this being the continuum which also occurs in the superposition law.

Let me say at the outset that I am not happy with this state of affairs in physical theory. The mathematical continuum has always seemed to me to contain many features which are really very foreign to physics. This point has been argued forcefully, particularly by Schrödinger [1] and also by a number of other physicists and philosophers [2]. If one is to accept the physical reality of the continuum, then one must accept that there are as

[0]This paper originally appeared in *Magic Without Magic*, edited by J. Klauder, Freeman, San Francisco, 1972, pp. 333–354.

many points in a volume of diameter 10^{-13} cm or 10^{-33} cm or 10^{-1000} cm as there are in the entire universe. Indeed, one must accept the existence of more points than there are rational numbers between any two points in space no matter how close together they may be. (And we have seen that quantum theory cannot really eliminate this problem, since it brings in its own complex continuum.) It seems clear that such "points" have actually very little to do with physical reality. Nevertheless, their postulated existence is virtually essential to contemporary physical theory. The concepts of open and closed sets, for example, depend vitally on this continuum of points and are essentially meaningless in strictly physical terms, but such concepts can be used to great effect when mathematical *theory* is applied to physics (For example, in proving global "singularity" theorems about space-times, etc. [3, 4, 5]).

I think it must be the case that the all-pervading use of the continuum in physics stems from its *mathematical utility* rather than from any essential physical reality that it may possess. However, it is not even quite clear that such use of the continuum is not, to some extent a *historical accident*. For although the essential mathematical ideas can be traced back to Eudoxus (4th century B.C.), it was not until sometime after Newton and Leibniz invented the calculus that it was felt to be necessary to formalize and make completely rigorous the mathematical continuum concept. But there are other "nonstandard" continua [6, 7] different from the one that has now become conventional, which could equally well have been adopted in order to make the calculus rigorous. In these nonstandard continua, "infinitesimal" and "infinite" elements are introduced and are treated as being just as "real" as the "real numbers" of conventional analysis. In fact, it has occasionally been argued that nonstandard analysis might really have been a more natural development of the ideas of Newton, Leibniz, and Euler than the actual analysis which Cauchy, Weierstrass, and others finally formalized. (In addition, nonstandard analysis allows one to define such concepts as Dirac delta-functions so that they become effectively "ordinary' functions [8].) If the history of mathematics had developed differently, then we might, by now, have formed a very different view from the one now prevalent of the nature of space and time, and of many other physical concepts.

I do not want to imply here that non-standard analysis ought to be employed in physical theory. I wish merely to point out the lack of firm foundation for assigning any physical reality to the *conventional* continuum concept. My own view is that ultimately physical laws should find their most natural expression in terms of essentially *combinatorial* principles, that is to say, in terms of finite processes such as counting or other basically simple manipulative procedures. Thus, in accordance with such a view, should emerge some form of discrete or combinatorial space-time. I do not mean that necessarily we should arrive at a space-time containing a discrete set of points, such as the lattice space-time of Schild [9], or some discrete causal space [10] or, for example, some structure based on Ahmavaara's large finite field [11]. I would expect, rather, that the concept of a space-time composed of *points* should *cease* to be an appropriate one—except in some kind of limiting sense. But if points are not to be the basic elements of the discrete space-time, then how are we to decide what these basic elements should in fact be?

It is my view that an essential insight into the nature of the appropriate combinatorial structure should actually emerge once the interrelations between quantum physics with the present-day view of space-time are fully appreciated.[1] But was I not arguing that quantum theory is of no value for eliminating the continuum, since it entails the use of the complex continuum right at the outset? This is certainly true as it stands. However, I think it is important to make the distinction here between quantum *theory* (which requires a complex continuum concept) and quantum *physics* (according to which certain physical quantities are recognized as being actually discrete, while having previously been taken to be continuous). My idea is to try to "reformulate" physical laws so that they may be expressed entirely in terms of quantities which are discrete according to quantum physics. These "reformulated" laws would, hopefully, be expressible entirely in combinatorial terms, even though they would be essentially re-expressing the content of conventional quantum theory, of space-time theory and, perhaps, of other aspects of physics as well. Thus, the quantum theory and space-time theory would be expected to arise together, out of some more primitive combinatorial theory.

In order to be more explicit as to the sort of "reformulation" that I have in mind, I should describe a certain model which I have referred to elsewhere [13, 14] which may be thought of as a prototype for this type of theory. The basic idea of the model is to take the concept of *total angular*

[1] I should remark here on the interesting program by D. Finkelstein [12] according to which quantum theory is to be built into the mathematical description of space-time even at basic level of set theory. To be in accordance with the view point I am expressing here, however, such a "quantization of set theory" would have to be accompanied by a suitable "combinatorialization of quantum theory."

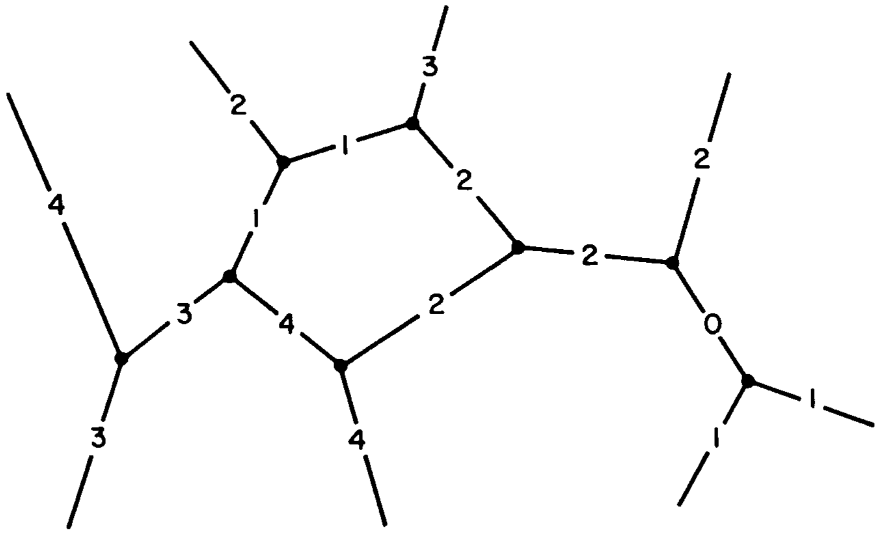

FIGURE 1.
An example of a spin-network. The numerals denote total spin values in units of $\frac{1}{2}\hbar$.

momentum and to regard this as the primary physical quantity; then the quantum mechanical rules for combining (nonrelativistic) total angular momenta can be re-expressed in purely combinatorial terms; finally, the concept of *space direction* is to be extracted and shown to agree, in the limit of large angular momenta, with the ordinary geometry of directions in a Euclidean three-dimensional space. The reason for fastening attention on the concept of total angular momentum in the first instance is that it seems to be more or less uniquely singled out by a number of criteria. For we have to choose something, which according to quantum physics, is *discrete*, preferably taking numerical values which are integer multiples of some basic unit (in this case $\frac{1}{2}\hbar$). Since we are interested in reconstructing a form of space out of this discrete physical quantity, we need something which is intimately related to spatial and directional properties. This suggests that we should use angular momentum, rather than other possible quantum numbers. Finally, since we wish to *construct* space rather than depend upon any assumption of pre-existing spatial directions, we must use *total* angular momentum (j-values) rather than the angular momentum in some preassigned direction (m-values).

