

## Theory of Quantized Directions

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1. Introduction The concept of the continuum (or real number system) is one of the most fruitful mathematical notions employed in physical science. Any ~~possible~~ <sup>All successful</sup> and reasonably comprehensive physical theories that ~~exist~~ have so far been proposed have rested heavily on this notion — if only because space and time according to our present ideas, form a continuum and are therefore to be represented by continuous coordinates. And quantum mechanics (as normally understood), although it has introduced a measure of discrete into physical theory, rests even more firmly on the continuum concept than did classical physics. For the array of complex numbers is called upon at the outset in connection with probability amplitudes and the superposition law. Nevertheless, there would appear to be strong reasons for believing that the continuum concept may eventually have to be abandoned as one of the basic ingredients of a fundamental physical theory. ~~Such~~ A theory could be based on purely combinatorial principle and only at a second stage; in the

of an approximation, or of a computation or conceptual aid would the continuum emerge. In this way, it has sometimes been suggested, the divergences and inconsistencies which plague quantum field theory might possibly be avoided. at a cut-off

Perhaps one of a significant reason for doubting that the continuum occupies a basic position in fundamental physics is a belief in the ultimate simplicity of nature. From a logical point of view, the continuum is not at all a simple structure. Also, <sup>conceptual</sup> ~~logical~~ difficulties arise in the definition of a real number which are of quite a different order than in the case of an integer (or a fraction). It is interesting to <sup>in context</sup> recall Kronecker's famous dictum: "God made the integers, all the rest is the work of man!" These ~~logical~~ difficulties have sometimes prompted various writers to try to replace the usual notion of the continuum by some other structure of a more constructive nature which would avoid these <sup>conceptual</sup> ~~logical~~ problems. However, it appears that the price paid is always an increase in the complexity of the structure involved.

As has been pointed out by Schrödinger, the continuum only seems to be a relatively simple idea because it is so familiar. The idea has developed historically from a mathematical idealization of a physical "straight line". Implicit in this is the assumption that such a "line" would

~~continuity~~  
the same

pear similar no matter how many times it magnified. But as we now know, the most precisely drawn material line would necessarily appear totally different if magnified by a factor of, say,  $10^{10}$ . It might be maintained, that the atomistic nature of matter in no way argues against continuity for a background space. But the fact remain that at the level at which continuity fails for matter, our initial motivation for believing in its applicability to space itself, ~~of the absolute space~~ now becomes of doubtful value. The <sup>present</sup> reasons for believing in the continuity of space and time amount to the fact that theories based on such a continuity appear to work. On the other hand this should not necessarily be taken as an indication that space and time "are" continuous at the level ~~at the level of~~ under consideration. It should be recalled that Newton's theory of gravitation which operates in a flat space and uses the concept of a gravitational force also "works". But these Newtonian ideas must be thought of as "wrong", when viewed from the conceptually more satisfying framework of general relativity. Furthermore Schrodinger argues forcefully that discontinuities are present.

in physical phenomena at the quantum level it is  
(i.e. wave mechanics)  
quantum theory that assumes a continuity is maintained  
taking place right up until the final step at which  
the "observation" is made.

What will be proposed here is a <sup>model</sup> ~~germ~~ of a theory whose rules are <sup>essentially simple and</sup> entirely combinatorial in character — and from which the beginnings of space-time geometry ~~will~~ emerge when structures of sufficient complexity are considered. At this stage no attempt is made at modifying the content of physical theory. Only a reformulation is <sup>suggested</sup> presented in which quantities normally requiring continuous coordinates for their description are eliminated <sup>from primary consideration. In particular,</sup> since space and time have therefore to be eliminated, a form of Mach's principle must be invoked: If a relationship of an object to <sup>some</sup> background space <sup>should not</sup> be considered ~~it is~~ only the relationships of objects to each other which can have significance.

What physical quantities, then, can be used as a starting point? In the first instance such a quantity must have a discrete spectrum, preferably taking values which are integer multiples of some elementary value. Secondly, we must expect that the rules for combining the

entities are essentially bound up with quantum processes. Finally, since we expect to be able to build up the idea of space from these rules, such a quantity must be intimately related to space-time properties — although it must be a scalar since space-time <sup>(direction)</sup> cannot appear at the outset. The one quantity which seems to emerge from all this ~~one~~ is ~~rest mass~~ and total angular momentum. ~~rest mass~~  
A more satisfactory alternative ~~Perhaps not satisfactory~~ might be to try to construct a theory using both ~~rest mass and total angular momentum as quantities as~~ <sup>rest mass</sup> and ~~quantities as~~ primary concepts. Unfortunately however, too little is known about the mass spectrum at present, and there is no suggestion as yet that mass is based on integral multiples of an elementary mass.

In order to make progress, we shall suppose here that the universe is analyzable into substructures each of which may be momentarily considered ~~and having a well-defined total angular momentum~~ as isolated from the rest of the universe. These substructures will be called units. They could be elementary particles or composite particles such as nuclei, atoms, or <sup>perhaps</sup> ~~sometimes~~ molecules, etc. Associated with each unit is a non-negative integer called its spin number. The spin number is twice the total

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angular momentum quantum number of the unit, (so that for an electron it is one, for a deuteron it is two, etc.) A unit of spin number  $n$  will be called an  $n$ -unit.

The only pieces of information about the universe that are allowed to be considered <sup>here</sup> at this stage are the network of interrelationships between the various units (i.e. the knowledge of which units combine or split up into which other units) and the associated spin numbers. Any such network with assigned spin numbers will be called a spin networks. Laws of physics then reduce to statements about frequencies of occurrence in the universe of the various spin networks.

The idea here is to construct a geometry of directions in space from <sup>the laws of</sup> spin networks. Now, according to quantum theory, a system of zero total angular momentum must be spherically symmetrical. Ref: Dirac To a 0-unit, therefore, any direction in space is equivalent to every other. In effect, only one distinguishable "direction" in space appears to a 0-unit to exist. Furthermore, a 1-unit ~~with axis~~ (e.g. an electron) can have only two orientations. Thus, only two external spatial "directions" exist for such a unit. More generally, only  $n+1$  possible "directions" appear to exist for any  $n$ -unit ~~of the spin networks~~. Thus, a continuum of possible directions can only be approached for units with large spin number. The suggestion is, therefore, to associate a direction - which may be thought of as the axis of spin - with each unit of large spin number. Such a unit will be called a large unit. It is possible to define angles between two such directions in the following way. An "exercis-

performed between two large units whereby a 1-unit detaches from one of the large units and is then united with the other one. The relative probabilities that the spin numbers <sup>? → as compared with each other</sup> of the large units increase or decrease & give a measure of an "angle" between their spin directions (consistent with the predictions of quantum mechanics). However, part of the probability may be due to "ignorance". This is manifest in insufficient connecting links between the two large units in the known spin networks. This possibility can be eliminated from consideration by means of a repetition of the above experiment. If the probabilities <sup>in the second experiment</sup> <sub>by the result of the first experiment</sub> remain essentially unchanged, then the "ignorance" factor is deemed to be insignificant and the "angle" between the large units well defined.

The main result of this paper is that if the laws for spin networks are taken to agree with the rules for combining non-relativistic angular momenta according to quantum mechanics, then the geometry of "directions" defined by large unit as described above agrees, in the limit, with the geometry of directions in a three-dimensional Euclidean space. In order to obtain four-dimension space-time, the corresponding relativistic laws would have to be employed which take into account the possibility of a relative "velocity" between units — or else some other method would have to be

3) employed. Also, with relative velocities present the mixing of orbital angular momentum with the spin should give rise to a notion of "distance" as well as of four-dimensional angle. With the present scheme distances do not emerge, and time is absent even to the extent that the time-ordering of events is totally irrelevant. But within the framework of the situations treated, the result obtained here is ~~surprisingly~~ reassuring. In fact, in at least one respect it would seem to be somewhat remarkable. For the initial assumption of being able to associate a direction of spin with any quantum mechanical system of large total angular momentum is by no means obviously justified. Moreover, it would appear at first sight to be false! Wave functions of high total angular momentum need single out no particular direction as an axis of spin (e.g. a state with spin number twenty, say, could have icosahedral symmetry, etc.). The reason the approach seems to work appears to be connected with the "Machian" nature of the theory: the geometry of directions that is built up is the one defined by the system itself and it may not agree with the geometry of some previously given (and unnecessary!) background space.

A second purpose of this paper is to introduce a new combinatorial ~~technique~~<sup>calculus</sup> for treating quantum-mechanical angular momenta. The results obtained are equivalent to a use of the conventional Wigner symbols etc. and the method seems well suited to a discussion of problems such as the one treated here. The calculus is somewhat related to the graphical techniques of Ord-Smith, Edmonds and Levinson, but appears to have a greater flexibility. The basic rules ~~of calculus~~ are essentially very simple. In an appendix it is also shown, here, how this calculus can be applied to obtain various known formulae. In particular, direct transformations are found which exhibit ~~deceptiveness of~~ Regge's 3-j and 6-j symmetries. Also, a formula is given which contains as special cases, Racah's explicit expressions for both the 3-j and 6-j symbols.

2. Spin Networks, Values and Norms. Any spin network can be represented as a diagram in which the units

(depicted as ~~small curved and pinching~~)  
are ~~denoted by~~ line segments/numbers

according to their respective spin numbers (fig.1). At one or both ends of each line segment

~~may be a~~ vertex which represents an "interaction"; i.e. the coming together of two units to form a third or the splitting up of a unit

units. Time ordering" is actually irrelevant but it may be thought of as proceeding from the bottom of the diagram to the top.

The spin networks considered here, exactly three line segments must terminate at each vertex and the total number of these units must be finite.

Any unit which is not terminated at both ends by a will be called an end unit. A spin network possessing no end unit

Fig. 1

An example of a spin network. The lines may be thought of as representing the world-lines of particles.

The associated integer is twice the total angular momentum quantum number.

The vertices represent interactions. Three edges must meet at each vertex,

(imposing) edges will be called closed. An example of

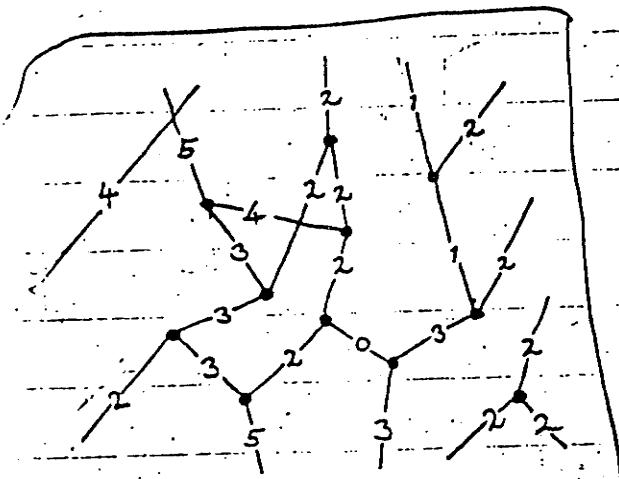
a closed spin network is given in fig. 2. Any

closed spin network determines, up to sign, a certain integer called its value.<sup>1</sup> The

value can be positive or negative (or zero)

and its sign is fixed once an orientation

at each vertex is specified which determines a cyclic order for the three units which meet



at each vertex. Every line segment connects two vertices.

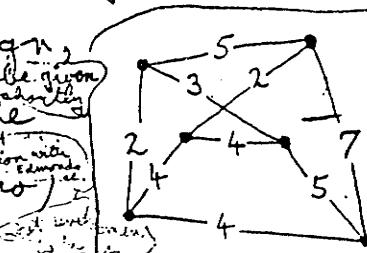


Fig. 2

A closed spin network  
Every line segment connects two vertices.

there. Once the spin network has been drawn on a plane such as in fig.2, conveniently determines the orientations to be (say) anticlockwise at each vertex.

