Higher-Dimensional Algebra
and Planck-Scale Physics

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Abstract

This is a nontechnical introduction to recent work on quantum gravity using ideas from higher-dimensional algebra. We argue that reconciling general relativity with the Standard Model requires a 'background-free quantum theory with local degrees of freedom propagating causally'. We describe the insights provided by work on topological quantum field theories such as quantum gravity in 3-dimensional spacetime. These are background-free quantum theories lacking local degrees of freedom, so they only display some of the features we seek. However, they suggest a deep link between the concepts of 'space' and 'state', and similarly those of 'spacetime' and 'process', which we argue is to be expected in any background-free quantum theory. We sketch how higher-dimensional algebra provides the mathematical tools to make this link precise. Finally, we comment on attempts to formulate a theory of quantum gravity in 4-dimensional spacetime using 'spin networks' and 'spin foams'.

1 Introduction

At present our physical worldview is deeply schizophrenic. We have, not one, but two fundamental theories of the physical universe: general relativity, and the Standard Model of particle physics based on quantum field theory. The former takes gravity into account but ignores quantum mechanics, while the latter takes quantum mechanics into account but ignores gravity. In other words, the former recognizes that spacetime is curved but neglects the uncertainty principle, while the latter takes the uncertainty principle into account but pretends that spacetime is flat. Both theories have been spectacularly successful in their own domain, but neither can be anything more than an approximation to the truth. Clearly some synthesis is needed: at the very least, a theory of *quantum gravity*, which might or might not be part of an overarching 'theory of everything'. Unfortunately, attempts to achieve this synthesis have not yet succeeded.

Modern theoretical physics is difficult to understand for anyone outside the subject. Can philosophers really contribute to the project of reconciling general relativity and quantum field theory? Or is this a technical business best left to the experts? I would argue for the former. General relativity and quantum field theory are based on some profound insights about the nature of reality. These insights are crystallized in the form of mathematics, but there is a limit to how much progress we can make by just playing around with this mathematics. We need to go back to the insights behind general relativity and quantum field theory, learn to hold them together in our minds, and dare to imagine a world more strange, more beautiful, but ultimately more *reasonable* than our current theories of it. For this daunting task, philosophical reflection is bound to be of help.

However, a word of warning is in order. The paucity of experimental evidence concerning quantum gravity has allowed research to proceed in a rather unconstrained manner, leading to divergent schools of opinion. If one asks a string theorist about quantum gravity, one will get utterly different answers than if one asks someone working on loop quantum gravity or some other approach. To make matters worse, experts often fail to emphasize the difference between experimental results, theories supported by experiment, speculative theories that have gained a certain plausibility after years of study, and the latest fads. Philosophers must take what physicists say about quantum gravity with a grain of salt.

To lay my own cards on the table, I should say that as a mathematical physicist with an interest in philosophy, I am drawn to a strand of work that emphasizes 'higher-dimensional algebra'. This branch of mathematics goes back and reconsiders some of the presuppositions that mathematicians usually take for granted, such as the notion of equality [8] and the emphasis on doing mathematics using 1-dimensional strings of symbols [12, 19]. Starting in the late 1980s, it became apparent that higher-dimensional algebra is the correct language to formulate so-called 'topological quantum field theories' [7, 20, 30]. More recently, various people have begun to formulate theories of quantum gravity using ideas from higher-dimensional algebra [6, 11, 16, 22, 23]. While they have tantalizing connections to string theory, these theories are best seen as an outgrowth of loop quantum gravity [24].