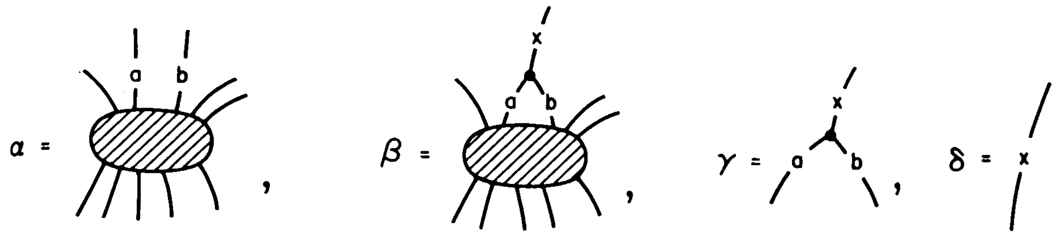

FIGURE 2.
The a- and the b-units come together to form an x-unit. With spin-networks labelled as above, the probability of the value x is given by Equation (1).

The model operates with structures I call *spin-networks.* An example is illustrated in Figure 1. The line segments are called *units* and are to be loosely interpreted as "world lines" of particles, or of simple systems which may be regarded as momentarily isolated from the rest of the universe. Each of these particles or simple systems has to be stationary (not really moving relative to the others, in this model) and possess a well-defined total angular momentum. The integer labelling each unit is its *spin-number.* This measures the total angular momentum of the unit, as a multiple of $\frac{1}{2}\hbar$. Exactly three units must come together at each (internal) vertex. Depending upon which way we choose to regard time as progressing in the diagram (but normally we choose upward), we may interpret the meanings of the vertices in different ways. We may think of a vertex as representing the combining of two units together to make a third, or as the splitting of a single unit into two separate units. A unit which is not terminated at both ends by a vertex is called an *end-unit.*

Given a spin-network, we may calculate, using *purely combinatorial means*, a certain nonnegative integer, called the norm of the spin network. I have described this calculational procedure elsewhere [14], and I do not propose to go into the details here. Suffice it to say that this combinatorially defined norm may be used to obtain the probability that the spin number x is the result, when two end-units of a given spin-network α, come together to form a new unit. With spin-networks labelled as in Figure 2, the formula turns out to be:

$$\begin{aligned}\text{probability of } x &= \frac{\text{norm}\beta\,\text{norm}\delta}{\text{norm}\alpha\,\text{norm}\gamma} \qquad (1)\\ &= \frac{(x+1)\,\{\frac{1}{2}(a+b+x)+1\}!\,\text{norm}\beta}{\{\frac{1}{2}(a+b-x)\}!\,\{\frac{1}{2}(b+x-a)\}!\,\{\frac{1}{2}(x+a-b)\}!\,\text{norm}\alpha}\end{aligned}$$

which is a rational number. This result is obtained from conventional nonrelativistic quantum mechanics, but things have to be reformulated considerably for use to be able to state the result in a reasonably simple purely combinatorial form. I should mention also that a spin-network represents a physical process which is forbidden (that is, zero probability) by the rules of nonrelativistic quantum mechanics if and only if its norm *vanishes*.

The proposal now is to extract a concept of space from this scheme—or, rather, the simpler concept of *directions* in space from this particular model. The basic idea stems from the quantum mechanical fact that a system with zero total angular momentum must be spherically symmetrical, and so cannot be used to define a direction in space; a system of total spin $\frac{1}{2}\hbar$ is not much better, since it "sees" but *two* alternative "directions" available to it as regards its state of spin; for spin $\hbar$ there are but three alternative "directions," and so on. Only for a system involving a comparatively large total spin value, can we expect that it could define a direction in any well-defined way. And once we have a large spin, we can envisage a fairly well-defined rotation axis as a convenient means of defining us a direction in space.

Next, *angles* between rotation axes can be defined in terms of certain simple "experiments." A spin $\frac{1}{2}\hbar$ unit is detached from some unit of large spin (called a *large unit*) and then reattached to another large unit. Let us suppose that the spin of the first large unit is reduced by $\frac{1}{2}\hbar$. Then the spin of the second large may increased by $\frac{1}{2}\hbar$—with probability p, say—or it may be reduced by $\frac{1}{2}\hbar$—with probability $1-p$. If there is a well-defined angle θ between the spin axes of the two units, then in accordance with standard quantum mechanics, we have (in the limit of large spin)

$$p = \cos^2 \frac{1}{2}\theta \qquad (2)$$

We can use (2) as the *definition* of the angle between two large units, provided that the two units *have*, in some appropriate sense, a well-defined angle between them. (Without such a proviso this definition would not lead to a reasonable geometry.)

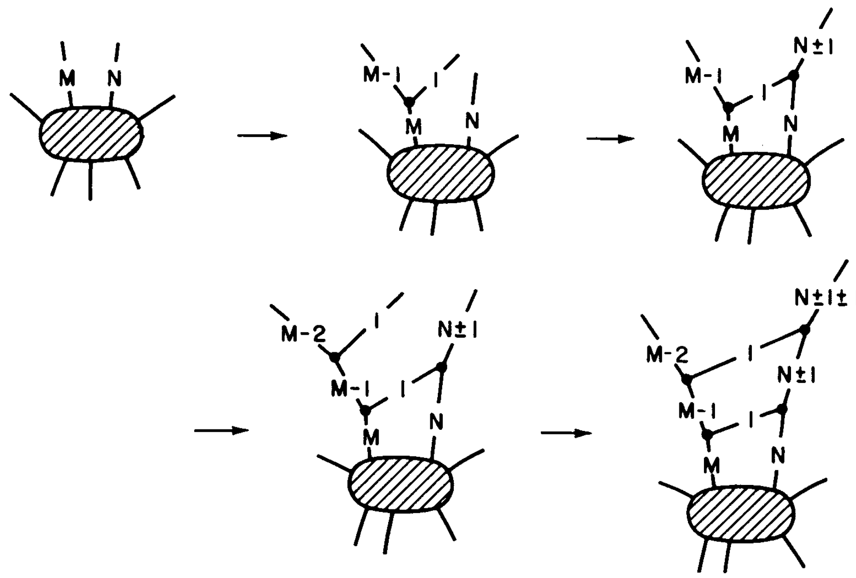

FIGURE 3.
An angle measuring experiment, repeated in order to eliminate the "ignorance factor."

Before considering the question as to when the angle between the spins of two large units can be considered as "well defined," I should first be a little more precise as to the interpretations of the probabilities arising here. These probabilities are always calculated starting from a *given* spin-network α (refer to Figure 2), which is supposed to represent some *known portion* of the universe. The spin-network α may occur again at various other places in the universe. At some places it may be part of a more extended spin-network β; at other places it may be part of β' (which is to differ from β only in that one of the spin-numbers is different: see x in Figure 2); at other places it may be part of other spin-networks. The calculated probability is then supposed to give us the relative frequency of β to β' (given α) in the universe.[2] Now, it may be that α contains two large end-units, M and N,

[2]This point of view serves to make the probability concept tolerably precise, for purposes of the model. It may possibly leave something to be desired for a physically more realistic model. For example, whether or not the units labelled a and b in Figure 2 could actually come together to form another unit might depend on whether or not a suitable

but without much connecting network between them. This is the situation where we "know" rather little about the relationship between M and N, so the probability p that we calculate in the above experiment cannot really be thought of as defining an "angle" according to (2), but is, rather, partly a measure of our ignorance of the relation between M and N. For the angle between M and N to be well defined, we should require this ignorance factor to be very small.