For any ~~open~~ (or closed) spin network a non-negative integer is defined called its norm. The norm of a spin network is defined as the modulus of the value of the closed ~~open~~ spin network which consists of two copies of the <sup>original</sup> network joined along the corresponding free ends. An example illustrating this is given in fig.3. (When drawing the diagrams on a plane, it is often convenient

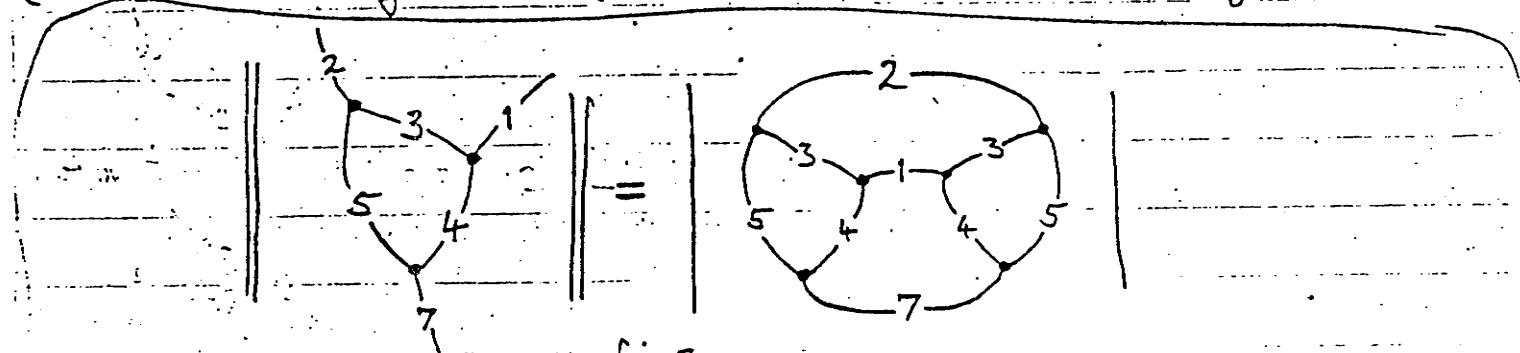


Fig.3

To determine the norm of the spin network on the left, two copies are drawn side by side and connected along their free ends. (The networks are drawn as mirror images of each other.) The modulus of the value of the resulting closed network is the required norm.

to represent the two networks as mirror images of each other. The implied orientations, of course, do not affect the norm.). The norm of a spin network  $\alpha$  is denoted by  $\|\alpha\|$ , while for a closed oriented spin network drawn on a plane, the network itself denotes its value.

The importance of the concept of the norm of a spin network, here, lies in the fact that it can be used to determine the probabilities of the various spin numbers value for a given network which has a suitable subset of its spin numbers already assigned. The rule will be stated <sup>(in full generality)</sup> here, but the proof is left until section 4. Effectively,

cial case of this rule will be the following.

(2.1) The norm of a spin network vanishes if and only if the physical situation it represents is forbidden by the rules of non-relativistic quantum mechanics.

For more general situations the norm gives a measure of the frequency of occurrence in the universe of the unknown ~~spin number values in~~ <sup>in an arbitrary</sup> specified circumstance. Consider a network  $w$ , a portion  $k$  of which consist of units whose spin numbers are known. A vertex of  $w$  is to belong to the (possibly disjointed) spin network  $x$  whenever all three units associated with the vertex belong to  $k$ . Thus, we can imagine the network  $w$  to be "cut" across some of its edges into two parts, one of which is  $k$ . Let  $\xi$  denote the remaining portion of  $w$ . Thus,  $\xi$  is a spin network containing all the vertices of  $w$  which are not in  $k$  and units corresponding to those of  $w$  which belong to these vertices. In order for the formula which follows to be applicable it will be necessary to assume that the network of  $\xi$  is a tree (i.e. a connected network containing no closed circuits of units) and that exactly one of the end units of  $\xi$  - say an  $x$ -unit - is an end unit of  $w$  (and not of  $k$ ). All the other end units of  $\xi$  will then also be end units of  $k$ . The situation is illustrated by an example in Fig. 1. T.A.

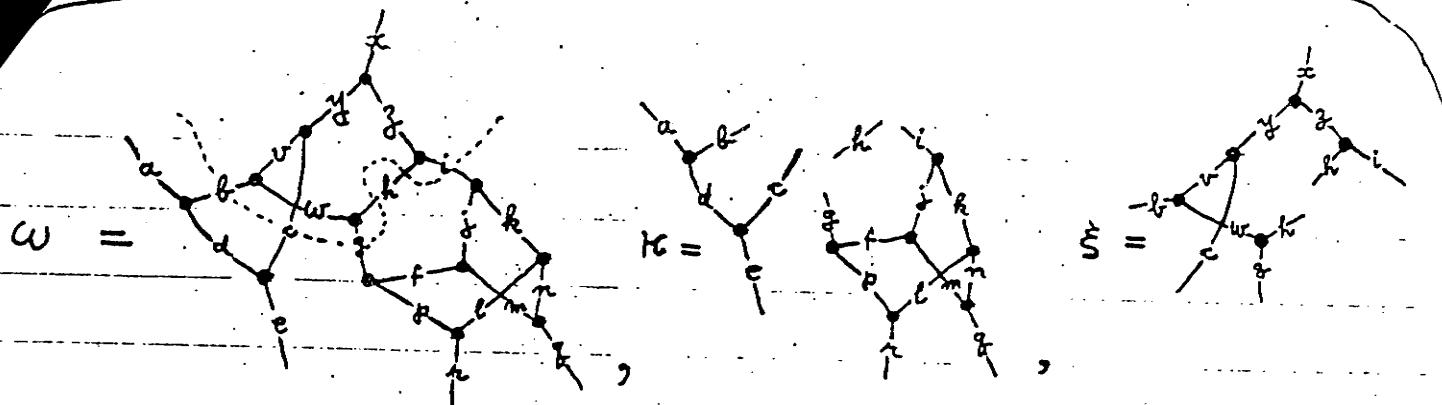


fig. 4

The type of situation to which formula (2.2) applies. The network structure of  $w$  is supposed known, as are the actual spin numbers for the (disjointed) portion  $K$ . The portion  $S$  must be a tree, exactly one of whose end units is an end unit of  $w$ . The probabilities for the values of  $v, w, x, y$  are then determined by (2.2).

probability for the spin values assigned to  $S$ , given  $K$ , is (assuming  $\|K\| \neq 0 \neq \|S\|$ , c.f. (2.1))

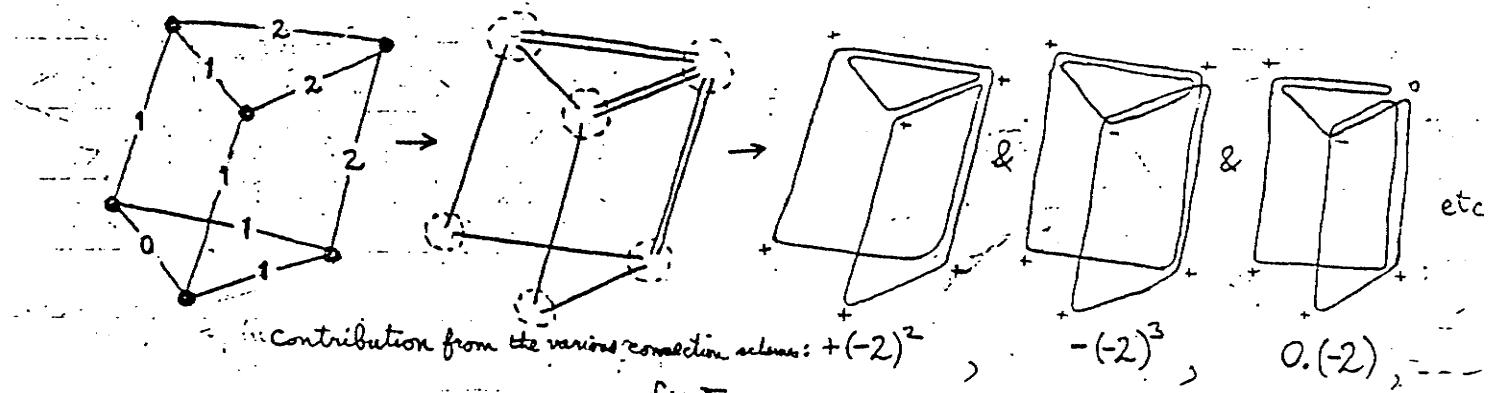
$$\text{probability} = \frac{(x+1)\|w\|}{\|K\| \cdot \|S\|} \quad 2.2$$

The special case when  $x$  is the only unknown in  $S$  will be the one of most interest to us here (see also (2.12)).

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~~Here, once the spin networks has been drawn on plane such as in fig.2, this conveniently determines the orientation (say anticlockwise) at each vertex combinatorially.~~

To define the value of a closed oriented spin network we proceed as follows. We imagine the line segment representing an s-unit to be composed of  $s$  parallel strands. At each vertex the ends of the strands can be joined together in pairs in various ways. Each such way of connecting the strands results in a number of closed loops (see fig.5). Each close



Each line of the closed network on the left is represented as  $n$  strands, where  $n$  is the corresponding spin number. The strands are connected in pairs at each vertex in all possible ways. If  $r$  complete circuits emerge, the term  $(-2)^r$  is to be associated with the connection scheme. Additional sign is included for each crossing point of one strand with another which occurs at a vertex. If a strand doubles back on itself at a vertex, the term is zero.

loop is assigned the value -2. The value of  $r$  closed loops is then  $(-2)^r$ . Each such system of closed loops contributes linearly to the total value  $N$  of the network:

This is

$$N = \prod \frac{1}{n_i} \times \sum \epsilon (-2)^r$$

fat deg of same up to sign much easier,  
i.e. when "1" is first written. Then prob. is:  
Then next section does sign of value.

were  $\epsilon$  is  $\pm 1$  or zero, depending on how the strands are connected at the vertices and where, for later convenience, a factor  $1/s!$  multiplies the whole expression, for each  $s$ -unit in the network. In ~~the beginning~~ fact  $N$  remains an integer, although this is not immediately obvious and will emerge from the discussion of section 5.  ~~$s^2$  unit will turn out to be irrelevant.  $N$  remains an integer!!~~ The definition of  $\epsilon$  is determined up to a sign for the whole network by the following two rules.

(2.4) If, at any vertex, two strands belonging to the same unit are connected to each other, then  $\epsilon = 1$

(2.5) Otherwise,  $\epsilon = \pm 1$  where the sign of  $\epsilon$  changes whenever the connections to two strand ends belonging to a unit are interchanged.

(Sufg 5) This is in fact sufficient for since the norm does not depend on the sign of the value. However, for the discussion to follow,

To precise determination of the sign of  $\epsilon$  is necessary. This is little more complicated <sup>since</sup> it depends on the orientation assigned to each vertex. Label the various units in the networks  $A, B, C, D, \dots$ , where  $A$  is an  $a$ -unit,  $B$  is a  $b$ -unit, etc. The  $a$  strands belonging to  $A$  can be labelled consecutively from  $A$ , up to  $A_a$ , but for consistency with later conventions, the strand ends instead will be so labelled, where the labelling of the strand ends at one end of <sup>the</sup> unit runs in the opposite direction from those at the other end of the unit, that is, if the label is  $A_j$  at one

Do graphical art. of v.

The orientation convention,

end of the strand, it is  $A_{a-j+1}$  at the other end ( $j=1, \dots$ ).  
The labeling of strand ends for  $B, C, \dots$  follows similar conventions.  
Consider a vertex at which the units  $A, B$  and  $C$  come  
together and suppose the assigned orientation at the vertex  
specifies the cyclic order  $ABC$  for the units. Consider a  
~~particular~~ any connection scheme of strand ends satisfying

(2.4) ~~is given by~~ pairings:

$$(A_1 B_1) \cdots (A_k B_k) (A_m C_n) \cdots (A_q C_r) (B_s C_t) \cdots (B_w C_x). \quad 2.6$$

Then we have  $\epsilon = +1$  or  $-1$  according as the ordering  
in (2.6) is an even or odd permutation of

$$A_1 A_2 \cdots A_a B_1 \cdots B_b C_1 \cdots C_c. \quad 2.7$$

← It should noted ~~fact~~ that the ordering of the  
pairs in (2.6) is irrelevant, since the interchange of two pairs  
is always an even permutation. Furthermore, (2.5) is clearly satisfied  
the fact that  $\epsilon$ , so defined, depends only on the cyclic order of  $A, B, C$   
will emerge in a moment. We observe that the number  
of strands connecting  $A$  to  $B$  plus the number of  
strands connecting  $A$  to  $C$  at the vertex must  
equal  $a$ . From this and the two similar relationships  
for  $B$  and  $C$ , we get:

$$\begin{aligned} \frac{1}{2}(a+b-c) &\text{ strands connect } A \text{ to } B \text{ at the vertex} \\ \frac{1}{2}(b+c-a) &\text{ " } \qquad \qquad \qquad B \parallel C \text{ " " " } \\ \frac{1}{2}(c+a-b) &\text{ " } \qquad \qquad \qquad C \parallel A \text{ " " " } \end{aligned} \quad \left\{ 2.8 \right.$$

These three expressions must represent non-negative  
integers, so for the possibility of a non-zero  $\epsilon$   
we must have the following two laws.

(2.9) The sum of the spin numbers of the three units coming together at any vertex must be even.

(2.10) The sum of the spin numbers of two units at a vertex cannot be smaller than the spin number of the third.