The plan of the paper is as follows. In Section 2, I begin by recalling why some physicists expect general relativity and quantum field theory to collide at the Planck length. This is a unit of distance concocted from three fundamental constants: the speed of light c, Newton's gravitational constant G, and Planck's constant $\hbar$. General relativity idealizes reality by treating Planck's constant as negligible, while quantum field theory idealizes it by treating Newton's gravitational constant as negligible. By analyzing the physics of c, G, and $\hbar$, we get a glimpse of the sort of theory that would be needed to deal with situations where these idealizations break down. In particular, I shall argue that we need a *background-free quantum theory with local degrees of freedom propagating causally.*

In Section 3, I discuss 'topological quantum field theories'. These are the first examples of background-free quantum theories. However, they lack local degrees of freedom. In other words, they describe imaginary worlds in which everywhere looks like everywhere else! This might at first seem to condemn them to the status of mathematical curiosities. However, they suggest an important analogy between the mathematics of spacetime and the mathematics of quantum theory. I argue that this is the beginning of a new bridge between general relativity and quantum field theory.

In Section 4, I describe one of the most important examples of a topological quantum field theory: the Turaev-Viro model of quantum gravity in 3-dimensional spacetime. This theory is just a warmup for the 4-dimensional case that is of real interest in physics. Nonetheless, it has some startling features which perhaps hint at the radical changes in our worldview that a successful synthesis of general relativity and quantum field theory would require.

In Section 5, I discuss the role of higher-dimensional algebra in topological quantum field theory. I begin with a brief introduction to categories. Category theory can be thought of as an attempt to treat processes (or 'morphisms') on an equal footing with things (or 'objects'), and it is ultimately for this reason that it serves as a good framework for topological quantum field theory. In particular, category theory allows one to make the analogy between the mathematics of spacetime and the mathematics of quantum theory quite precise. But to fully explore this analogy one must introduce 'n-categories', a generalization of categories that allows one to speak of processes between processes between processes... and so on to the nth degree. Since n-categories are purely algebraic structures but have a natural relationship to the study of n-dimensional spacetime, their study is sometimes called 'higher-dimensional algebra'.

Finally, in Section 6 I briefly touch upon recent attempts to construct theories of 4-dimensional quantum gravity using higher-dimensional algebra. This subject is still in its infancy. Throughout the paper, but especially in this last section, the reader must turn to the references for details. To make the bibliography as useful as possible, I have chosen references of an expository nature whenever they exist, rather than always citing the first paper in which something was done.

2 The Planck Length

Two constants appear throughout general relativity: the speed of light c and Newton's gravitational constant G. This should be no surprise, since Einstein created general relativity to reconcile the success of Newton's theory of gravity, based on instantaneous action at a distance, with his new theory of special relativity, in which no influence travels faster than light. The constant c also appears in quantum field theory, but paired with a different partner: Planck's constant $\hbar$. The reason is that quantum field theory takes into account special relativity and quantum theory, in which $\hbar$ sets the scale at which the uncertainty principle becomes important.

It is reasonable to suspect that any theory reconciling general relativity and quantum theory will involve all three constants c, G, and $\hbar$. Planck noted that apart from numerical factors there is a unique way to use these constants to define units of length, time, and mass. For example, we can define the unit of length now called the 'Planck length' as follows:

$$\ell_P = \sqrt{\frac{\hbar G}{c^3}}.$$

This is extremely small: about $1.6 \cdot 10^{-35}$ meters. Physicists have long suspected that quantum gravity will become important for understanding physics at about this scale. The reason is very simple: any calculation that predicts a length using only the constants c, G and $\hbar$ must give the Planck length, possibly multiplied by an unimportant numerical factor like 2π.

For example, quantum field theory says that associated to any mass m there is a length called its Compton wavelength, ℓ_C, such that determining the position of a particle of mass m to within one Compton wavelength requires enough energy to create another particle of that mass. Particle creation is a quintessentially quantum-field-theoretic phenomenon. Thus we may say that the Compton wavelength sets the distance scale at which quantum field theory becomes crucial for understanding the behavior of a particle of a given mass. On the other hand, general relativity says that associated to any mass m there is a length called the Schwarzschild radius, ℓ_S, such that compressing an object of mass m to a size smaller than this results in the formation of a black hole. The Schwarzschild radius is roughly the distance scale at which general relativity becomes crucial for understanding the behavior of an object of a given mass. Now, ignoring some numerical factors, we have

$$\ell_C = \frac{\hbar}{mc}$$

and

$$\ell_S = \frac{Gm}{c^2}.$$

These two lengths become equal when m is the Planck mass. And when this happens, they both equal the Planck length!