But how are we to decide whether a probability arises partly from ignorance or entirely from angle? Consider the above experiment repeated in the way depicted in Figure 3. The second experiment gives a probability p', which could be affected by the result of the first experiment. If p' is *not* significantly affected by the result of the first experiment (and $p' \doteqdot p$) then we can say that the ignorance factor is small and the angle between M and N is well defined. (If the angle between M and N is not well defined in this sense, then we can generally make the angle better defined by carrying out a fairly large number of similar experiments in succession. The probability normally settles down to some fixed value.) Now, the following theorem can be proved: *if a spin-network has a number of large end units such that the angle between any two of them is well defined in the above sense, then these angles can be consistently interpreted as angles between directions in a Euclidean three-dimensional space.*

This result is very satisfactory, but in a way it is perhaps too satisfactory! It involves a peculiar feature which I want to emphasize particularly in this article. Suppose we had set up the normal quantum mechanical formalism for the description of the situation; that is, by giving states in terms of wave functions involving, say, coordinates r, θ, and ϕ, these being ordinary spherical polar coordinates for a preassigned background Euclidean space. Can we identify the directions (of spin-axes) that we end up with *as* the directions in this background Euclidean space? No, we certainly cannot! The condition, given above, that angles between spin-axes of units be well defined, by no means ensures that the spin-axes themselves correspond to well-defined directions. (For example, a state with $m = 0$ does not define a good background direction for its spin-axis, no matter how large j is.) I propose to take the attitude, however, that in this model it is the geometry of "spin-axes" of the large units which is the *real* geometry. The background space,

"particle" with spin-number x really existed. This would influence the probabilities in a way not taken into consideration here.

with its spherical polar r, θ, and ϕ, has no actual physical meaning and is introduced merely as a convenience for calculations (that is, if one chooses to use the conventional wave-function description, rather than a combinatorial procedure). Thus, *the system itself defines the geometry* and the background space is really an irrelevance.

It appears that we may think of the relation between the background space and the real space as being given by a *unitary transformation in Hilbert space*. Thus, states which give well-defined directions in one space might correspond to linear superpositions of states having well-defined directions in the other. This idea is very close to a suggestion by Aharonov [15], according to which one can pass from one concept of geometry to another by applying a Hilbert space unitary transformation. In this view, an electron moving through two slits, for example, does not "feel itself to be split" since the geometry "felt" by the electron is not quite the same as that of the slits. According to the electron's geometry (which would be related to the background geometry by a unitary transformation) the electron would remain intact while the background space would be "split" in a certain sense.

Also it is not actually necessary that the "real" geometry should fit together globally to give a space of the same kind as the background space. Let me illustrate what I have in mind by reference to the model. Consider a spin-network α which consists of two portions, λ and μ, each of which has a number of large end-units with a well-defined angle between them (Figure 4). Thus, each of λ and μ defines its own Euclidean geometry of directions in a well-defined way. Now suppose that the connections between λ and μ in α are not sufficient (or not of the right type) to ensure that the geometry defined by λ and that defined by μ are consistent with one another (that is, the angle between a large unit of λ and a large unit of μ need *not* be well defined). Then the large units of λ need not correspond to states of spin with well-defined rotation axes according to the μ geometry, and *vice versa*. Thus, we may expect that the relation between the λ-geometry and the μ-geometry should occur via something like a unitary transformation in Hilbert space.

The circumstance whereby each part of a structure can define its own local Euclidean geometry, but where these local geometries need not quite patch together to make a global Euclidean geometry, is reminiscent of the situation occurring with Riemannian geometry, as used in general relativity. In fact, it is one of my main contentions in this article that the *curvature of space-time* may indeed have its origins in an effect of the kind just described. But how are we to relate, in any meaningful way, the effects of space-time

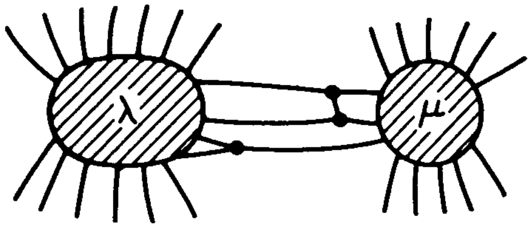

FIGURE 4.
The spin-networks λ and μ each define their own geometry, but owing to inadequate connections between λ and μ, the two geometries are not consistent with one another.

curvature to the effects of a unitary Hilbert space transformation? On a classical level, we would expect such unitary transformations to show up as canonical transformations in suitable classical variables. Thus, it must be possible to relate space-time curvature to canonical transformations in a suitable manner. I shall show how this can be done, first, in a way which is closely related to (but not quite the same as) the way which I believe to have the most significance. I shall describe the way I prefer afterward.

Let M be a space-time manifold[3] and let C be its cotangent bundle [16, 17]. Thus, each point of C represents a point x of M together with some covariant vector p_a (referred to as a *momentum* vector) *at* the point x. The eight-dimensional manifold C has a natural symplectic (or "Hamiltonian") structure defined by the two-form [17, 18]

$$\omega = dp_a \wedge dx^a \ . \tag{3}$$

Of interest also is the naturally defined one-form

$$\varphi = p_a \, dx^a \tag{4}$$

of which ω is the exterior derivative: $\omega = d\varphi$. The *inverse* of the tensor (on

[3] A four-dimensional, pseudo-Riemannian $(+ - - -)$, time-oriented Hausdorff manifold with a C^2-metric.

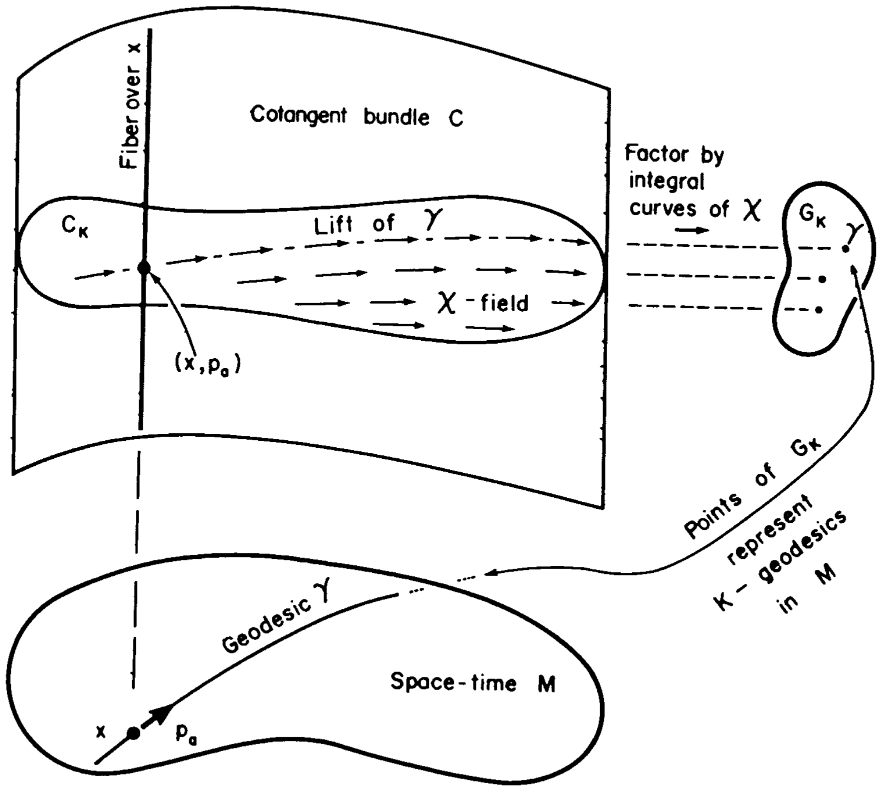

FIGURE 5.
The space G_K of k-geodesics in M inherits a symplectic structure from that of the cotangent bundle C.