Once (2.9) and (2.10) are satisfied a connection scheme ~~with  $\epsilon \neq 0$~~  with  $\epsilon \neq 0$  will be possible.

The law (2.9) will be recognized as a statement of the familiar fact that the total number of fermions (in and out) involved in an interaction must be even. The law (2.10) is a well-known "triangle inequality" in the theory of angular momentum. (No relative velocities!) The norm of any spin network not satisfying (2.9), (2.10) must vanish. This is to be expected in view of the physical interpretation (2.1). The spin networks of figs. 1, 2 and 3 have all been drawn in accordance with (2.9) and (2.10).

To see that  $\epsilon$  depends only on the cyclic order of A, B, c at the vertex, change the ordering ABC into BCA in (2.6) and (2.7). This results in  $a(b+c)$  interchanges in (2.7) and in  $a^2$  interchanges in (2.6) since the ordering of the pairs is irrelevant. But  $a(b+c)$  and  $a^2$  have the same parity because of (2.9). Hence  $\epsilon$  remains unchanged, whence the sign of the value of any closed spin network is fixed once a cyclic orientation is assigned at each vertex. To see the effect

(5)

of reversing the orientation at any one vertex, replace the ordering ABC in (2.6), (2.7) by BAC. We get ab-interchanges in (2.7) and  $\frac{1}{2}(a+b-c)$  interchange in (2.6) (see (2.8)). Thus, the value of the closed spin network is changed by a factor

$$(-1)^{\frac{1}{2}(a+b-c)-bc} = (-1)^{\frac{a(a-1)}{2} + \frac{b(b-1)}{2} + \frac{c(c-1)}{2}} \quad (2.11)$$

the second formula being obtained from the first by a little manipulation using (2.9). It shows, incidentally, that a reversal of the orientation of all the vertices in a closed spin network leaves its value unaltered. One further fact that follows from (2.7) is that for  $\epsilon \neq 0$ , the connection scheme at each vertex is unique up to a permutation of the strand ends belonging to each unit, since the number of strands from each unit to each other is fixed. It follows from this that rule (2.5) does, in fact, determine the sign of  $\epsilon$  up to an overall ambiguity for the whole (unoriented) spin-network. In the next section a diagrammatic method for determining the sign of each term in (2.3) will next be given, which depends on a representation of the spin network on a plane<sup>(fig. 5)</sup>. This somewhat simplifies the appearance of the definition of the value of a spin networks and will be especially useful in relation to ~~some of the~~ later calculations.

~~It becomes difficult, however to give the signs for simpler definition completed in a manifestly combinatorial form, however, which is the reason for the approach prescribed here.~~

In order to complete the definitions for values and norms of spin networks, one final convention is necessary.

(2.12) The value of the closed "spin networks" which consist of a single s-unit forming a closed loop is  $(-1)^s (s+1)$ .

Although this is, perhaps, not strictly a spin network in the sense intended initially, the definition is consistent with the other conventions, as will emerge later. Also, the definition is necessary for determining the norm of the (open) spin network which consists of a single s-unit, the norm being  $s+1$ . Any spin network (see K in fig. 4) which has an (open) s-unit as a disconnected part has, therefore, a norm which is  $s+1$  times the norm of the remainder.

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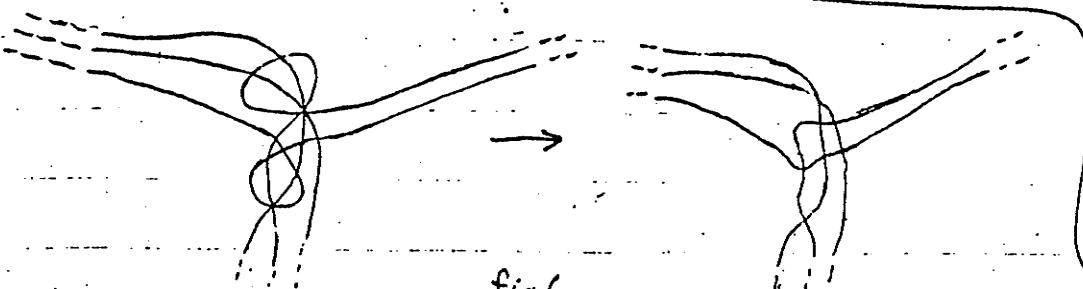
3. Diagrammatic Notation: Binors Let  $\alpha$  be an oriented spin network represented on a plane, as in fig. 1, so that the orientation at each vertex is anticlockwise. The units are represented as smooth curves which may cross each other in the diagram, some points other than at the vertices. These crossing points will be called spurious intersections as they have no physical significance. They are necessary in order for some networks (such as the one of fig 2.) to be representable on a plane. The diagram should be drawn so that the spurious intersections do not occur at vertices. To represent each term in (2.3) we draw  $s$  parallel strands (as in fig. 3) corresponding to each  $s$ -unit. No intersections except for the spurious ones are to occur between strands, until the connections are made at the vertices.

Each connection scheme at each vertex must be drawn (fig. 6) according to the following two conventions.

(3.1) No connecting strand shall be self-intersecting

(3.2) All intersection points must be simple.

A simple intersection occurs when exactly two strands cross, with distinct tangents there. *The term*



*connecting strand refers to the part of strand at the vertices only.*

fig.6.

The left-hand diagram contains self-intersecting connecting strands and non-

ing thus drawn a diagram for a given connection scheme which satisfies (2.4) (i.e.  $\epsilon \neq 0$ ) the rule for (2.3) is as follows:

(3.3)  $\epsilon = +1$  or  $-1$  according as the total number of intersections between the strands - not counting spurious intersections - is even or odd.

To see that (3.3) agrees with that given by (2.6), (2.7), we observe, first, that the sign given by (3.3) is independent of the particular way in which the connections are drawn provided the conventions (3.1) and (3.2) are adhered to in the drawing: simple topological considerations show that allowed intersections can only appear or disappear in pairs. Now draw the vertex with the unit lines entering from the bottom of the diagram in the order A.B.C (fig.7). The strands can then be depicted entering from the bottom in

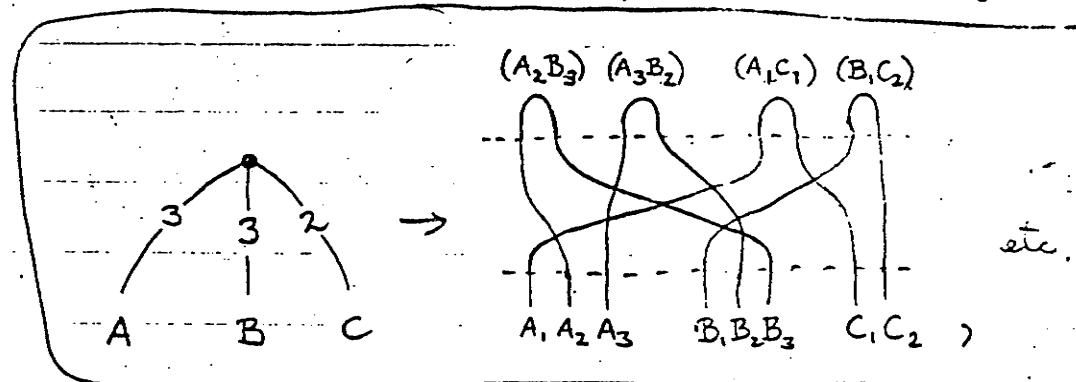


fig.7

A connection scheme for the vertex on the left is given on the right. The portion between the broken lines is a Aitken diagram for the permutation  $A, A_2, A_3, B, B_2, B_3, C, C_2 \rightarrow A_2 B_3 A_3 B_2 A, C, B, C_2$

the order  $A, A_2, \dots, A_n, B_1, \dots, B_n, C_1, \dots, C_n$  (see 2.7). This will be consistent with the labeling for the other vertices because of the rule that the labeling is in the reverse order at the other end of each unit. (This is necessary if the strands are not to cross over along the unit.) At the top of the diagram the pairings (2.6) can be represented as a row of loops, the ends of which can be directly

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relevant

connected to the strand ends  $A_1, \dots, C_n$ . The part of the diagram which gives the connections between the loops ends and the strand ends (fig.7) is an Aitken diagram for the required permutation — transforming (2.7) into (2.6). Each intersection point represents a simple transposition, so it follows that the permutation is even or odd according as the number of intersections is even or odd. Combining the results for all the vertices we arrive at the rule (3.3).

Although (3.3) simplifies the appearance of the expression (2.3) for the value of a closed spin network, this formula is still not very practical for evaluating anything but the simplest of closed spin networks. A very simple example is evaluated in fig.8. Some

$$\begin{aligned}
 & \text{Diagram showing a closed loop with two strands labeled 1 and 2.} \\
 & = \frac{1}{2!} \left\{ \text{Diagram } P - \text{Diagram } P - \text{Diagram } P + \text{Diagram } P \right\} \\
 & = \frac{1}{2} \left\{ (-2) - (-2)^2 - (-2)^2 + (-2) \right\} = -6
 \end{aligned}$$

The evaluation of a simple closed spin network according to the formula (2.3) and the rules (2.4) and (3.3). Any closed loop denotes the integer  $-2$ .

more practical methods will be developed in section 5.

As a preliminary to this and for the various proofs to be given, it will be convenient to develop the notation further. We may, in fact, regard any vertex, at which (say) an  $a$ -unit and a  $b$ -unit <sup>come together, as a</sup> formal sum, with the appropriate sign

is checked, of all the different possible strand connection for which  $\epsilon \neq 0$ . The case  $a=3, b=2, c=1$  is illustrated in fig. 9, where the sum consists of six terms. For the

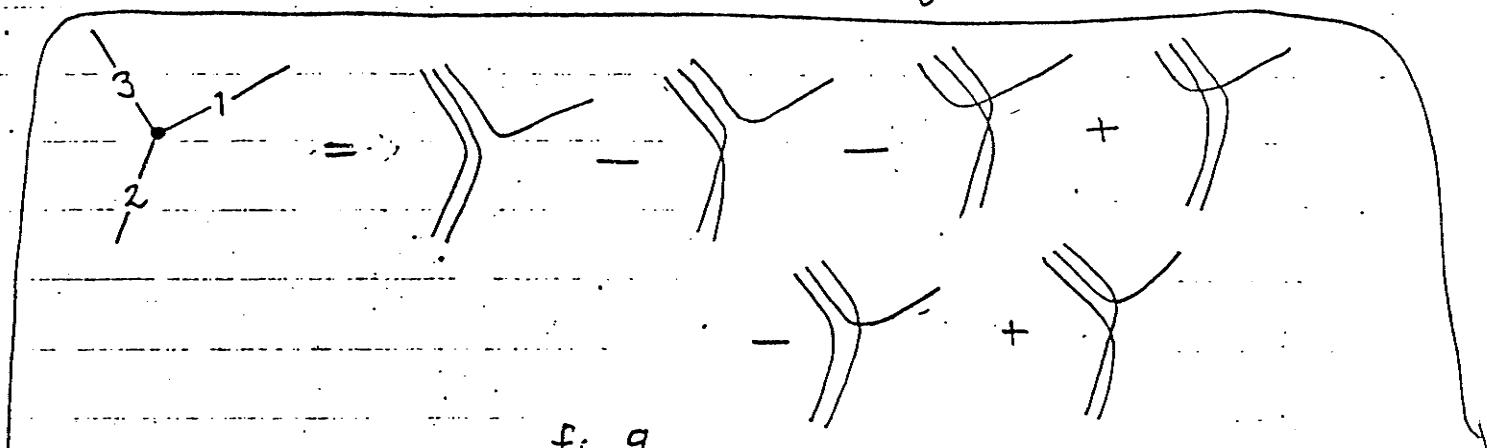


fig. 9.

The representation of a vertex as a formal linear combination of strand connection schemes. This formal sum is a simple example of a binor.

general case there will be

$$a! b! c!$$

$$\left\{ \frac{1}{2}(a+b-c) \right\}! \left\{ \frac{1}{2}(b+c-a) \right\}! \left\{ \frac{1}{2}(c+a-b) \right\}!$$

3.4

terms in all (see 2.8). A formal sum of strand connection such as this, with integer (or perhaps rational) coefficient will be called a binor. A more complete definition of a binor will be given shortly.

If all the vertices of a closed spin network are expanded (distributively) in this way and if each of the resulting closed strand loops is counted as -2, then according to formula (2.3), the result will be just  $Ts!$  times the value of the spin network. In order to eliminate this factor, the notational convention will be

(compare fig. 9)

moreover

20)

employed that, in any spin network diagram, each  $s$ -unit which has a vertex at both ends denotes not just  $s$  strands, but  $s$  strands multiplied by the factor  $\frac{1}{s!}$ . (For

$$= \frac{1}{2!} \left\{ (\text{strand 1}) - (\text{strand 2}) - (\text{strand 3}) + (\text{strand 4}) \right\} = (\text{strand 1}) - (\text{strand 2})$$

fig.10

The binor representing a spin network has a factor  $\frac{1}{s!}$  incorporate for each  $s$ -unit which has a vertex at both ends. In the case of tree diagram this removes a redundancy in the expansion.