At least naively, we thus expect that *both* general relativity and quantum field theory would be needed to understand the behavior of an object whose mass is about the Planck mass and whose radius is about the Planck length. This not only explains some of the importance of the Planck scale, but also some of the difficulties in obtaining experimental evidence about physics at this scale. Most of our information about general relativity comes from observing heavy objects like planets and stars, for which $\ell_S \gg \ell_C$. Most of our information about quantum field theory comes from observing light objects like electrons and protons, for which $\ell_C \gg \ell_S$. The Planck mass is intermediate between these: about the mass of a largish cell. But the Planck length is about 10^{-20} times the radius of a proton! To study a situation where both general relativity and quantum field theory are important, we could try to compress a cell to a size 10^{-20} times that of a proton. We know no reason why this is impossible in principle. But we have no idea how to actually accomplish such a feat.

There are some well-known loopholes in the above argument. The 'unimportant numerical factor' I mentioned above might actually be very large, or very small. A theory of quantum gravity might make testable predictions of dimensionless quantities like the ratio of the muon and electron masses. For that matter, a theory of quantum gravity might involve physical constants other than c, G, and $\hbar$. The latter two alternatives are especially plausible if we study quantum gravity as part of a larger theory describing other forces and particles. However, even though we cannot prove that the Planck length is significant for quantum gravity, I think we can glean some wisdom from pondering the constants c, G, and $\hbar$ — and more importantly, the physical insights that lead us to regard these constants as important.

What is the importance of the constant c? In special relativity, what matters is the appearance of this constant in the Minkowski metric

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

which defines the geometry of spacetime, and in particular the lightcone through each point. Stepping back from the specific formalism here, we can see several ideas at work. First, space and time form a unified whole which can be thought of geometrically. Second, the quantities whose values we seek to predict are localized. That is, we can measure them in small regions of spacetime (sometimes idealized as points). Physicists call such quantities 'local degrees of freedom'. And third, to predict the value of a quantity that can be measured in some region R, we only need to use values of quantities measured in regions that stand in a certain geometrical relation to R. This relation is called the 'causal structure' of spacetime. For example, in a relativistic field theory, to predict the value of the fields in some region R, it suffices to use their values in any other region that intersects every timelike path passing through R. The common way of summarizing this idea is to say that nothing travels faster than light. I prefer to say that a good theory of physics should have *local degrees of freedom propagating causally.*

In Newtonian gravity, G is simply the strength of the gravitational field. It takes on a deeper significance in general relativity, where the gravitational field is described in terms of the curvature of the spacetime metric. Unlike in special relativity, where the Minkowski metric is a 'background structure' given a priori, in general relativity the metric is treated as a field which not only affects, but also is affected by, the other fields present. In other words, the geometry of spacetime becomes a local degree of freedom of the theory. Quantitatively, the interaction of the metric and other fields is described by Einstein's equation

$$G_{\mu\nu} = 8\pi G T_{\mu\nu},$$

where the Einstein tensor $G_{\mu\nu}$ depends on the curvature of the metric, while the stress-energy tensor $T_{\mu\nu}$ describes the flow of energy and momentum due to all the other fields. The role of the constant G is thus simply to quantify how much the geometry of spacetime is affected by other fields. Over the years, people have realized that the great lesson of general relativity is that a good theory of physics should contain no geometrical structures that affect local degrees of freedom while remaining unaffected by them. Instead, all geometrical structures — and in particular the causal structure — should themselves be local degrees of freedom. For short, one says that the theory should be *background-free*.