C) defining ω is the bivector which defines the *Poisson bracket*:

$$\{\,,\,\} = \frac{\partial}{\partial p_i} \otimes \frac{\partial}{\partial x^i} - \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial p_i}\,. \tag{5}$$

Any smooth transformation of C to itself which preserves the symplectic structure ω—or equivalently, which preserves the Poisson bracket (5)—will be a *canonical transformation.* However, these are not quite the transformations that concern us here. We must first construct a reduced phase space by a standard procedure [19]. Let C_K be the seven-dimensional submanifold of C

for which the squared "Hamiltonian "

$$H \equiv g^{ab}\, p_a\, p_b \tag{6}$$

takes the constant value $H = K$. The Poisson bracket now defines a vector field χ (or "flow") on C_K by

$$\chi(f) = \{H, f\} \tag{7}$$

(f being an arbitrary smooth function on C). The integral curves of χ are the "lifts" of geodesics [19, 20] in the space-time M (see Figure 5), the tangent vector to the geodesics being $p^a = g^{ab}\, p_b$. Now, the space C_K, being odd-dimensional, does not possess a symplectic structure. (The form ω induces a degenerate two-form called a pre-symplectic structure on C_K). However, we can (locally, at least) factor out C_K by the integral curves of χ to obtain a symplectic six-dimensional manifold G_K. Each *point* of G_K represents a geodesic γ in M whose (parallelly propagated) tangent vector p^a has squared length $p^a\, p_a = K$. Call such a geodesic a K-geodesic.[4] The sympletic structure on G_K is that induced by the two-form ω.

It is worthwhile to examine the geometric meaning [19, 20, 21] of this symplectic structure on G_K, and also the meaning of φ in relation to G_K. In fact, both φ and ω represent integrals of the Jacobi equation for geodesics. That is to say, they represent properties of neighboring geodesics which can be calculated at any one point but which are actually constant along the geodesics. Let us consider φ, first, as given by (4). We may think of p^a as the tangent vector to some K-geodesic γ and dx^a as a "connecting vector" which connects a points x on γ to a corresponding neighboring[5] point x' on a neighboring K-geodesic γ' to γ (see Figure 6). The fact $p_a\, dx^a$ is constant[6] along γ is a well-known property of Lie derivatives [16]. Now consider the

[4] We can, of course, restrict attention to the three values $K = 1, 0, -1$ if desired, since all other cases are related to these by scalings.

[5] For the purposes of such geometrical descriptions it is convenient to adopt an "old-fashioned" attitude to "dx^a" and to talk about "neighboring" points and curves. There is actually no real conflict between the "old-fashioned" and "contemporary" viewpoints here. An equation such as $Q_a\, dx^a = 0$ can be interpreted either as "the infinitesimal vector dx^a connecting two neighboring points x and x' is orthogonal to Q_a," or as "the one-form $Q_a\, dx^a$ maps to zero the vector at x with which we are concerned."

[6] If $K = 0$ this does *not* require x and x' to occur at corresponding parameter values on γ and γ'. Thus, the one-form φ carries over to G_K if $K = 0$.

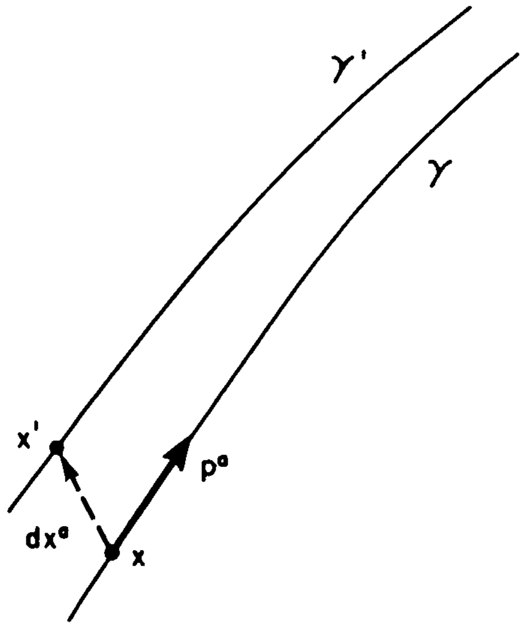

FIGURE 6.
The quantity $\varphi = p_a \, dx^a$ is constant along γ (where if $K \neq 0$, x' has the same parameter value on γ' as x has on γ).

interpretation of ω. In "old-fashioned" notation we write

$$\omega = \frac{1}{2} \left(dp_a \, \delta x^a - \delta p_a \, dx^a \right) \tag{8}$$

where dx^a is as before (but where parameter values need not now correspond) and where δx^a is viewed as a "connecting vector" which connects the point x on γ to any neighboring point x'' on a *third* neighboring geodesic γ''. The tangent vectors to γ' and γ'' are to be, respectively, $p^a + dp^a$ and $p^a + \delta p^a$ (see Figure 7). The fact that (8) is constant along γ is sometimes referred to as the *Lagrange identity* [16]. It is a consequence of the interchange symmetry of the Riemann tensor: $R_{abcd} = R_{cdab}$. We may think of ω as defining a measure of the *rotation* of three neighboring geodesics about one another.

Let us return to the question of the relation between space-time curvature and canonical transformations. The canonical transformations are actually to

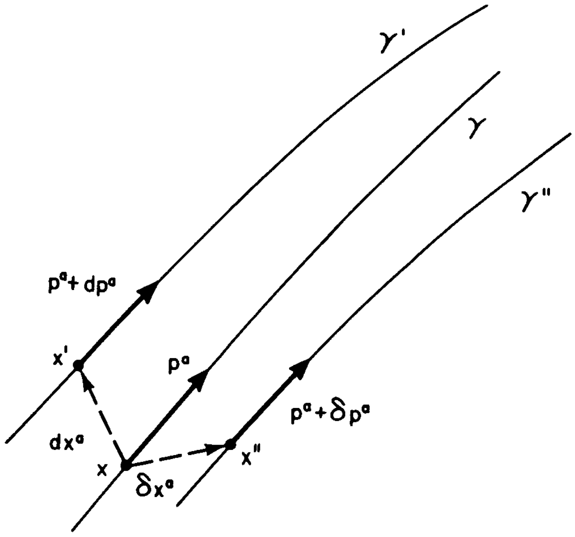

FIGURE 7.
The geometric meaning of the symplectic structure of G_K: the rotation measure $\frac{1}{2}\omega = (dp_a\, \delta x^a - \delta p_a\, dx^a)$ is constant along γ (Lagrange identity).

be applied to the *space of K-geodesics* G_K (for some fixed K), rather than to the space-time M. (Thus, we must expect that if a space is to be constructed out of some combinatorial principles in acordance with the general idea of the model described earlier, then this space should, in the first instance, be more like G_K than directly like M.) We can imagine that it should be possible to reconstruct the geometry of M once sufficient structure of the space G, is known, for we may associate each point x of M with the system $G_K(x)$ of K-geodesics through x. Each $G_K(x)$ may be viewed as a *three-dimensional submanifold* of G_K, these submanifolds forming a four-dimensional family within G_K. If we know this family of submanifolds, then we should be able to reconstruct the space-time M from G_K.