(example, see fig.10) It turns out that the resulting binor will still have integral coefficients. In the case of spin networks

which are simple trees this is <sup>in fact</sup> fairly obvious, since each factor  $\frac{1}{s!}$  removes a redundancy among the terms in the expansion (see fig.10).

(Any permutation applied to the strand ends at one end of an  $s$ -unit is equivalent, as regards the resulting connections, to a corresponding permutation applied to the strand ends at the other end of the same  $s$ -unit.) ~~any~~ The

general argument is left to section 5?, however.

For a more complete discussion of binors and their properties it will be useful to introduce another notation which brings out ~~a~~ remarkable similarity between binor algebra and certain tensor algebras. A single-strand starting from a point  $\alpha$  and ending at a point  $\beta$  will be denoted by  $S_{\alpha\beta}$ . If  $\alpha, \beta, \gamma, \delta, \dots, \mu$  are distinct points and strands are depicted connecting  $\alpha$  to  $\beta$ ,  $\gamma$  to  $\delta$ ,  $\dots$ ,  $\mu$  to  $\nu$ , then the corresponding binor is  $S_{\alpha\beta} S_{\gamma\delta} \dots S_{\mu\nu}$ . This is to be thought of as a product.

outer product — of the elementary binors  $S_{\alpha\beta}, \dots, S_{\mu\nu}$ . In this case the strands are supposed to be drawn so that no portion of a strand passes through  $\alpha, \dots, \nu$  — except for the one strand end which terminates there. The general binor is a linear combination with integer (or perhaps rational) coefficients of such outer products, each term having the same set of indices  $\alpha, \dots, \nu$  but permuted in some order. Thus, the strand connections for each term are made between the same set of points, but different connections may be made.

$$S_{\alpha\gamma} S_{\beta\delta} S_{\epsilon\zeta} = S_{\alpha\gamma} S_{\beta\delta} + 2 S_{\alpha\delta} S_{\gamma\beta} S_{\epsilon\zeta} = \text{Diagram 1} - \text{Diagram 2} + 2 \text{Diagram 3}$$

Fig. 11

Two notations for a binor are equated here, the indices in the first case denoting the strand ends in the second case. The actual paths of the strands are not to be significant — only the resulting topological connections are. In each case it is only the connection scheme which is significant; not the actual paths of the strands. Since the strands are not oriented, we have the symmetry

$$S_{\alpha\beta} = S_{\beta\alpha}.$$

3.5

Furthermore the implicit assumptions of commutativity and associativity have been made for outer products:

$$A_{\alpha\dots\gamma} B_{\delta\dots\zeta} = B_{\delta\dots\zeta} A_{\alpha\dots\gamma} \quad 3.6$$

$$A_{\alpha\dots\gamma} (B_{\delta\dots\zeta} C_{\lambda\dots\nu}) = (A_{\alpha\dots\gamma} B_{\delta\dots\zeta}) C_{\lambda\dots\nu} \quad 3.7$$

where  $A_{\alpha\dots\gamma}, B_{\delta\dots\zeta}, C_{\lambda\dots\nu}$  stand for outer products of element binors. Extending the definition of outer product to all lines by means of the distributive laws

$$A_{\alpha-\gamma}(B_{\lambda-\nu} \pm C_{\lambda-\nu}) = A_{\alpha-\gamma}B_{\lambda-\nu} \pm A_{\alpha-\gamma}C_{\lambda-\nu},$$

we have 3.6, 3.7, 3.8 holding for all binors with the specified sets of indices. Outer products can occur only between binors with indices in common, whereas sums and differences can occur only between binors whose (unordered) index sets are identical. We must have:

(3.9) for each unordered set of indices  $(\alpha, \dots, \gamma)$  the binors  $A_{\alpha-\gamma}$  form an abelian group with respect to addition.

To these may be added the substitution law:

(3.10) in any transitive relation between binors the substitution for  $\alpha, \beta, \gamma, \delta, \dots$  of some permutation of  $\alpha, \beta, \gamma, \delta, \dots$  results in an equivalent relation between binors.

A ~~new~~ <sup>such as in (3.10)</sup> binor resulting from a substitution will generally not be equal to the original binor, although it can be if a symmetry is present. Moreover two binors are never equal if the unordered sets of indices are distinct in the two cases (e.g.  $S_{\alpha\beta} \neq S_{\alpha\gamma}$ ).

Finally there is the operation of contraction between binors. The notation is the same as for cartesian tensors: any two index letters may be put equal throughout an expression or both replaced by some new letter whereby they become dummy indices and do not contribute to the index structure of the resulting binor. Thus,  $A_{\alpha-\gamma\lambda-\nu\beta-\tau} = A_{\alpha-\gamma\lambda-\nu\theta-\tau} = B_{\alpha-\gamma\lambda-\nu\theta-\tau}$  is a contraction of  $A_{\alpha-\gamma\lambda-\nu\eta\beta-\tau}$ .

In any term of a sum, no index may occur more than twice, and if it occurs twice it is a dummy. Any equation is supposed to remain true if any pair of index letters is replaced, throughout the expression by dummy. In terms of the diagrams, a contraction corresponds to an identification of the two relevant end-points in each outer product diagram. Corresponding identifications must be made throughout every term in the

either joins a pair of strands together or else it produces a closed loop. In the first case the joined pair simply produces a new strand, whence

$$S_{\alpha\beta} S_{\gamma\delta} = S_{\alpha\gamma}.$$

3.11

In the second case, since a closed loop counts as -2, we have

$$S_{\alpha\alpha} = -2.$$

3.1

In view of (3.11) we may also think of a contraction between  $\alpha$  and  $\beta$  as the drawing of a new strand in the diagram, which joins the point  $\alpha$  to  $\beta$  (since  $S_{\alpha\alpha} S_{\alpha\beta} S_{\beta\beta} = S_\gamma$ ). Thus we may write

$$A_{\dots \gamma \dots \gamma \dots} = S_{\xi\eta} A_{\dots \xi \dots \eta \dots}$$

3.1

whenever  $A_{\dots \xi \dots \eta \dots}$  is an outer product of elementary binors.

Implicit in this is the assumption that (3.6) and (3.7) still remain true for products of elementary binors when some of the indices are dummies. Furthermore, extend the definition of contraction from the case of outer products to any binor, we can maintain (3.13) when  $A_{\dots \xi \dots \eta \dots}$  is a general binor. (See fig. 12 for a diagram illustrating a contraction of a binor.) We also have the distributive property

$$S_{\xi\eta} (A_{\dots \xi \dots \eta \dots} \pm B_{\dots \xi \dots \eta \dots}) = A_{\dots \xi \dots \xi \dots} \pm B_{\dots \xi \dots \xi \dots} \quad 3.1$$

(See fig. 12 for a diagram illustrating a contraction applied to a binor.)

Note also the following rule, which is now effectively a special case of (3.13):

$$S_{\xi\eta} A_{\alpha \dots \eta \dots} = A_{\alpha \dots \xi \dots}.$$

3.15

$$A_{\alpha\beta\gamma\delta\epsilon\xi} = \underbrace{\alpha \xi}_{\beta \gamma} - \underbrace{\alpha \xi}_{\beta \gamma} + 2 \underbrace{\alpha \xi}_{\beta \gamma} \Rightarrow A_{\alpha\beta\gamma\delta\epsilon\xi} = \underbrace{\alpha \xi}_{\beta \gamma} - \underbrace{\alpha \xi}_{\beta \gamma} + 2 \underbrace{\alpha \xi}_{\beta \gamma}$$

fig. 12.

The binor of fig. 11 is contracted over  $\alpha, \xi$ . The final result is  $-S_{\alpha\xi} S_{\xi\epsilon}$ .

24)

This represents a strand extension from the point  $\eta$  to the point  $\xi$ .  
 The usual notations for contracted products will be used, e.g.

$$A_{\dots\xi\dots} B_{\dots\xi\dots} = S_{\xi\eta}(A_{\dots\xi\dots} B_{\dots\eta\dots}). \quad 3.16$$

The commutative, associative and distributive laws

It is not the purpose, here, to give a formal axiomatic discussion of binor algebra. It is hoped <sup>on the other hand,</sup> that the foregoing description has been adequate for the present purposes. The essential point about the algebra is that all the binor rules (cf. (3.5), ..., (3.16)), with the single exception of (3.12), are just the rules of cartesian tensor algebra where  $S_{\alpha\beta}$  plays the part of an  $n$ -dimensional Kronecker delta. The binors correspond to "isotropic" tensors, that is, tensors which are numerically invariant under the full  $n$ -dimensional orthogonal group, since these tensors are the ones built up from  $S_{\alpha\beta}$  (and scalars) by means of outer products, index permutations and sums. (The Levi-Civita symbols are excluded here since they change sign on reflection.) The key difference between the algebra of binors and the algebra of isotropic tensors lies in (3.12):  $S_{\alpha\alpha} = -2$ . For the contraction of a Kronecker delta we have  $S_{\alpha\alpha} = n$  where  $n > 0$  is the dimension of the space. The binors cannot, in fact, be represented in terms of components in the same

inner as are tensors since the dimensionality of the space would have to turn out to be negative (but see section 4 for an alternative representation).

There is one final rule which is convenient (although not absolutely essential) to impose on the binors in order to define the system completely. This is the equation

$$S_{\alpha\beta} S_{\gamma\delta} + S_{\alpha\gamma} S_{\delta\beta} + S_{\alpha\delta} S_{\beta\gamma} = 0. \quad 3.17$$

The diagram for (3.17) is drawn in fig. 13.  
 (The symbol 0, written without indices, denotes, as is the usual practice, the identity element of the relevant additive abelian group.) We observe that any contraction of the left-hand side of (3.17) vanishes identically by virtue of (3.12). It follows that any expression

$$(S_{\alpha\beta} S_{\gamma\delta} + S_{\alpha\gamma} S_{\delta\beta} + S_{\alpha\delta} S_{\beta\gamma}) A_{\alpha\beta\gamma\delta}$$

must also vanish identically where  $A_{\alpha\beta\gamma\delta}$  is a binor, since upon expanding out  $A_{\alpha\beta\gamma\delta}$  in terms of elementary binors, we get simply a linear combination of contractions of the bracketed expression, so that each term must vanish identically. Equation (3.17) therefore adds nothing to the theory if it is ultimately the scalars of the theory that are of interest. More important, this indicates that (3.17) will not lead us to a contradiction if we desire to impose it. Equation

ideal

26)

(3.17) will, in fact, be imposed here as it is useful in reduction formulae. (It should be recalled that isotropic  $n$ -dimensional tensors also satisfy certain identities. An outer product of  $n+1$  Kronecker deltas, skew-symmetrized over their first indices must necessarily vanish.)

For any binor  $A_{\alpha \dots \gamma}$  we can define a binor norm:

$$\text{norm}(A_{\alpha \dots \gamma}) = A_{\alpha \dots \gamma} A_{\alpha \dots \gamma}, \quad 3.18$$

which is a scalar. For the case of a binor representing a spin network the binor norm (3.18) differs by an unimportant non-zero factor from the norm  $\|A \dots\|$  defined in section 2. In fact

$$\|A \dots\| = \frac{1}{a! b! \dots d!} |\text{norm}(A \dots)| \quad 3.19$$

where  $a, b, \dots, d$  are the spin numbers of the end units of the spin network. (This factor emerges here because of the convention that  $\frac{1}{a!}$  is introduced into the definition of the binor representing a spin network, every time a unit acquires a vertex at both ends.)<sup>1</sup> In the next section, the following results will be proved:  
here  $A \dots$  has  $2n$  indices and  $\text{norm}(A \dots) \geq 0$

$$\therefore \text{norm}(A \dots) = 0 \text{ if and only if } A \dots = 0. \quad 3.20$$

For (3.21) it is essential that (3.17) hold. In fact, if we use the vanishing

$$)( + \cancel{\times} + ) = 0$$

fig 13.

in p 25)

The fundamental binor identity 3.17 in diagrammatic form.

the binor norm as a criterion for the vanishing of the binor,  
 (see section 4.1)  
 turns out that the only binor identities are those which  
 result from (3.17). If any other identity were to be imposed  
 an inconsistency would result. In any case, in view of (3.24)  
 we now have an algorithm for deciding whether or not a binor expression represents  
 zero. ~~In addition to (3.20), we will show~~ (A different algorithm will  
~~(3.20) become (3.22)~~  
~~whether a binor has zero indices.~~ be given in section 5.)