The struggle to free ourselves from background structures began long before Einstein developed general relativity, and is still not complete. The conflict between Ptolemaic and Copernican cosmologies, the dispute between Newton and Leibniz concerning absolute and relative motion, and the modern arguments concerning the 'problem of time' in quantum gravity — all are but chapters in the story of this struggle. I do not have room to sketch this story here, nor even to make more precise the all-important notion of 'geometrical structure'. I can only point the reader towards the literature, starting perhaps with the books by Barbour [9] and Earman [15], various papers by Rovelli [25, 26, 27], and the many references therein.

Finally, what of $\hbar$? In quantum theory, this appears most prominently in the commutation relation between the momentum p and position q of a particle:

$$pq - qp = -i\hbar,$$

together with similar commutation relations involving other pairs of measurable quantities. Because our ability to measure two quantities simultaneously with complete precision is limited by their failure to commute, $\hbar$ quantifies our inability to simultaneously know everything one might choose to know about the world. But there is far more to quantum theory than the uncertainty principle. In practice, $\hbar$ comes along with the whole formalism of complex Hilbert spaces and linear operators.

There is a widespread sense that the principles behind quantum theory are poorly understood compared to those of general relativity. This has led to many discussions about interpretational issues. However, I do not think that quantum theory will lose its mystery through such discussions. I believe the real challenge is to better understand why the mathematical formalism of quantum theory is precisely what it is. Research in quantum logic has done a wonderful job of understanding the field of candidates from which the particular formalism we use has been chosen. But what is so special about this particular choice? Why, for example, do we use complex Hilbert spaces rather than real or quaternionic ones? Is this decision made solely to fit the experimental data, or is there a deeper reason? Since questions like this do not yet have clear answers, I shall summarize the physical insight behind $\hbar$ by saying simply that a good theory of the physical universe should be a *quantum theory* — leaving open the possibility of eventually saying something more illuminating.

Having attempted to extract the ideas lying behind the constants c, G, and $\hbar$, we are in a better position to understand the task of constructing a theory of quantum gravity. General relativity acknowledges the importance of c and G but idealizes reality by treating $\hbar$ as negligibly small. From our discussion above, we see that this is because general relativity is a background-free classical theory with local degrees of freedom propagating causally. On the other hand, quantum field theory as normally practiced acknowledges c and $\hbar$ but treats G as negligible, because it is a background-dependent quantum theory with local degrees of freedom propagating causally.

The most conservative approach to quantum gravity is to seek a theory that combines the best features of general relativity and quantum field theory. To do this, we must try to find a *background-free quantum theory with local degrees of freedom propagating causally*. While this approach may not succeed, it is definitely worth pursuing. Given the lack of experimental evidence that would point us towards fundamentally new principles, we should do our best to understand the full implications of the principles we already have!

From my description of the goal one can perhaps see some of the difficulties. Since quantum gravity should be background-free, the geometrical structures defining the causal structure of spacetime should themselves be local degrees of freedom propagating causally. This much is already true in general relativity. But because quantum gravity should be a quantum theory, these degrees of freedom should be treated quantum-mechanically. So at the very least, we should develop a quantum theory of some sort of geometrical structure that can define a causal structure on spacetime.

String theory has not gone far in this direction. This theory is usually formulated with the help of a metric on spacetime, which is treated as a background structure rather than a local degree of freedom like the rest. Most string theorists recognize that this is an unsatisfactory situation, and by now many are struggling towards a background-free formulation of the theory. However, in the words of two experts [18], "it seems that a still more radical departure from conventional ideas about space and time may be required in order to arrive at a truly background independent formulation."

Loop quantum gravity has gone a long way towards developing a background-free quantum theory of the geometry of space [1, 28], but not so far when it comes to spacetime. This has made it difficult to understand dynamics, and particular the causal propagation of degrees of freedom. Work in earnest on these issues has begun only recently. One reason for optimism is the recent success in understanding quantum gravity in 3 spacetime dimensions. But to explain this, I must first say a bit about topological quantum field theory.