Let us consider how this can be done in the special case when M is Minkowski space-time. The flat geometry of M assigns considerably more structure to G_K than just the symplectic geometry of G_K. I do not want to go into much detail here, but the essential point is that the *shear* of a congruence of geodesics in M can, owing to the flatness of M, be defined in a way which refers to the geodesics in their entirety and does not depend on a choice of a particular point on the geodesic. It follows that the shear concept can be interpreted in terms of some local structure on G_K; in fact, in terms of some tensor field σ on G_K. One way of achieving this is by means of a covariant tensor on G_K, of valence four, which, using a notation similar to that of (8) can be expressed as:

$$\sigma = dx^a\, dp_a\, \delta p^b\, \delta p_b - \delta x^a\, \delta p_a\, dp^b\, dp_b\ . \tag{9}$$

In order to fix ideas, let us suppose that σ is, in fact, defined by equation (9), even though this is not completely satisfactory for later purposes. The form ω already (satisfactorily) achieves a corresponding interpretation with regard to the rotation of a congruence of geodesics (but, in this case, independent of the flatness).

The importance of being able to interpret both the shear and rotation on G_K is that σ and ω may now be used to locate the $G(x)$ submanifolds in G_K and hence to reconstruct space-time points. For if x is a point in M then the congruence of K-geodesics through x (now straight lines) has the property that its shear and rotation both *vanish*. (I am now supposing that $K \neq 0$, in order to keep the discussion simple.) Conversely, *any* congruence with vanishing shear and rotation (and nonvanishing divergence) is a $G_K(x)$ system and hence defines a unique point x in M.

But how does the effect of *curvature* in M show up in G_K? Let us first consider a very idealized situation. (This is not really essential, but it serves to clarify matters.) Suppose that M contains two (geodesically convex) regions F_1, and F_2, of *flat* space-time and suppose that some open set E in G_K, represents K-geodesics passing through the interiors of *both* F_1, and F_2. The flat geometry of F_1, assigns some "shear structure" σ_1, to E; similarly, the flat geometry of F_2, assigns some "shear structure" σ_2 to E. But, in general, we shall have $\sigma_1 \neq \sigma_2$. This is because a congruence of geodesics shear-free in F_1, will normally begin to pickup shear as soon as it leaves F_1, and enters a region of curved space-time. Only in very exceptional circumstances would all the shear exactly cancel out by the time the geodesics finally enter the region F_2.

Now the K-geodesics through x, where x is some point in the interior of F_1, will have the property that while within F_1, they constitute a shear-free, rotation-free congruence. This fact could be recognized in E, by reference to σ_1, and ω. But there will be many other shear-free, rotation-free congruences of K-geodesic segments in F_1. Some of these will appear to be converging

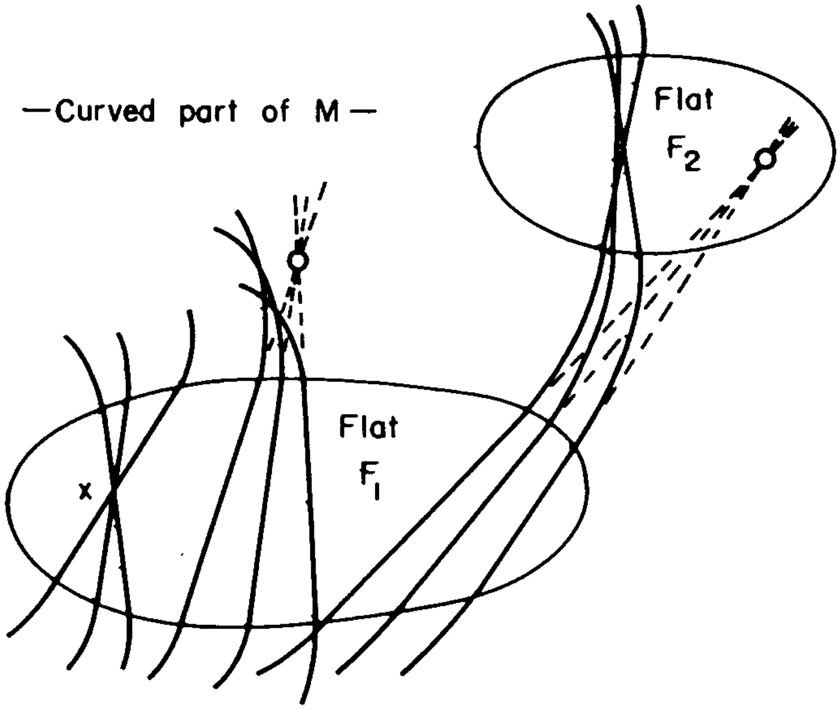

FIGURE 8.
Points in F_1 may be located in terms of shear-free rotation-free congruences of geodesics, but on this basis, points outside F_1 would appear to be "fuzzed out."

on points which lie just *outside* the F_1 region (see Figure 8). As soon as the geodesics enter the curved region they will begin to pick up shear, so they will not normally converge cleanly on any actual point of M. Correspondingly, a congruence of K-geodesics which actually *does* converge on a point y just outside F_1, will normally possess a certain amount of shear while *in* F_1.

Thus, we see that the "shear structure" σ_1 of E can be used to help locate the points of F_1, but that if we are not careful we will also "locate" things which appear to be points outside F_1, but which are actually not points of M at all. Similarly, σ_2 helps to locate the points of F_2, but points which are not in F_2 are incorrectly "located" by this means. In particular, σ_1, locates points in F_1, correctly and points in F_2 incorrectly, while with σ_2

the situation is just the reverse. In a sense, the points of F_2 appear to be "fuzzed out" when viewed from F_1 (that is, using σ_1) and *vice versa.* This "fuzzing out" may be validly thought of as being the result of a canonical transformation of E. Such a canonical transformation would preserve ω (the symplectic structure must be preserved by definition of "canonical") and could transform σ_1 to σ_2. To achieve such a transformation explicitly, we can set up ordinary Minkowski coordinates in F_1. We can then set up a related canonical coordinate system in a standard way, for the relevant part of the cotangent bundle C, and thence arrive at canonical coordinates for $E \subset G_K$. Similarly, ordinary Minkowski coordinates in F_2 will give rise to different canonical coordinates for E. These two coordinate systems for E will be related by a canonical transformation, the components of σ_1 with respect to the first being the same as the components of σ_2, with respect to the second.

Let us pass, now, to the case of a general curved space-time M. for simplicity, suppose that $K \geq 0$ (so that the geodesics are all time-like or all null) and also that M is *globally hyperbolic* [22, 23]. Then M admits slicings by certain space-like hypersurfaces, with the property that any one of them could be used globally as a Cauchy hypersurface for M [24]. Each hypersurface S will then intersect each K-geodesic once and once only. Thus, given S, we can examine the extent of shearing of a congruence of K-geodesics at the intersections of these geodesics with S. This gives us a definition of "shear structure" σ_s, for G_K, *relative* to the hypersurface S. If we wish to use (9) for the definition of σ in flat space-time, then we can still use (9) in *curved* space-time to define σ_s by choosing dx^a and δx^a to connect neighboring points lying *within* the hypersurface S (so dx^a and δx^a are tangent to S).