(B.P.D.)

and (2.1) we see that:

(3.22) the binor representing a spin network vanishes if and only if the physical situation represented is excluded by non-relativistic quantum mechanics.

In view of (3.22) we will lose no physics from the theory if we think of a spin network as being equated to the binor which represents it.

To conclude this section, some further notation will be introduced, which will extend the utility of the binor diagrams considerably. Define

$$S_{\alpha\beta,\gamma\delta} = S_{\alpha\gamma} S_{\beta\delta} - S_{\alpha\delta} S_{\beta\gamma} \quad 3.22$$

$$\begin{aligned} S_{\alpha\beta\gamma,\delta\epsilon\phi} &= S_{\alpha\delta} S_{\beta\epsilon} S_{\gamma\phi} + S_{\alpha\epsilon} S_{\beta\phi} S_{\gamma\delta} + S_{\alpha\phi} S_{\beta\delta} S_{\gamma\epsilon} \\ &\quad - S_{\alpha\delta} S_{\beta\phi} S_{\gamma\epsilon} - S_{\alpha\phi} S_{\beta\epsilon} S_{\gamma\delta} - S_{\alpha\epsilon} S_{\beta\delta} S_{\gamma\phi} \end{aligned} \quad 3.23$$

and generally

$$S_{\alpha\dots\gamma,\kappa\dots\nu} = \begin{vmatrix} S_{\alpha\kappa} & \dots & S_{\alpha\nu} \\ \vdots & \ddots & \vdots \\ S_{\gamma\kappa} & \dots & S_{\gamma\nu} \end{vmatrix} \quad 3.24$$

28)

The diagrammatic notation for  $S_{\alpha-\delta, \kappa-\nu}$  is  $r$  parallel strands with a thicker bar drawn across. (see fig.14).

$$S_{\alpha\beta-\delta, \kappa\lambda-\nu} = \begin{array}{c} \alpha\beta\dots\delta \\ \hline \dots \\ \kappa\lambda\dots\nu \end{array}; \quad | = |, \quad \{ = || - X,$$

$$\{ \{ \{ = ||| + X + X - IX - X - XI, \text{ etc.}$$

fig 14

The binor notation for the expressions (3.22), (3.23), (3.24). Each term in each expansion is an Aitken diagram, so that the sign can be determined as the parity of the number of intersections appearing between strands (drawn according to (3.1) and (3.2)).

Each term in the diagrammatic expansion of  $S_{\alpha-\delta, \kappa-\nu}$  appears as an Aitken diagram (cf. fig.7) for the relevant term in the determinant, so the sign of the term is determined by the parity of the number of strand intersections appearing on the plane (assuming (3.1) and (3.2)).

We can represent the binor corresponding to a spin network vertex in terms of these  $S$ 's. Thus if an  $a$ -unit, with strand ends labelled  $\alpha, \beta, \gamma$ ; a  $b$ -unit,  $\pm$  labelled  $\delta, \epsilon, \zeta$ ; and a  $c$ -unit,  $\mp$  labelled  $\theta, \nu, \kappa$  come together at a vertex, the binor for that vertex is

$$\frac{1}{\{\frac{1}{2}(a+b+c)\}! \{\frac{1}{2}(b+c-a)\}! \{\frac{1}{2}(c+a-b)\}!} S_{\alpha..,\delta, \lambda..,\nu..,\tau} S_{\delta..,\zeta, \tau..,\rho..,\psi} S_{\theta..,\kappa, \psi..,\lambda}. \quad 3.25$$

This should become clearer from the example given in fig.15.  
The factorial expression in the denominator removes

$$3 \quad 4 \\ = \frac{1}{4! 2! 3!} \quad \text{Diagram showing three strands merging into two strands, with labels 3, 4, 5 at the top and 1, 2, 3 at the bottom.}$$

fig. 15

The binor representing a vertex can be expressed as a contracted product of the binors of fig. 16. The factorials in the denominator remove a redundancy in the expansion.

the redundancy that the ordering of the connecting strands is irrelevant (and see (2.8), (3.4)). The coefficients in (3.25) are thus  $\pm 1$ .

From the definition of  $S_{\alpha, \gamma, \delta}$  and in  $S_{\delta, \gamma, \beta}$  we see that it is skew-symmetric!

(and symmetrical between the groups). Hence

$$S_{\alpha, \gamma, \delta} S_{\delta, \gamma, \beta} = r! S_{\alpha, \gamma, \beta} \quad \text{3.26}$$

where  $\alpha, \gamma, \delta$  are  $r$  in number. More generally

$$S_{\alpha, \gamma, \delta} S_{\gamma, \beta, \tau} = r! S_{\alpha, \gamma, \beta, \tau} \quad \text{3.27}$$

(see fig. 16).

The binors (3.25) corresponding to the

$$\text{Diagram showing two strands merging into one strand, with labels 1, 2, 3, 4, 5 at the top and 1, 2, 3 at the bottom.} = - \text{Diagram showing two strands merging into one strand, with labels 1, 2, 3, 4, 5 at the top and 1, 2, 3 at the bottom.}, \text{etc.}; \quad \text{Diagram showing three strands merging into one strand, with labels 1, 2, 3, 4, 5 at the top and 1, 2, 3 at the bottom.} = 5! \text{Diagram showing three strands merging into one strand, with labels 1, 2, 3, 4, 5 at the top and 1, 2, 3 at the bottom.}, \text{etc.}; \quad \text{Diagram showing four strands merging into one strand, with labels 1, 2, 3, 4, 5 at the top and 1, 2, 3 at the bottom.} = 3! \text{Diagram showing four strands merging into one strand, with labels 1, 2, 3, 4, 5 at the top and 1, 2, 3 at the bottom.}, \text{etc.}$$

fig. 16

Some simple properties of  $S_{\dots, \dots}$  in diagrammatic form.

various vertices of a spin network may be contracted together to give the binor for the whole spin network, multiplied by  $r! \dots t!$  where  $r, \dots, t$  are the spin numbers of all the internal units

30) in the spin network. By (3.26), and (3.25) this factor is removed if we write only one  $S_{\dots}$  for each unit instead of two (one from each vertex). This is illustrated in fig. 17.

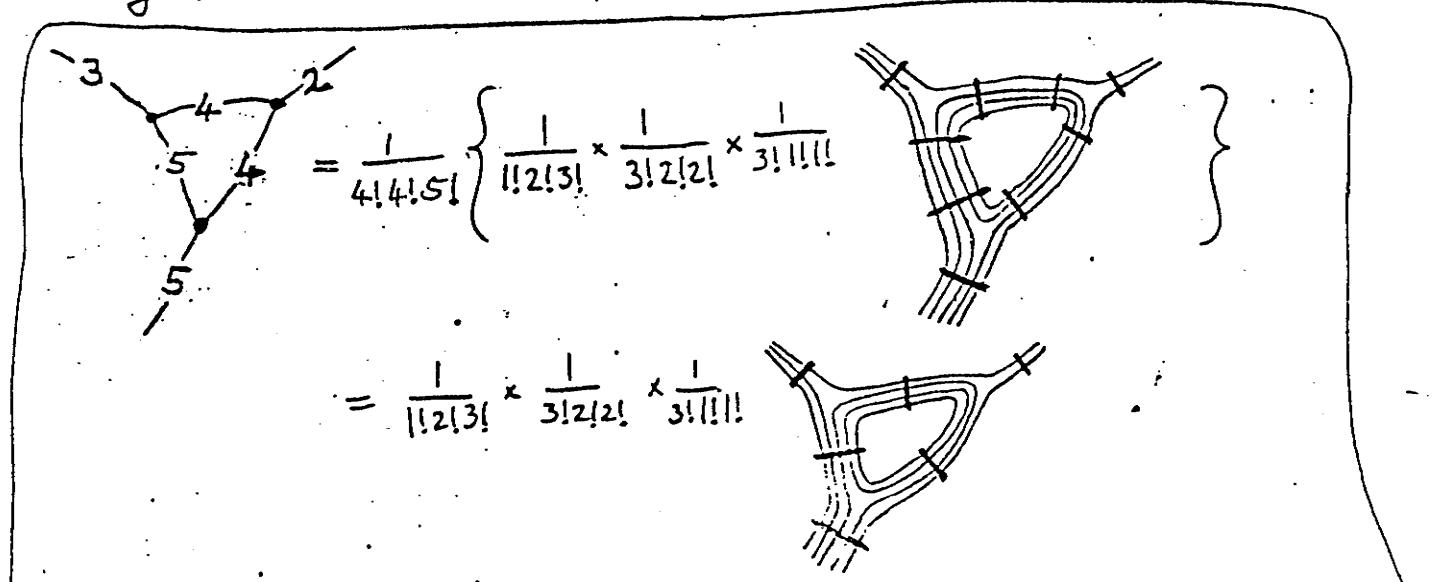


fig.17.

A simple spin network diagram represented as a binor. The factorials introduced in the denominator corresponding to each internal unit are removed by writing only one  $S_{\dots}$  for each unit.

The other factorials, from (3.25), will remain and it will be convenient to incorporate them into anotation for binors.— as has already been done in the case of spin network lines. A line with a non-negative integer  $n$  attached denotes, as before, a set of  $n$  parallel strands. The integer 1 may be omitted when a single strand is denoted. The line may be omitted altogether if the integer is zero. The thick bar may be used to denote the expression  $S_{\dots}$ , and blobs may still be used to denote vertices. We shall use, in addition, the following convention:

28) Any line segment with associated integer  $n$  and which is terminated at both ends, either by a vertex blob or by a thick bar, denotes not just  $n$  strands but  $n$  strands times  $1/n!$

With this convention, we can remove the factorials from figs 15, 16 and 17. This is illustrated somewhat more generally in fig. 18.

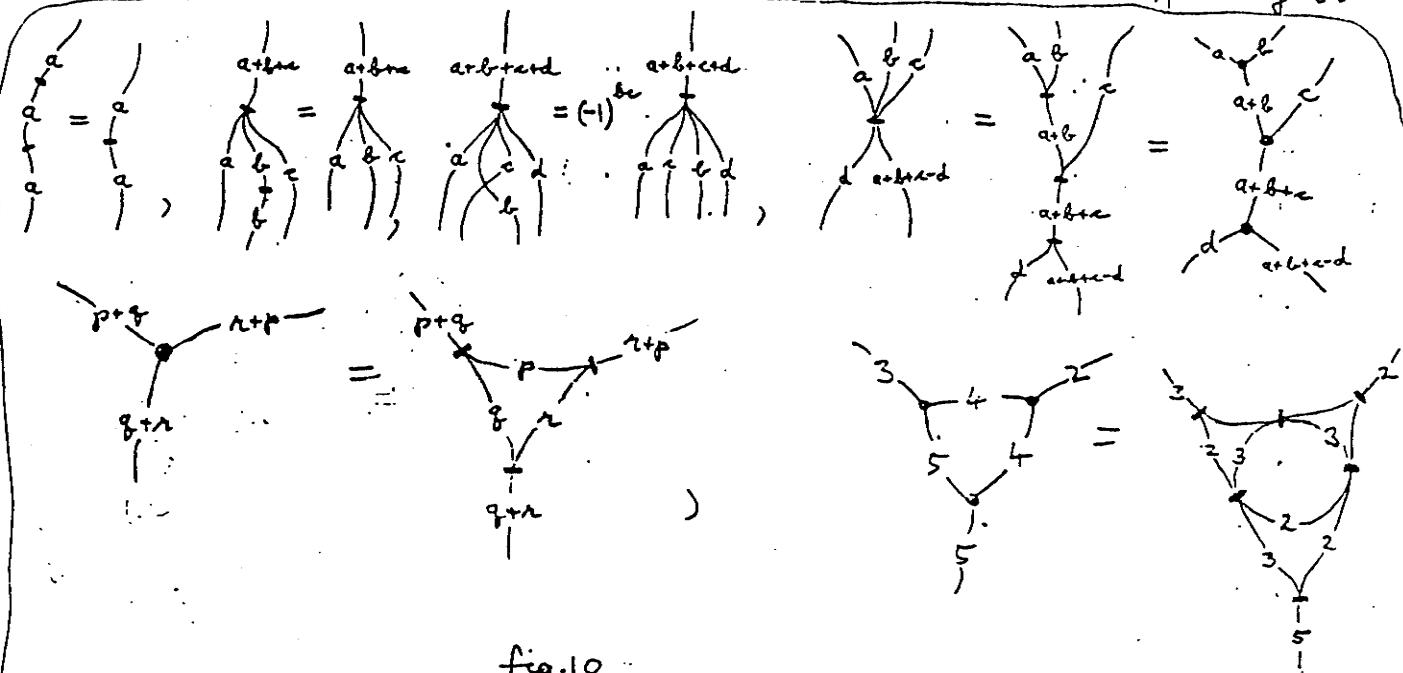


fig. 18

The notations of figs. 15, 16, 17 can be written more generally and a little more concisely if a set of  $n$  strands taken together are denoted by a single line labeled by  $n$ . Such a line when terminated at both ends, either by a vertex blob or a thick bar, denotes, instead,  $1/n!$  times  $n$  strands. (If  $n=1$  the label may be omitted.) Henceforth, the right-hand side of the first equation above is the simplest diagram that will be used to denote an isolated  $a$ -unit. The fourth equation illustrates how any thick bar can be written in terms of vertex blobs, whereas the fifth describes the converse operation. The final diagram denotes the strand networks for fig 17. In every case, the total of the strand numbers must be the same at the two sides of each bar.