3 Topological Quantum Field Theory

Besides general relativity and quantum field theory as usually practiced, a third sort of idealization of the physical world has attracted a great deal of attention in the last decade. These are called topological quantum field theories, or 'TQFTs'. In the terminology of the previous section, a TQFT is a *background-free quantum theory with no local degrees of freedom*[1].

A good example is quantum gravity in 3-dimensional spacetime. First let us recall some features of *classical* gravity in 3-dimensional spacetime. Classically, Einstein's equations predict qualitatively very different phenomena depending on the dimension of spacetime. If spacetime has 4 or more dimensions, Einstein's equations imply that the metric has local degrees of freedom. In other words, the curvature of spacetime at a given point is not completely determined by the flow of energy and momentum through that point: it is an independent variable in its own right. For example, even in the vacuum, where the energy-momentum tensor vanishes, localized ripples of curvature can propagate in the form of gravitational radiation. In 3-dimensional spacetime, however, Einstein's equations suffice to completely determine the curvature at a given point of spacetime in terms of the flow of energy and momentum through that point. We thus say that the metric has no local degrees of freedom. In particular, in the vacuum the metric is flat, so every small patch of empty spacetime looks exactly like every other.

The absence of local degrees of freedom makes general relativity far simpler in 3-dimensional spacetime than in higher dimensions. Perhaps surprisingly, it is still somewhat interesting. The reason is the presence of 'global' degrees of freedom. For example, if we chop a cube out of flat 3-dimensional Minkowski space and form a 3-dimensional torus by identifying the opposite faces of this cube, we get a spacetime with a flat metric on it, and thus a solution of the vacuum Einstein equations. If we do the same starting with a larger cube, or a parallelipiped, we get a different spacetime that also satisfies the vacuum Einstein equations. The two spacetimes are *locally* indistinguishable, since locally both look just like flat Minkowski spacetime. However, they can be distinguished *globally* — for example, by measuring the volume of the whole spacetime, or studying the behavior of geodesics that wrap all the way around the torus.

Since the metric has no local degrees of freedom in 3-dimensional general relativity, this theory is much easier to quantize than the physically relevant 4-dimensional case. In the simplest situation, where we consider 'pure' gravity without matter, we obtain a background-free quantum field theory with no local degrees of freedom whatsoever: a TQFT.

I shall say more about 3-dimensional quantum gravity in Section 4. To set the stage, let me sketch the axiomatic approach to topological quantum field theory proposed by Atiyah [2]. My earlier definition of a TQFT as a 'background-free quantum field theory with no local degrees of freedom' corresponds fairly well to how physicists think about TQFTs. But mathematicians who wish to prove theorems about TQFTs need to start with something more precise, so they often use Atiyah's axioms.

An important feature of TQFTs is that they do not presume a fixed topology

[1]It would be nicely symmetrical if TQFTs involved the constants G and $\hbar$ but not c. Unfortunately I cannot quite see how to make this idea precise.

for space or spacetime. In other words, when dealing with an n-dimensional TQFT, we are free to choose any $(n-1)$-dimensional manifold to represent space at a given time[2]. Moreover given two such manifolds, say S and S', we are free to choose any n-dimensional manifold M to represent the portion of spacetime between S and S'. Mathematicians call M a 'cobordism' from S to S'. We write $M\colon S \to S'$, because we may think of M as the process of time passing from the moment S to the moment S'.

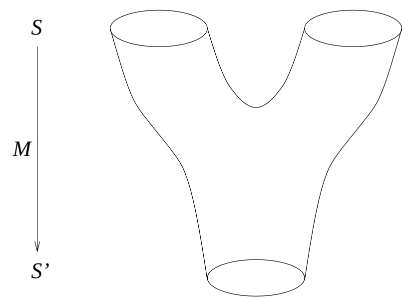

Figure 1: A cobordism

For example, in Figure 1 we depict a 2-dimensional manifold M going from a 1-dimensional manifold S (a pair of circles) to a 1-dimensional manifold S' (a single circle). Crudely speaking, M represents a process in which two separate spaces collide to form a single one! This may seem outré, but these days physicists are quite willing to speculate about processes in which the topology of space changes with the passage of time. Other forms of topology change include the formation of a wormhole, the appearance of the universe in a 'big bang', or its disappearance in a 'big crunch'.