The six-dimensional manifold G_K will now possess the following structure. In the first place, it will have a permanent symplectic structure defined by the two-form ω (and, if $K = 0$, the one-form Φ with $\omega = d\Phi$). In the second place, it will have a shifting "shear structure" σ_s, which depends on the location of the hypersurface S in M. If S is moved over a region of *flat* space-time in M, then σ_s will *not change.* If S is moved over curved regions in M, on the other hand, then σ_s will shift, its rate of change being actually governed by the Riemann tensor in M at the points over which S moves. It is σ_s (in relation to ω) which carries information specifying the actual *geometry* of M. The form ω by itself conveys no information as to the metric structure of the space-time. Given σ_s the points of S can be realized in G_K as three-dimensional submanifolds $G_K(x)$ whose tangent vectors annihilate σ_s and

ω (the shear-free and rotation-free conditions). But the points of M which lie *off* will, due to the curvature in M, generally appear to be "fuzzed out" from the point of view of the σ_s structure, in the sense that the $G_K(x)$ cannot now be recognized as shear-free, with respect to σ_s. The farther away the points are from S, the greater, in general, will be this "fuzzing out." Since ω does *not* shift on G_K, we can think of this "fuzzing out" as a *canonical* transformation effect.

I want, now, to modify this picture somewhat, so as to bring it more into line with some other features which I feel should be involved, in connection with a quantized space-time. As a first step let me specialize to the case $K = 0$. I have several reasons for wishing to do this. One of these is that it turns out that the "shear structure" of G_K, can now be expressed in a particularly significant form, namely as a *complex analytic* structure for a closely related eight-dimensional manifold T (strictly speaking it is an "almost complex" structure [25] in the most general case). The interplay between complex analytic structure (that is, "analyticity") and unitary structure (which, on the classical level becomes canonical; that is, symplectic structure) seems to play an important role in modern theory of particles [26, 27], so there could be some significance in exhibiting such an interplay also at the level of a space-time analysis.

A second reason for desiring the specialization to $K = 0$ is that *null* geodesics are *conformally invariant*; that is to say, they depend only on the light cone structure of M and not on its metric. There are various reasons for believing that conformal invariance may actually have some basic role to play in physics, and that conformally invariant formalisms could have special significance as "background formalisms" for physical theory. The notion of "causality" inasmuch as this refers to the location of the light cones in space-time (and, therefore, to conformal structure) would seem to have a particular physical importance, more so than the actual space-time metric. In addition, all zero rest-mass free fields are, or can be made, conformally invariant [28, 29, 30, 31, 32, 33]. This applies, in particular, to gravitation, if we interpret "free" to mean that we are considering linear theory, or at least the propagation of curvature in empty space (Bianchi identities) [33] rather than the nonlinear response of curvature (or other fields) *to* curvature (Ricci identities). A number of *interactions* are also conformally invariant (for example, electromagnetic interactions or the non-linear self-coupling in the "ϕ^4 scalar theory") but it is sometimes a little difficult to separate the conformal invariance of the "pure" interaction from the conformally noninvariant effects of the presence of rest-mass. It is even conceivable that *all* conformal invariance breaking is connected in some important way with the presence of mass. It is *rest-mass* which (apparently) is responsible for defining the scale of phenomena and hence the metric of space-time [5, 34]; it provides us with the most obvious obstacle to a belief in a universal validity of conformal invariance in nature. The other most obvious obstacle in which conformal invariance is broken is in gravitational interactions: Again it is mass which is involved in an essential (but now different) way. Indeed, it is tempting to believe that there may be a common origin to these two aspects of conformal invariance breaking. In any case, to think of basic physical processes in terms of either conformal invariance, or the breaking of conformal invariance, seems to be a fruitful point of view. To this end, it is very useful to employ a formalism which makes this conformal invariance manifest wherever it *is* present.

The particular choice of formalism that I have in mind, namely *twistor theory* [21, 35, 36], is motivated partly by considerations of this kind. Twistors are, in fact, the spinors [37, 38] for the fifteen parameter conformal group [39], which is the space-time symmetry group for the zero rest-mass free-field equations (including linearized gravitation). Any finite-dimensional representation of the conformal group is equivalent to a twistor representation. Also, infinite-dimensional representations can be conveniently described in terms of functions of twistors. Thus, twistors play a role with regard to the conformal group analogous to the role played by two-component spinors with regard to the Lorentz group [40] or rotation group. Now, the combinatorial model that I was diffussing earlier for the description of nonrelativistic angular momentum was based on the representations of the rotation group. In fact, the particular combinatorial rules that I had in mind for evaluating the norm of a spin-network [14] were derived directly from the two-component spinor algebra. In a corresponding way we might expect to be able to derive a combinatorial calculus based on twistors, where the conformal group now takes over the role previously enjoyed by the rotation group.

It was, in fact, the hope of generalizing the spin-network model to make it more realistic which provided other parts of the motivation for the original introduction of twistor theory. The two most obvious respects in which the spin-network model is unrealistic are that it is a *nonrelativistic* scheme and that there is no provision for the mixing of spin with *orbital* angular momentum. These two inadequacies are related to one another; indeed, they are both aspects of the fact that any relative velocity between the different units has been neglected. But it was always clear that the removal of these inadequacies in the model would involve considerably more than just the simple substitution of one group by another. Once the scheme is made relativistic then we must encounter some of the difficulties involved in passing from a quantum theory to a quantum field theory; once orbital angular momentum is brought in, then we have to contend not only with *directions* in space-time) and *angles*, but also with *locations* and *distances*.

Accepting that there must be some essential new features arising, the twistors do seem to provide the right kind of generalization of the nonrelativistic, $SU(2)$, two-component spinors which formed the basis for the spin-network theory. Twistors are genuinely spinorial objects and so can still handle half-odd spin values, they are completely relativistic (in the sense of special relativity), and they adequately mix together the concepts of angular and translational displacements (so that spin and orbital angular momenta will combine together in the appropriate way). The use of the conformal group—and hence the locally isomorphic "twistor group" $SU(2,2)$—rather than the Poincaré group, arises partly from technical mathematical reasons, connected with the semi-simplicity of $SU(2,2)$, and partly from reasons mentioned above concerning physical importance of conformal invariance. (In any case, Poincaré invariance is easily extracted from a framework designed to handle conformal invariance.) The main essentially *new* feature which is involved arises from the fact that the conformal group, and $SU(2,2)$, possess *infinite-dimensional* irreducible representations. Most particularly, the zero rest-mass free fields provide such representations. This implies that the twistor algebra must be employed more subtly than in the direct way in which the two-component spinor algebra generated the spin-network theory. Thus, it may prove to be difficult to reduce the resulting twistor calculations (which at present involve contour integration) to a set of purely combinatorial rules. There seems to be nothing *in principle* against the possibility of doing this, however.

But how is twistor theory to be reconciled with *general* relativity? The conformal group refers only to symmetries of conformally *flat* space-time. Gravitation on the other hand implies the existence of conformal *curvature*. My point of view with regard to this question is really the one that I have been trying to stress throughout this article. Imagine that we have been able to develop the twistor theory to the point at which calculation can be expressed in terms of certain combinatorial rules. By analogy with the spin-network theory, we might expect that these rules would enable us to calculate the probability of occurrence of certain types of (graphically defined) situations in the universe. From these probabilities we should be able to extract geometrical concepts which would emerge as "well defined" under suitable circumstances. This would then lead to a concept of *local geometry* which would be of a *Minkowskian* character (assuming that conformal invariance breaking has been adequately incorporated into the theory—otherwise we should presumably only obtain a local conformal geometry). The Minkowski geometries that we extract "locally" might not be consistent with one another over the whole universe. The concept of space-time "point" that we extract in one region would then appear to be "fuzzed out" from the point of view of the geometry defined in some other region.[7] This "fuzzing out" would be of the nature of that obtainable by a unitary transformation in Hilbert space. On the classical level, this would appear as the result of a canonical transformation applied to a suitable space (namely twistor *space*—closely related to G_K with $K = 0$). From this space we do the best we can to extract a concept of space-time "point" which has some form of universal validity, but we find that having done this, the space-time that we finally construct is no longer conformally flat, the conformal curvature being directly relatable to this "fuzzing out" of points as "viewed " from distant regions.