The fourth ~~and fifth~~ equation depicted in fig. 18 illustrates how any thick bar (i.e.  $S_{\dots\dots}$ ) in a diagram may be written, instead, in terms of special ~~special~~ spin network vertices. The fifth

32)

equation shows how the general spin network vertex can be written in terms of the ~~bars~~ bars (c.f. (3.25)). A diagram built up solely of the bars and (multiple) strand lines — i.e. a contracted product of the  $S_{\dots,\dots}$ 's — will be called a strand network. From the above, we see that strand networks and spin networks are in a sense equivalent, since for any strand network there is a spin network that is equal to it and conversely. However, the operations which suggest themselves are different in the two cases. It will prove very convenient to be able to work here primarily with the strand networks rather than with the spin networks (c.f. the Regge symmetry transformation given in figs ?, ?). The one point to note when drawing strand networks is the following.

(3.29) The sum of the strand numbers for the lines coming from one side of any bar must equal the sum of the strand numbers at the other side of the bar.

This takes the place of the two laws (2.9), (2.10) for spin networks.

One final convention should be adopted in the representation of spin networks.

(3.30) The binor representing an isolated n-unit is  $S_{\alpha,\dots,\beta,\dots,\gamma}$ , where the strands at one end of the unit are labelled  $\alpha,\dots,\gamma$  and at the other end,  $\beta,\dots,\gamma$ .

Thus, the simplest diagram that should be used for an isolated a-unit is thus the right-hand side of the first equation of fig. 18.

This is in fact consistent with (2.12), see (4.19).)

Proof of Validity of the Binor Calculus. A first step in establishing the validity of binors is to set up a kind of "isomorphism" between binor diagrams (drawn on a plane) and expressions built up from the two-dimensional Kronecker deltas and Levi-Civita symbols:  $\delta_A^B$ ;  $\epsilon_{AB}$ ,  $\epsilon^{AB}$ . (Capital Roman indices take on the values 1, 2. The summation convention is used throughout. The Levi-Civita and Kronecker symbol components are given by  $\epsilon_{AB} = B-A = \epsilon^{AB}$ ,  $\delta_A^B = 1 - (B-A)$ .) We have

$$\epsilon_{AB} = -\epsilon_{BA}, \quad \epsilon^{AB} = -\epsilon^{BA}, \quad 4.1$$

$$\delta_A^B \delta_B^C = \delta_A^C, \quad \delta_A^B \epsilon_{BC} = \epsilon_{AC}, \quad \delta_A^B \epsilon^{AC} = \epsilon^{BC}, \quad \epsilon_{A3} \epsilon^{AC} = \delta_3^C, \quad 4.2$$

$$\delta_A^A = 2 = \epsilon_{AB} \epsilon^{AB}. \quad 4.3$$

Also, the identity

$$\epsilon_{AB} \epsilon_{CD} + \epsilon_{AD} \epsilon_{BC} + \epsilon_{AC} \epsilon_{DB} = 0 \quad 4.4$$

holds, together with the equivalent identities obtained by raising indices of (4.4) with Levi-Civita symbols, e.g.

$$\delta_A^B \delta_C^D - \delta_A^D \delta_C^B = \epsilon_{AC} \epsilon^{BD}. \quad 4.5$$

It is a classical result of invariant theory that every identical relation satisfied by the  $\epsilon$ 's and  $\delta$ 's can be built up from these relations. A similarity between (4.4) and the binor identity (3.17) is evident — the difference being that  $\epsilon_{AB}$  is skew-symmetric!

(4.1) whereas  $\delta_{AB}$  is symmetrical (3.5). The difference between (4.3) and (3.12) is also one of sign. Therefore if some form of isomorphism between the two systems is to be established, a suitable convention to take care of the

34) sign differences must be employed. This can, in fact, be done very simply in terms of ~~the~~ binor diagrams. The diagrams must first be drawn suitably on a plane which will be thought of as vertical so we can refer to the "top" and "bottom" ends of the plane. (as in (3.2)) We must draw the diagram with smooth curves so that

(4.6) all intersection points between strands are simple and, in addition,

(4.7) the strands are vertical only at isolated points

at which no intersections must occur;

(4.8) the strand ends come out vertically at the extreme top and bottom ends of the plane.

In fig. 19, a diagram is drawn according to these rules.

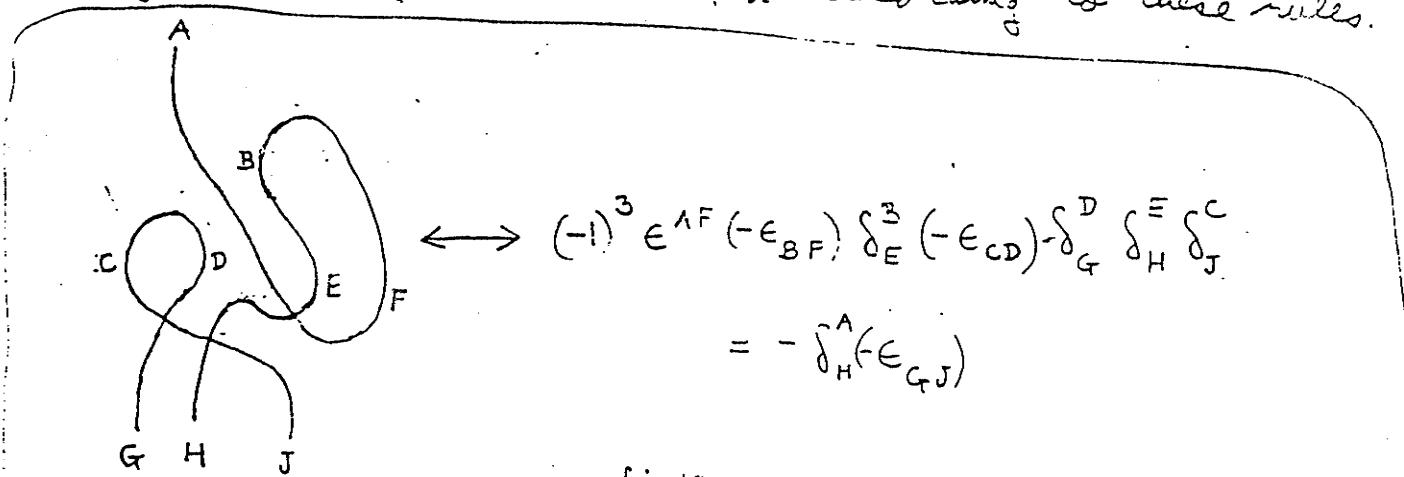


fig. 19

The binor diagram on the left is drawn according to the rules (4.6), (4.7), (4.8). Each point at which a strand is vertical is labelled by a capital letter. The corresponding  $\delta, \epsilon$  expression has these letters for indices - covariant if the relevant strand portion lies just above the point and contravariant if it lies just below it. A minus sign is introduced for each intersection point in the diagram and for each covariant  $\epsilon$ .

Each point at which a strand is vertical is now labelled by a capital Roman letter. These points divide the strands into a number of portions each of which is vertical at no other point. Three essentially distinct situations can now arise with such a strand portion. If P and Q label

~~downwards to Q, where Q lies to the right of P in the diagram~~

the two end-points of the portion, we may suppose that Q lies to the right of P. (The situation of P and Q lying vertically one above the other cannot arise.) Then if P and Q are both lower end-points of the portion,

we represent this by  $\epsilon_{PQ}^{PQ}$ . If P is a lower end-point and Q an upper end-point, we use  $\delta_P^Q$  - whether or not Q lies to the right of P.

If P and Q are both upper end-points we use  $+ \epsilon_{PQ}^{PQ}$  (or  $\epsilon_{QP}^{QP}$ ). If P is a lower end-point and Q an upper end-point, we use  $\delta_P^Q$  - whether or not Q lies to the right of P. The appropriately contracted product of all these quantities is formed and then multiplied by  $(-1)^r$ , where r is the total number of intersection points in the diagram.

The essential virtue of this representation lies in the following two facts.

(4.9) Any continuous deformation of the strand paths ~~with fixed end-points~~ into another allowable diagram leaves the value of the resulting  $\delta, \epsilon$  expression unchanged.

(4.10) Any binor identity between diagrams corresponds to a  $\delta, \epsilon$  identity and vice-versa.

36) The proof of (4.9) depends on showing that each one of a set of elementary topological deformations - from which the general deformation can be built up - corresponds to a valid  $\delta, \epsilon$  equation (4.1), (4.2). This is done in fig. 20. As for the proof of (4.10), fig. 21 shows that (3.12)

$$\begin{array}{c}
 \text{Diagram: } A \cup B = A \cup \underset{c}{\circ} \underset{d}{\circ} B = A \cup \underset{c}{\circ} \underset{d}{\circ} = A \cup \underset{c}{\circ} = A \cup B \\
 \downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow \\
 \epsilon^{AB} = -\delta^A_D \delta^B_c \epsilon^{cd} = -\epsilon^{AD}(-\epsilon_{cd}) \epsilon^{CB} = \delta^A_c \epsilon^{CB} = \epsilon^{AC} \delta^B_c
 \end{array}$$

$$\begin{array}{c}
 \text{Diagram: } A \overset{\uparrow}{\cup} B = C \overset{\uparrow}{\cup} D = A \overset{\uparrow}{\cup} \underset{c}{\circ} \underset{d}{\circ} B = A \overset{\uparrow}{\cup} \underset{c}{\circ} \underset{d}{\circ} = A \overset{\uparrow}{\cup} \underset{c}{\circ} = A \overset{\uparrow}{\cup} z = A \overset{\uparrow}{\cup} z \\
 (-\epsilon_{AB}) = -(-\epsilon_{cd}) \delta^c_B \delta^d_A = -(-\epsilon_{AD}) \epsilon^{cd} (-\epsilon_{cb}) = \delta^c_A (-\epsilon_{cb}) = (-\epsilon_{AC}) \delta^c_B \\
 \downarrow \qquad \downarrow \\
 \delta^A_B = \delta^A_c \delta^c_B = \epsilon^{AC} (-\epsilon_{cb}) = (-\epsilon_{bc}) \epsilon^{CA} = -\epsilon^{AC} (-\epsilon_{bc}) = -(-\epsilon_{cb}) \epsilon^{CA}
 \end{array}$$

$\left\{ \text{Diagram } 1 = \text{Diagram } 2, \text{ Diagram } 3 = \text{Diagram } 4 \right\} \leftrightarrow \text{no sign change}$

fig. 20

Continuous deformations may be performed on any bivector diagram provided the resulting figure satisfies (4.6), (4.7) and (4.8). Although such a deformation may change the form of the corresponding  $\delta, \epsilon$  expression it will not change the resulting value of the expression (assuming the <sup>individual</sup> orderings of the upper and of the lower strand ends are unaltered by the deformation).

equivalent to (4.3) and that (3.17) is equivalent to (4.5).  
The rest is implicit in the relations expressed in fig.20.

(For example, the symmetry of  $S_{\alpha\beta}$  is expressed in the skew-symmetry of  $\epsilon^{AB}$  by the first equation in fig.20.)  
If some of the indices of a valid  $\delta, \epsilon$  equation are raised or lowered (by  $\epsilon$ 's), or permuted, this does not affect

$$\text{Diagram: } A \text{ (curly line)} = -2 = \text{Diagram } B + \text{Diagram } C + \text{Diagram } D = 0$$

$\downarrow \quad \downarrow \quad \downarrow$

$$(-\epsilon_{AB}) \epsilon^{AB} = -2 = -\delta_A^B \delta_B^A, \delta_C^A \delta_D^B - \delta_D^A \delta_C^B + \epsilon^{AB} (-\epsilon_{CD}) = 0$$

fig.21.

The two relations which most significantly characterize the binor algebra find more familiar representations in the  $\delta, \epsilon$  system.

the validity of the corresponding binor diagram equation.  
In each case the operation may be achieved by drawing in more lines. This cannot alter the validity of a diagram equation. Also, the number of new intersection points introduced has the same parity for each term of a sum, whence the relative signs of the corresponding  $\delta, \epsilon$  expression remain unaffected.