There are various important operations one can perform on cobordisms, but I will only describe two. First, we may 'compose' two cobordisms $M\colon S \to S'$ and $M'\colon S' \to S''$, obtaining a cobordism $M'M\colon S \to S''$, as illustrated in Figure 2. The idea here is that the passage of time corresponding to M followed by the passage of time corresponding to M' equals the passage of time corresponding to $M'M$. This is analogous to the familiar idea that waiting t seconds followed by waiting t' seconds is the same as waiting $t+t'$ seconds. The big difference is that in topological quantum field theory we cannot measure time in seconds, because there is no background metric available to let us count the passage of time! We can only keep track of topology change. Just as ordinary addition is associative, composition of cobordisms satisfies the associative law:

$$(M''M')M = M''(M'M).$$

[2]Here and in what follows, by 'manifold' I really mean 'compact oriented smooth manifold', and cobordisms between these will also be compact, oriented, and smooth.

However, composition of cobordisms is not commutative. As we shall see, this is related to the famous noncommutativity of observables in quantum theory.

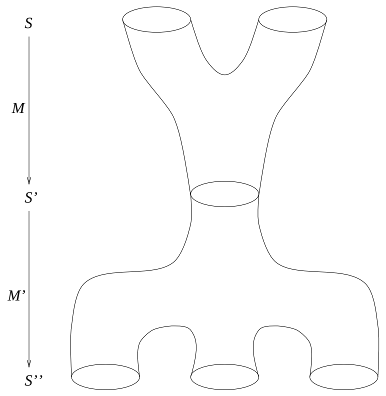

Figure 2: Composition of cobordisms

Second, for any $(n-1)$-dimensional manifold S representing space, there is a cobordism $1_S\colon S \to S$ called the 'identity' cobordism, which represents a passage of time during which the topology of space stays constant. For example, when S is a circle, the identity cobordism 1_S is a cylinder, as shown in Figure 3. In general, the identity cobordism 1_S has the property that for any cobordism $M\colon S' \to S$ we have

$$1_S M = M,$$

while for any cobordism $M\colon S \to S'$ we have

$$M 1_S = M.$$

These properties say that an identity cobordism is analogous to waiting 0 seconds: if you wait 0 seconds and then wait t more seconds, or wait t seconds and then wait 0 more seconds, this is the same as waiting t seconds.

These operations just formalize the notion of 'the passage of time' in a context where the topology of spacetime is arbitrary and there is no background metric. Atiyah's axioms relate this notion to quantum theory as follows. First, a TQFT must assign a Hilbert space $Z(S)$ to each $(n-1)$-dimensional manifold S. Vectors in this Hilbert space represent possible states of the universe given that space is the manifold S. Second, the TQFT must assign a linear operator $Z(M)\colon Z(S) \to Z(S')$ to each n-dimensional cobordism $M\colon S \to S'$. This operator describes how states change given that the portion of spacetime between S and S' is the manifold M. In other words, if space is initially the manifold S and the state of the universe is ψ, after the passage of time corresponding to M the state of the universe will be $Z(M)\psi$.