To a considerable extent, the above program is speculation. Nevertheless, the present state of twistor theory does have a number of points of contact with it. To illustrate something of this, I should be more explicit about the nature of twistors. Let us, in the first instance, choose M to be Minkowski space-time. Choose an origin O and consider a classical special-relativistic system whose total momentum P_a is *null* and future-pointing and whose total angular momentum tensor M^{ab} has the property that the *spin vector* constructed from it (and P_a) is proportional to the momentum P_a:

$$P^d\, M^{bc}\, e_{abcd} = 2\, s\, P_a\ . \tag{10}$$

this is a normal requirement [41] for the momentum and angular momentum structure for a zero rest-mass particle (e_{abcd} being the alternating tensor). The quantity s is the spin-helicity; that is to say, $|s|$ is the intrinsic spin, while the sign of s measures the helicity. Translating into a two-spinor notation

[7]This is necessary because the local Minkowski geometries are not really tangent spaces. They have to *merge* one into the other when our point of view changes as we move around the universe. In a sense, they are all really the *same* "space."

for P_a and M^{ab} we obtain

$$P_{AA'} = \overline{\pi}_A \, \pi_{A'} \ , \tag{11}$$

for some $\pi_{A'}$, whence from 10

$$M^{AA'BB'} = i\,\omega^{(A}\,\overline{\pi}^{B)}\,\epsilon^{A'B'} - i\,\overline{\omega}^{(A'}\,\pi^{B')}\,\epsilon^{AB} \tag{12}$$

for some ω^A (where the round brackets indicate symmetrization). The pair of spinors $(\omega^A, \pi_{A'})$ defines P_a and M^{ab} uniquely, while P_a and M^{ab} define $(\omega^A, \pi_{A'})$ up to the combined phase transformation

$$\omega^A \rightarrow e^{i\theta}\,\omega^A, \qquad \pi_{A'} \rightarrow e^{i\theta}\,\pi_{A'} \tag{13}$$

(θ real). Relative to the origin O, this pair of spinors represents a *twistor* [35, 36] Z^α:

$$Z^\alpha \leftrightarrow (\omega^A, \pi_{A'}) \ . \tag{14}$$

If we pass to a new origin $\tilde{O}$ whose position vector relative to O is h^a, then relative to $\tilde{O}$ we must represent the twistor Z^α by $(\tilde{\omega}^A, \tilde{\pi}_{A'})$ where

$$\tilde{\omega}^A = \omega^A - i\,\pi_{A'}\,h^{AA'}, \qquad \tilde{\pi}_{A'} = \pi_{A'} \tag{15}$$

this being consistent with the transformation of momentum and angular momentum:

$$\tilde{P}_a = P_a, \qquad \tilde{M}^{ab} = M^{ab} - h^a\,P^b + P^a\,h^b \ . \tag{16}$$

If we pass from the space-time metric g_{ab} to a conformally related metric $\hat{g}_{ab}$, according to the *conformal rescaling* [5]

$$\hat{g}_{ab} = \Omega^2\,g_{ab}, \qquad \hat{\epsilon}_{AB} = \Omega\,\epsilon_{AB} \ ; \tag{17}$$

then Z^α must be represented by $(\hat{\omega}^A, \hat{\pi}_{A'})$ where

$$\hat{\omega}^A = \omega^A, \qquad \hat{\pi}_{A'} = \hat{\pi}_{A'} + i\,\omega^A\,\Omega^{-1}\,\nabla_{AA'}\,\Omega \ . \tag{18}$$

The two transformations (15) and (18), and also any homogeneous Lorentz transformation, have the property that they are linear transformations (of unit determinant) which leave the form

$$\begin{aligned} Z^\alpha\,\overline{Z}_\alpha &= \omega^A\,\overline{\pi}_A + \pi_{A'}\,\overline{\omega}^{A'} \\ &= -2\,s \end{aligned} \tag{19}$$

invariant; where the complex conjugate $\overline{Z}_\alpha$, of the twistor Z^α is represented as

$$\overline{Z}_\alpha \leftrightarrow (\overline{\pi}_A, \overline{\omega}^{A'}) \tag{20}$$

relative to the origin O. Since the Hermitian form (19) has signature $(++--)$, such transformations generate the group $SU(2,2)$, this being $4-1$ isomorphic with the connected component of the conformal group.

Twistors for which $Z^\alpha \overline{Z}_\alpha = 0$, are called *null twistors*. A null twistor Z^α defines the *null straight line* ζ in M, which is the locus of points whose position vectors x^a satisfy

$$i\,\pi_{A'}\, x^{AA'} = \omega^A \tag{21}$$

(see equation 15). We may think of ζ as *the world line of zero rest-mass particle of momentum P_a and zero intrinsic spin*, this particle having angular momentum M^{ab} relative to O. The *non-null* twistors (that is, $Z^\alpha \overline{Z}_\alpha \neq 0$) may be thought of as describing zero rest-mass particles with *non-zero intrinsic spin*, but they cannot be uniquely localized in terms of a single world-line in space-time.

Two null straight lines η and ζ will *intersect* (possibly at conformal infinity) if and only if their corresponding twistors Y^α and Z^α are *orthogonal*:

$$Y^\alpha \overline{Z}_\alpha \neq 0\,; \tag{22}$$

in which case the *light cone* whose vertex is the intersection q of η and ζ, is generated by null lines ξ described by null twistors

$$X^\alpha = \lambda\, Y^\alpha + \mu\; Z^\alpha\;. \tag{23}$$

Since any point q can be represented by its light cone, the linear set (23), denoted by $T(q)$, gives a twistor way of realizing the point q. If desired, we can use a twistor

$$Q^{\alpha\beta} = Y^\alpha\, Z^\beta - Z^\alpha\, Y^\beta \tag{24}$$

to represent $T(q)$, and hence the point q. (Generally, twistors of higher valence can be constructed from twistors like Z^α or $\overline{Z}_\alpha$, by means of the usual tensor rules.)

According to a theorem by Kerr [33], the condition that a congruence of null straight lines be *shear-free* can be stated very elegantly in twistor terms as the fact that the congruence be representable as the null solutions of an equation

$$\Phi(Z^\alpha) = 0 \tag{25}$$

(or by the limiting case of such a construction) where Φ is a complex analytic (holomorphic) function of the components of the twistor Z^α. Thus, the shear-free condition is interpreted, in twistor terms, as essentially the *Cauchy-Riemann equations* $\partial\phi/\partial\overline{Z}_\alpha = 0$. In other words, the "shear-structure" for the space T of twistors Z^α, may be thought of as a *complex (analytic) structure* on T. We can use this structure to locate the $T(q)$ manifolds, and hence the points of M (since light cones are the only nonshearing null hypersurfaces in M, apart from null hyperplanes).