Having thus established our "isomorphism", a number of properties follow at once. Most important of these is:

(4.11) the binor algebra is consistent,  
although this is, in fact, not hard to see from more direct considerations.  
Also, (3.20) and (3.21) are immediate. For let the binor  $A_{\lambda\mu\nu}$  correspond to the contravariant  $\delta, \epsilon$  expression  $\phi^{L..N}$  and let  $\psi^{L..N}$  be the corresponding covariant  $\delta, \epsilon$  expression (i.e. each  $\epsilon^{ij}$  in

38)  $\phi^{L-N}$  is replaced by  $-\epsilon_{..}$  to get  $\psi_{L-N}$ ). Then comp. by component, we have  $\psi_{L-N} = (-1)^r \phi^{L-N}$  where  $\phi^{L-N}$  has  $2r$  indices. Thus,  $(-1)^r \phi^{L-N} \psi_{L-N} > 0$  unless  $\psi_{L-N} = 0$ . That is,  $(-1)^r A_{\lambda-\nu} A_{\lambda-\nu} > 0$  unless  $A_{\lambda-\nu} = 0$ , as required. Another result, related to this, which follows from the corresponding trivial result is the fact that no further binomial identities, beyond (3.17) can result, except for those built up from (3.17) itself.

Note that the minus sign introduced for each strand intersection results in a curious reversal of the roles of symmetry and skew-symmetry. The diagrammatic expansion for  $S_{\alpha-\beta, \lambda-\nu}$  in terms of Fitten diagrams given in fig. 14, involves a minus sign each time the number of intersections is odd. Thus, corresponding to  $S_{\alpha-\beta, \lambda-\nu}$ , the expression  $\sigma_{A-C}^{L-N}$  built up out of  $\delta_i$ 's is (apart from a factor) a symmetrizing operator rather than a skew-symmetrizing operator:

$$\sigma_{A-C}^{L-N} = \sum_P \delta_A^{P(L)} \cdots \delta_C^{P(N)}, \quad 4.12$$

the summation extending over all permutations  $P$  of  $L, \dots, N$ . On the other hand, since an expression  $X_{A-C}$  cannot be skew-symmetrical in more than two of its indices unless it is zero, we have the result:

(4.13) any binor which is symmetrical in more than two indices must vanish.

we consider pairs of binor indices, then symmetry and anti-symmetry revert to their normal roles, since any odd permutation of pairs of elements is still an even permutation of the individual elements. Now, pairs of symmetrical two-dimensional indices may be thought of as constituting a three-dimensional compound index: the set of symmetrical expressions  $\Theta_{AB}$  (~~not associated~~, constructed from  $\epsilon$ 's and  $\delta$ 's) is three-dimensional. The role of the Kronecker delta is then taken over by  $\frac{1}{2} \sigma_{AB}^{CD}$ . It follows from the three-dimensionality that

$$\begin{vmatrix} \sigma_{AB}^{RS} & \sigma_{AB}^{TU} & \sigma_{AB}^{VW} \\ \sigma_{CD}^{RS} & \sigma_{CD}^{TU} & \sigma_{CD}^{VW} \\ \sigma_{EF}^{RS} & \sigma_{EF}^{TU} & \sigma_{EF}^{VW} \end{vmatrix} = k \lambda_{AB,CD,EF}^{RS,TU,VW}, \quad \begin{vmatrix} \sigma_{A_3}^{RS} & \sigma_{A_3}^{TU} & \sigma_{A_3}^{VW} & \sigma_{AB}^{XY} \\ \sigma_{CD}^{RS} & \sigma_{CD}^{TU} & \sigma_{CD}^{VW} & \sigma_{CD}^{XY} \\ \sigma_{EF}^{RS} & \sigma_{EF}^{TU} & \sigma_{EF}^{VW} & \sigma_{EF}^{XY} \\ \sigma_{GH}^{RS} & \sigma_{GH}^{TU} & \sigma_{GH}^{VW} & \sigma_{GH}^{XY} \end{vmatrix} = 0 \quad 4.14$$

where  $\lambda^{RS, TU, VW}$  and  $\lambda_{AB, CD, EF}$  are effectively the three-dimensional Levi-Civita symbols, being defined up to proportionality by the skew-symmetry in their pairs of indices,  $k$  being any convenient constant.

The binor equivalents of (4.14) must also hold, where the expressions now have pairs of skew-symmetrical indices:

$$\begin{vmatrix} S_{\alpha\beta,\rho\sigma} S_{\alpha\beta,\tau\nu} S_{\alpha\beta,\phi\chi} \\ S_{\gamma\delta,\rho\sigma} S_{\gamma\delta,\tau\nu} S_{\gamma\delta,\phi\chi} \\ S_{\epsilon\zeta,\rho\sigma} S_{\epsilon\zeta,\tau\nu} S_{\epsilon\zeta,\phi\chi} \end{vmatrix} = k L_{\rho\sigma,\tau\nu,\phi\chi} L_{\alpha\beta,\gamma\delta,\epsilon\zeta}, \quad \begin{vmatrix} S_{\alpha\beta,\rho\sigma} S_{\alpha\beta,\tau\nu} S_{\alpha\beta,\phi\chi} S_{\alpha\beta,\gamma\delta} \\ S_{\gamma\delta,\rho\sigma} S_{\gamma\delta,\tau\nu} S_{\gamma\delta,\phi\chi} S_{\gamma\delta,\epsilon\zeta} \\ S_{\epsilon\zeta,\rho\sigma} S_{\epsilon\zeta,\tau\nu} S_{\epsilon\zeta,\phi\chi} S_{\epsilon\zeta,\gamma\delta} \\ S_{\eta\theta,\rho\sigma} S_{\eta\theta,\tau\nu} S_{\eta\theta,\phi\chi} S_{\eta\theta,\gamma\delta} \end{vmatrix} = 0. \quad 4.15$$

It follows from (2.11) that the binor representing the spin network vertex at which three 2-units come together, is skew-symmetrical

in its three pairs of skew-symmetrical indices. Thus, from (3.25), we can put

$$L_{\alpha\beta,\gamma\delta,\epsilon\zeta} = S_{\alpha\beta,\mu\nu} S_{\gamma\delta,\nu\zeta} S_{\epsilon\zeta,\xi\mu} \quad 4.16$$

from which it then turns out that  $\kappa = -\frac{1}{4}$ . These relations will be needed later.

More generally, we can consider  $n$ -tuples of symmetrical capital indices and, equivalently,  $n$ -tuples of skew-symmetrical binor indices. The role of the Kronecker delta is taken over, respectively, by

$$\frac{1}{n!} \sigma_{A \dots D}^{R \dots T} \quad \text{and} \quad \frac{1}{n!} S_{\alpha \dots \delta, \rho \dots \tau} \quad 4.17$$

in the two cases. (Assuming  $(A, \dots, D; \alpha, \dots, \delta)$  are each  $n$  in number). Since the symmetrical expressions  $\Theta_{A \dots D}^R$  (not constructed from  $\epsilon$ 's and  $\delta$ 's) form an  $(n+1)$ -dimensional vector space, it follows that the trace of the corresponding "Kronecker delta" must be  $n+1$ , whence

$$\sigma_{A \dots D}^{A \dots D} = (n+1) \cdot n! = (n+1)! \quad 4.18$$

Thus

$$S_{\alpha \dots \delta, \alpha \dots \delta} = (-1)^n (n+1)! \quad 4.19$$

(The contraction can be performed diagrammatically with no additional intersection lines if  $n+1$   $\epsilon$ 's and  $n+1$   $(-\epsilon)$ 's are used. Hence the factor  $(-1)^n$ .) (An alternative derivation of (4.19) is given in section 5.) The convention (2.12) is now seen to be consistent with (3.26) and (3.30).

It will be necessary, in section 6, to be able to treat approximate binor equations. The notation for binors

$$A_{\alpha \dots \gamma} \sim B_{\alpha \dots \gamma}$$

4.20

will be used to denote

$$|\text{norm}(A_{\alpha \dots \gamma} - B_{\alpha \dots \gamma})| \ll |\text{norm}(A_{\alpha \dots \gamma})| \quad 4.21$$

4.21

In the discussion given earlier, which was used to prove (3.20), (where " $\ll$ " means that the ratio of the left-hand to the right-hand expression is very small). Let the  $\delta, \epsilon$  expressions  $\alpha_{A \dots c}, \beta_{A \dots c}$  correspond to  $A_{\alpha \dots \gamma}, B_{\alpha \dots \gamma}$ , respectively. In the discussion given earlier, which was used to prove (3.20), it emerged that  $|\text{norm}(A \dots)|$  was simply the sum of the squares of the components of  $\alpha \dots$ . Thus (4.21) states that the "vectors"  $\alpha_{A \dots c}$  and  $\beta_{A \dots c}$  in the appropriate  $2^r$  dimensional space are very nearly equal in direction and have magnitudes whose ratio is very nearly unity. It follows that (4.20) is symmetric:

$$A \dots \sim B \dots \Rightarrow B \dots \sim A \dots ;$$

4.22

transitive:

$$A \dots \sim B \dots, B \dots \sim C \dots \Rightarrow A \dots \sim C \dots ; \quad 4.23$$

4.23

and appropriately additive:

$$A \dots \sim B \dots, C \dots \sim D \dots, A \dots + C \dots \Rightarrow A \dots + C \dots \sim B \dots + D \dots . \quad 4.24$$

4.24

Also, suppose the elements of the matrix  $P_{\alpha \dots \gamma, \lambda \dots \nu}$  all have moduli less than or equal to unity, i.e.

Now suppose the binor  $P_{\alpha \dots \gamma, \lambda \dots \nu}$  (with the same number of indices in each of its two groups) is such that for any binor  $X_{\alpha \dots \gamma}$ , we have

$$|\text{norm}(P_{\alpha \dots \gamma, \lambda \dots \nu} X_{\lambda \dots \nu})| \leq n |\text{norm}(X_{\alpha \dots \gamma})| \quad 4.25$$

4.25

42)

where  $n$  is fixed independently of  $X \dots$  and is chosen to be not large compared with unity. (In fact  $n=2$  will do.) Then

$$P_{\alpha-\gamma, \lambda-\nu} A_{\lambda-\nu} \sim A_{\alpha-\gamma} \sim Q_{\alpha-\gamma, \lambda-\nu} A_{\lambda-\nu}$$

note that  $A \sim B \Rightarrow P_A \sim P_B$

$$\Rightarrow P_{\alpha-\gamma, \lambda-\nu} Q_{\lambda-\nu, \rho-\tau} A_{\rho-\tau} \sim A_{\alpha-\gamma} \quad 4.26$$

To prove this, assume the left-hand side of the implication holds and substitute  $Q_{\lambda-\nu, \rho-\tau} A_{\rho-\tau} - A_{\lambda-\nu}$  for  $X_{\lambda-\nu}$  in (4.25). Then

$$|\text{norm}(P_{\alpha-\gamma, \lambda-\nu} Q_{\lambda-\nu, \rho-\tau} A_{\rho-\tau} - P_{\alpha-\gamma, \lambda-\nu} A_{\lambda-\nu})| \leq n |\text{norm}(Q_{\lambda-\nu, \rho-\tau} A_{\rho-\tau} - A_{\lambda-\nu})| \\ \leq |\text{norm}(A_{\lambda-\nu})|$$

and also

$$|\text{norm}(P_{\alpha-\gamma, \lambda-\nu} A_{\lambda-\nu} - A_{\alpha-\gamma})| \leq |\text{norm}(A_{\lambda-\nu})|.$$

Combining these two results we get the right-hand side of (4.26), as required.

Another implication of the binor to  $\delta, \epsilon$  correspondence is Schwarz' inequality for binors:

$$A_{\alpha-\gamma} A_{\alpha-\gamma} B_{\lambda-\nu} B_{\lambda-\nu} \geq A_{\alpha-\gamma} B_{\alpha-\gamma} A_{\lambda-\nu} B_{\lambda-\nu}. \quad 4.27$$

Both sides are positive (see (3.20)),  $A \dots$  and  $B \dots$  having the same number of indices. Equality in (4.27) implies  $A \dots$  and  $B \dots$  are proportional. Thus approximate equality implies approximate proportionality:

~~$$A_{\alpha-\gamma} A_{\alpha-\gamma} B_{\lambda-\nu} B_{\lambda-\nu} \geq A_{\alpha-\gamma} B_{\alpha-\gamma} A_{\lambda-\nu} B_{\lambda-\nu}$$~~

$$\text{norm}(A \dots) \text{norm}(B \dots) \sim (A_{\alpha-\gamma} B_{\alpha-\gamma})^2 \Rightarrow A_{\alpha-\gamma} \sim \kappa B_{\alpha-\gamma} \quad 4.28$$

for some  $\kappa$ . (To see this, we may argue from continuity, after first normalizing  $A \dots$  and  $B \dots$ , if desired.)