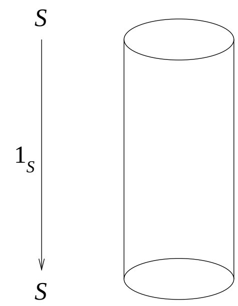

Figure 3: An identity cobordism

In addition, the TQFT must satisfy a list of properties. Let me just mention two. First, the TQFT must preserve composition. That is, given cobordisms $M: S \to S'$ and $M': S' \to S''$, we must have

$$Z(M'M) = Z(M')Z(M),$$

where the right-hand side denotes the composite of the operators $Z(M)$ and $Z(M')$. Second, it must preserve identities. That is, given any manifold S representing space, we must have

$$Z(1_S) = 1_{Z(S)}.$$

where the right-hand side denotes the identity operator on the Hilbert space $Z(S)$.

Both these axioms are eminently reasonable if one ponders them a bit. The first says that the passage of time corresponding to the cobordism M followed by the passage of time corresponding to M' has the same effect on a state as the combined passage of time corresponding to $M'M$. The second says that a passage of time in which no topology change occurs has no effect at all on the state of the universe. This seems paradoxical at first, since it seems we regularly observe things happening even in the absence of topology change. However, this paradox is easily resolved: a TQFT describes a world quite unlike ours, one without local degrees of freedom. In such a world, nothing local happens, so the state of the universe can only change when the topology of space itself changes[3].

The most interesting thing about the TQFT axioms is their common formal character. Loosely speaking, they all say that a TQFT maps structures in differential topology — by which I mean the study of manifolds — to corresponding structures in quantum theory. In coming up with these axioms, Atiyah took advantage of a

[3] Actually, while perfectly correct as far as it goes, this resolution dodges an important issue. Some physicists have suggested that the second axiom may hold even in quantum field theories *with* local degrees of freedom, so long as they are background-free [10]. Unfortunately a discussion of this would take us too far afield here.

powerful analogy between differential topology and quantum theory, summarized in Table 1.

DIFFERENTIAL TOPOLOGY	QUANTUM THEORY
$(n-1)$-dimensional manifold (space)	Hilbert space (states)
cobordism between $(n-1)$-dimensional manifolds (spacetime)	operator (process)
composition of cobordisms	composition of operators
identity cobordism	identity operator

Table 1. Analogy between differential topology and quantum theory

I shall explain this analogy between differential topology and quantum theory further in Section 5. For now, let me just emphasize that this analogy is exactly the sort of clue we should pursue for a deeper understanding of quantum gravity. At first glance, general relativity and quantum theory look very different mathematically: one deals with space and spacetime, the other with Hilbert spaces and operators. Combining them has always seemed a bit like mixing oil and water. But topological quantum field theory suggests that perhaps they are not so different after all! Even better, it suggests a concrete program of synthesizing the two, which many mathematical physicists are currently pursuing. Sometimes this goes by the name of 'quantum topology' [3, 30].

Quantum topology is very technical, as anything involving mathematical physicists inevitably becomes. But if we stand back a moment, it should be perfectly obvious that differential topology and quantum theory must merge if we are to understand background-free quantum field theories. In physics that ignores general relativity, we treat space as a background on which states of the world are displayed. Similarly, we treat spacetime as a background on which the process of change occurs. But these are idealizations which we must overcome in a background-free theory. In fact, the concepts of 'space' and 'state' are two aspects of a unified whole, and likewise for the concepts of 'spacetime' and 'process'. It is a challenge, not just for mathematical physicists, but also for philosophers, to understand this more deeply.

4 3-Dimensional Quantum Gravity

Before the late 1980s, quantum gravity was widely thought to be just as intractable in 3 spacetime dimensions as in the physically important 4-dimensional case. The situation changed drastically when physicists and mathematicians developed the tools for handling background-free quantum theories without local degrees of freedom. By now, it is easier to give a complete description of 3-dimensional quantum gravity than most quantum field theories of the traditional sort!

Let me sketch how one sets up a theory of 3-dimensional quantum gravity satisfying Atiyah's axioms for a TQFT. Before doing so I should warn the reader that there are a number of inequivalent theories of 3-dimensional quantum gravity [13]. The one I shall describe is called the Turaev-Viro model [30]. While in some ways this is not the most physically realistic one, since it is a quantum theory of Riemannian rather than Lorentzian metrics, it illustrates the points I want to make here.