Let us turn to the case when M is a (globally hyperbolic) *curved* space-time. The concept of a *null* twistor $\mathcal{Z}$ can be adapted without difficulty from the Minkowski case: we simply interpret $\mathcal{Z}$ as a null geodesic ζ in M at each of whose points is a spinor $\pi_{A'}$, parallelly propagated along ζ, such that the "momentum vector" P_a, given by $P_{AA'} = \overline{\pi}_A \pi_{A'}$, is tangent to ζ. On the other hand, there appears to be no way of uniquely associating a *non-null* twistor with some well-defined structure on M. However, it is useful to *postulate* the existence of a space of non-null twistors, into which the space T_0 of null twistors is to be embedded as a hypersurface. This gives us an eight-dimensional manifold T, the space T_0 being a seven-dimensional submanifold.

The space T is supposed to possess a symplectic structure, namely a real two-form, which I shall still denote by ω, this being the exterior derivative $\omega = d\varphi$ of a *complex* one-form φ on T. The imaginary part of φ is to be the exterior derivative of a scalar field $-s$ on T, the hypersurface T_0 being defined by $s = 0$. If a region of T_0 refers to null geodesics which enter some region of *flat* space in M, then we can use the representation of twistors given in (14), etc., and be more explicit as to the definition of these forms. We can set [21]

$$\omega = i\, dZ^\alpha \wedge d\overline{Z}_\alpha, \qquad \varphi = i\, Z^\alpha\, d\overline{Z}_\alpha \qquad \text{and} \qquad s = -\frac{1}{2}\, Z^\alpha\, \overline{Z}_\alpha \tag{26}$$

(compare with equation 19). In the case when $s = 0$ (vanishing of intrinsic spin) we can substitute $Z^\alpha \leftrightarrow (i\, x^{AA'}\, \pi_{A'}, \pi_{A'})$ (compare with equation 21) into the right-hand sides and verify directly the formal equivalence with (3) and (4). If the null geodesics pass through *two* regions of flat space, then the definitions (26) arising from each will agree with one another [21]. We can also use (26) if the null geodesics do not enter any region of flat space, provided we are concerned only with T_0 and interpret the dZ^α's suitably.

The space G_0 (that is, G_K with $K = O$) of null geodesics that we considered earlier, can be regarded as obtainable from T by a process analogous to that by which G_0 was previously obtained from the tangent bundle C. The (squared) "Hamiltonian" is now the quantity $s = -\frac{1}{2}\, Z^\alpha\, \overline{Z}_\alpha$. We are concerned with the seven-dimensional submanifold T_0 of T given by $s = 0$. The vector field corresponding to χ is now

$$-\frac{i}{2}\, Z^\alpha \frac{\partial}{\partial Z^\alpha} + \frac{i}{2}\, \overline{Z}_\alpha \frac{\partial}{\partial \overline{Z}_\alpha}\ , \tag{27}$$

which generates the transformations $Z^\alpha \to e^{i\theta}\, Z^\alpha$ (see equation 13). We factor out by these phase transformations to pass from T_0 to the six-dimensional space G_0 (see Figure 9). This process enables us to carry over the symplectic structure ω, and also φ, from T, now, to the space G_0. The result agrees with the previous construction [20]. Thus, the symplectic structure of T can, when restricted to T_0, be given some real geometrical significance.

It is actually possible to go considerably farther than this in explicitly exhibiting the form of canonical transformation on T, which is induced, for example, by the presence of a gravitational wave in M. An interplay between analytic functions and the symplectic structure of T appears again in a surprising way [36]. I do not want to go into all this here. The main point I wanted to make is that conformal curvature shows up classically on the space T in terms of *canonical* transformations

$$Z^\alpha \to \tilde{Z}^\alpha\, (Z^\beta, \overline{Z}_\gamma),$$

the form ω being preserved [21]. Such a transformation *shifts* the complex structure of T. It is the complex structure of T (being its "shear structure") which serves to "locate" the *points* in M. Such a shift causes "good" null cones to be transformed into cones which do not focus cleanly at a proper vertex. In this way it has the effect of "fuzzing out" points which had previously appeared to be "good" points and *vice versa*. Quantum mechanically, we would expect this effect to result from a unitary transformation

$$Z^\alpha \to \tilde{Z}^\alpha\, (Z^\beta, \partial/\partial Z^\gamma)\ .$$

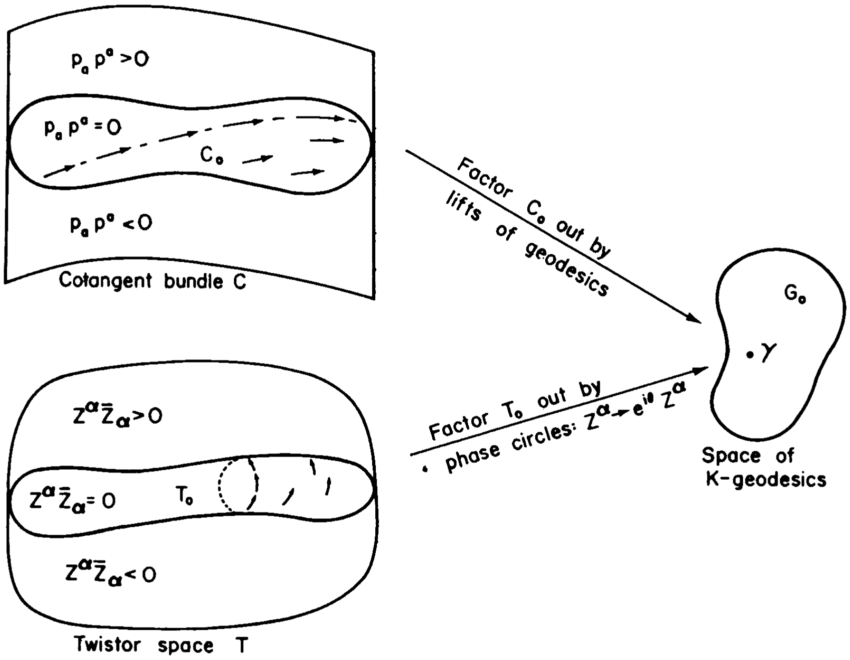

FIGURE 9.
The symplectic structure of G_0 arises from that of C or from that of T in a similar way.

In my descriptions up to now I have essentially ignored the question of the topology of M. I can, however, see no real objection to applying these ideas to quite general space-time manifolds with complicated topology. The restriction to a globally hyperbolic M allows one to avoid the more serious problems which might arise. But it also precludes any possibility of having a space-time with a *changing* topology, for example. (Global hyperbolicity is equivalent [24] to the existence of a Cauchy hypersurface for M.) If global hyperbolicity is dropped, then T_0 and G_0 could exhibit certain pathologies. If M is assumed to be strongly causal [3, 4, 5] then the worst of these pathologies can be avoided. But there still remains the possibility that T_0 and G_0 could turn out to be non-Hausdorff manifolds under such circumstances.

Finally, I should say something about the status of the *details* of twistor theory at the moment [36]. The theory allows computations to be carried out using *contour integrals*, which are intended to represent (cross sections for) scattering processes involving electromagnetic and gravitational interactions, etc. (It is such computations as these which one would hope to be able ultimately to re-express in a purely combinatorial form.) At the time of writing, there is some evidence that those computations which represent the conformally invariant part of the theory (for example, electromagnetic interactions without rest-mass) are substantially correct. On the other hand, it has not yet been possible to find the correct way of handling the conformally noninvariant parts of the theory (for example, rest-mass and gravitational interactions). But there appears to be no insuperable obstacle eventually to achieving this.

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