Finally, the relevance of the binor calculus to physics must be established, namely (2.1) and the more general probability law (2.2). Consider a spin network representing some physical situation. It will be convenient to draw the spin networks and the corresponding binor on a plane according to the conventions of (4.6), (4.7) and (4.8). (Of course this implies no such restriction on the actual physical situation represented!) The time-ordering of events is conceived of as proceeding from the bottom of the diagram to the top. Thus, the incoming end units are those depicted as entering at the bottom of the diagram and the outgoing end units, those depicted leaving at the top. It will simplify considerations here if, moreover, we can assume that all the incoming units have spin number zero. Then all the strand ends occur at the top of the diagram. In fact this does not affect the generality of the situations considered since each incoming unit with non-zero spin number  $s$  can be considered as having originated from a 0-unit as one of a pair of  $s$ -units, ~~connected from a 0-unit~~, the other of which plays no further part in the interaction scheme. In terms of diagrams, since 0-unit lines can be ignored, this has the effect that free ends are turned away from the bottom of a diagram and they terminate instead at the top.

44) The initial wave function corresponds to zero angular momentum and can therefore be represented by a scalar  $\psi$ . Suppose, next, two s-units are produced out of this zero angular momentum state. Each s-unit individually would have a wave function which could be represented as a symmetric s-index spinor, but we must consider the two together here. Thus the wave function for the two s-units together can be represented as a spinor  $\psi^{A-C, D-F}$  which is symmetrical in each of its two sets of s indices. However  $\psi^{A-C, D-F}$  represents a state with zero total angular momentum and it therefore still amounts effectively to a scalar. Hence

$$\psi^{A-C, D-F} = \phi \epsilon^{AR} \dots \epsilon^{CT} \sigma_{R\dots T}^{D\dots F} \quad (4.29)$$

where the  $\sigma_{\dots \dots}$  factor (c.f.(4.12)) ensures symmetry in both A, ..., c and D, ..., F. We can incorporate the introduction of further pairs of s-units, t-units, etc. by extending (4.29) to include further factors  $\epsilon^{KV} \dots \epsilon^{MW} \sigma_{V\dots W}^{N\dots P}$ , etc., one for each pair of units introduced. Thus we have a wave function

$\psi^{A-C, \dots, K-M, \dots}$  which has a group of s symmetrical spinor indices for each s-unit which is present at this stage and which, apart from a scalar multiplier is the s, e expression corresponding to the (rather trivial) spin network so far (see example in fig.22).

$$\psi_{ABC,DEF,G,HJ,K,LM} \leftrightarrow \chi \begin{array}{c} ABC \\ DEF \\ G HJ \\ K \\ LM \end{array} = \chi \begin{array}{c} \frac{1}{3} & \frac{1}{3} & \frac{1}{2} & \frac{1}{2} \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 3 & 2 & 2 \end{array}$$

Fig.22

The required "incoming" units (here a 3-unit, a 1-unit and a 2-unit) are considered to have arisen initially from 0-units. The wave function  $\psi_{\dots}$  is then a scalar  $\chi$ , times the S,E expression which corresponds to the spin network, or bivector diagram, representing the production of these units.

Next, suppose that an r-unit and an s-unit combine together to form a t-unit. Let the symmetrical sets of indices D..J and P..U in  $\psi_{\dots, D..J, \dots, P..U, \dots}$  correspond, respectively, to this r-unit and this s-unit. The production of the t-unit corresponds to the reduction of the spinor - with respect to these two sets of indices only - to one which, for fixed values of the remaining indices, is irreducible, corresponding to the spin value  $t = \frac{1}{2}h$ . Thus, the new wave function is proportional to

$$\psi_{\dots, D..FG..J, \dots, P..RS..U, \dots} \propto \epsilon_{GP} \dots \epsilon_{JR} \sigma_{D..FS..U}^{X\dots Z} \quad (4.30)$$

where the symmetrical set  $X\dots Z$  corresponds to the new t-unit. The corresponding similar <sup>(right-hand)</sup> expression is then contracted on, for any other pair of units which combine together. Conversely, whenever an r-unit splits up into two other units, the process is just the reverse of (4.30). Thus, if

46) the symmetrical set D...J corresponds to the r-unit, the new wave function is proportional to

$$\psi \dots, D \dots F G \dots J, \dots \in^{NQ} \dots \in^{PS} \sigma_{D \dots F N \dots P}^U \sigma_{G \dots J Q \dots S}^X \dots Z \quad (4.31)$$

where the symmetrical sets U...W and X...Z correspond to the two new units appearing. This process is illustrated in fig. 23. The final wave function is seen to be, apart

$$((\psi_{ABC, DEF, G, HJ, K, LM} \in_{FH} \sigma_{DEJ}^{NPG}) \in_{RS} \sigma_{SPQ}^{TU} \sigma_{VWX}^{YZ}) \in_{AG} \sigma_{VW}^{YZ}$$

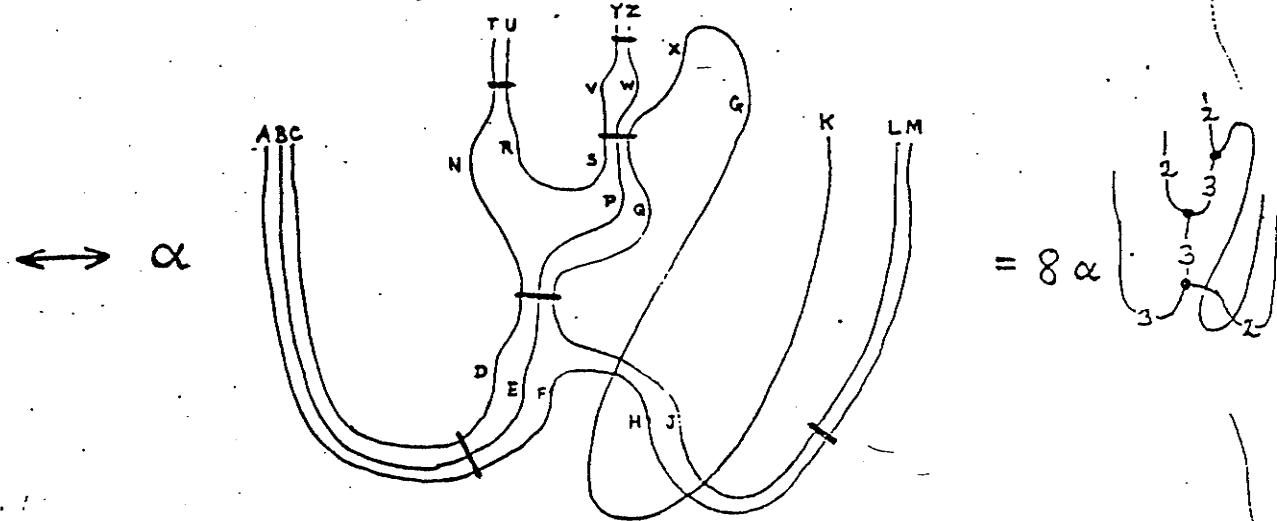


fig. 23

To the initial wave function given in fig 22 is applied a succession of three operations. The first represents the combining of a 3-unit and a 2-unit to form a 3-unit, the second represents a subsequent splitting of this 3-unit into a 2-unit and another 3-unit which combines in the final operation with a 1-unit to form a 2-unit. The final wave function is proportional to the S, e expression which corresponds to the relevant spin network.

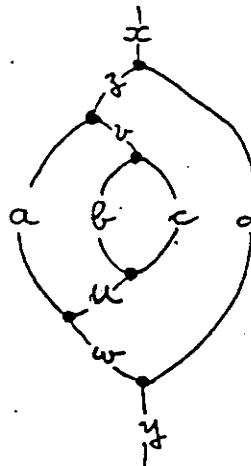
on a non-zero scalar multiplier, simply the expression corresponding to the binor of the relevant spin networks. It follows that this resulting binor vanishes if and only if the corresponding physical situation has zero probability. This establishes (3.22) and hence also (2.1).

In order to establish the more general probability law (2.2) some simple properties of spin networks binors must be noted. First, <sup>(the binor of)</sup> any spin network which has just two end units — an  $r$ -unit and an  $s$ -unit — must necessarily vanish unless  $r=s$ . This is conservation of total angular momentum, but it is also easy enough to see directly from the expansion of the  $S_{...,...}^r$ 's for the binor. (If  $r > s$ , and only the  $S_{...,...}$  belonging to the end  $r$ -unit is not expanded out, then it follows that every term involves  $S_{...,...a...a...} = 0$ .) If  $r=s$ , then similar reasoning shows that the binor must be a multiple of  $S_{...,...}$  — the binor of an isolated  $r$ -unit.

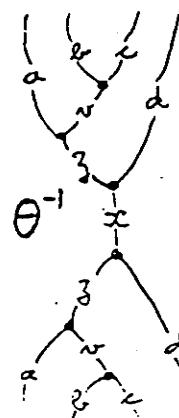
~~It follows from this that~~ Now, consider any spin network  $\xi$  which is a tree, with end units with respective spin numbers  $a, b, \dots, d, x$ . Consider, also, a similar spin network  $\eta$ , which has the same network structure <sup>as  $\xi$</sup>  and corresponding end units with spin numbers  $a, b, \dots, d, y$ . Join  $\xi$  to  $\eta$  at the corresponding  $a$ -unit,  $b$ -unit,  $\dots$ ,  $d$ -unit leaving only the  $x$ -unit and the  $y$ -unit as end units. Then it follows from the above that the

49)

resulting binor must vanish unless  $\xi$  and  $\eta$  are, in fact, identical (see fig. 24), in which case it is, say,  $\Theta$  times the binor of an isolated  $x$ -unit. Consider, next, the spin network  $\Phi$  which is obtained from two copies of  $\alpha$  by joining along the  $x$ -unit only. Denote the corresponding



$$\text{d} = \Theta \delta_{xy} \delta_{zw} u v$$



$$\Theta^{-1} \xi^{-1} = \Theta^{-2}$$

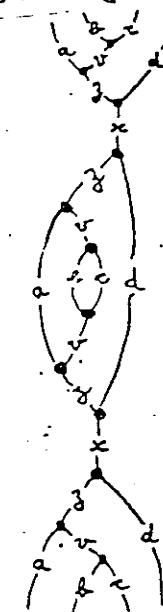


fig.24

The equation on the left expresses conservation of total angular momentum in a simple case. The right-hand expression is effectively an idempotent.

binor by  $F_{\alpha \dots s, \kappa \dots v}$  where  $\alpha, \dots, s$  correspond to one set of end  $a, b, \dots, d$ -units and  $\kappa, \dots, v$  to the other set taken in the corresponding order. From the above considerations it follows that

$$F_{\alpha \dots s, \kappa \dots v} F_{\kappa \dots v, \pi \dots \tau} = \zeta F_{\alpha \dots s, \pi \dots \tau} \quad 4.32$$

where

$$\zeta = a! b! \dots d! \Theta \quad 4.33$$

the factorials arising because of convention (3.28). We can evaluate  $\Theta$  by contracting the left-hand expression in fig. 24.

result being

$$\theta = \frac{\|\xi\|}{x+1}$$

4-34

(see (2.12)). (The explicit evaluation of  $\|\xi\|$  can, in fact, be easily achieved; see section 5.)

From (4.32), we see that  $\zeta^{-1} F_{\alpha-s, \kappa-v}$  is an idempotent. Also, the <sup>various</sup> corresponding idempotents obtained by changing the internal spin number values must all be orthogonal to each other because of the above remarks. Denoting the binor of the known spin network  $K$  (see section 2; fig.4) by  $K_{\alpha-s; p, r}$  (where  $p, r$  correspond to end units not connected to  $\xi$ ), it follows that the probability of the spin number values given by  $\xi$  is

$$\frac{|K_{\alpha-s; p, r} : (\zeta^{-1} F_{\alpha-s, \kappa-v}) K_{\kappa-v; p, r}|}{|K_{\alpha-s; p, r} K_{\alpha-s; p, r}|}$$

4-35

Now,  $|K_{\alpha-s; p, r} F_{\alpha-s, \kappa-v} K_{\kappa-v; p, r}| = a! b! \dots d! / \text{norm}(G_{\alpha-s; p, r})$ , where  $G_{\alpha-s; p, r}$  is the binor of the spin network  $w$  (see section 2; fig.4) which is the known network  $K$  with the unknown "tree" part  $\xi$  attached. The result (2.2) now follows from (4.35) by (4.34), (3.19), (3.18).