To get a TQFT satisfying Atiyah's axioms we need to describe a Hilbert space of states for each 2-dimensional manifold and an operator for each cobordism between 2-dimensional manifolds. We begin by constructing a preliminary Hilbert space $\tilde{Z}(S)$ for any 2-dimensional manifold S. This construction requires choosing a background structure: a way of chopping S into triangles. Later we will eliminate this background-dependence and construct the Hilbert space of real physical interest.

To define the Hilbert space $\tilde{Z}(S)$, it is enough to specify an orthonormal basis for it. We decree that states in this basis are ways of labelling the edges of the triangles in S by numbers of the form $0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, \frac{k}{2}$. An example is shown in Figure 4, where we take S to be a sphere. Physicists call the numbers labelling the edges 'spins', alluding to the fact that we are using mathematics developed in the study of angular momentum. But here these numbers represent the *lengths* of the edges as measured in units of the Planck length. In this theory, length is a discrete rather than continuous quantity!

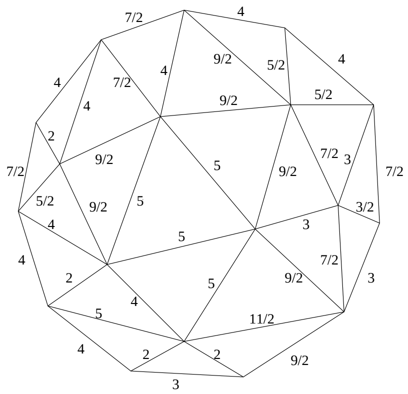

Figure 4: A state in the preliminary Hilbert space for 3-dimensional quantum gravity

Then we construct an operator $\tilde{Z}(M)\colon \tilde{Z}(S) \to \tilde{Z}(S')$ for each cobordism $M\colon S \to S'$. Again we do this with the help of a background structure on M: we choose a way to chop it into tetrahedra, whose triangular faces must include among them the triangles of S and S'. To define $\tilde{Z}(M)$ it is enough to specify the transition amplitudes $\langle \psi', \tilde{Z}(M)\psi \rangle$ when ψ and ψ' are states in the bases given above. We do this as follows. The states ψ and ψ' tell us how to label the edges of triangles in S and S' by spins. Consider any way to label the edges of M by spins that is compatible with these labellings of edges in S and S'. We can think of this as a 'quantum geometry' for spacetime, since it tells us the shape of every tetrahedron in M. Using a certain recipe we can compute a complex number for this geometry, which we think of as its 'amplitude' in the quantum-mechanical sense. We then sum these amplitudes over all such geometries to get the total transition amplitude from ψ to ψ'. The reader familiar with quantum field theory may note that this construction is a discrete version of a 'path integral'.

Now let me describe how we erase the background-dependence from this construction. Given an identity cobordism $1_S\colon S \to S$, the operator $\tilde{Z}(1_S)$ is usually not the identity, thus violating one of Atiyah's axioms for a topological quantum field theory. However, the next best thing happens: this operator maps $\tilde{Z}(S)$ onto a subspace, and it acts as the identity on this subspace. This subspace, which we call $Z(S)$, is the Hilbert space of real physical interest in 3-dimensional quantum gravity. Amazingly, this subspace doesn't depend on how we chopped S into triangles. Even better, for any cobordism $M\colon S \to S'$, the operator $\tilde{Z}(M)$ maps $Z(S)$ to $Z(S')$. Thus it restricts to an operator $Z(M)\colon Z(S) \to Z(S')$. Moreover, this operator $Z(M)$ turns out not to depend on how we chopped M into tetrahedra. To top it all off, it turns out that the Hilbert spaces $Z(S)$ and operators $Z(M)$ satisfy Atiyah's axioms.

In short, we started by chopping space into triangles and spacetime into tetrahedra, but at the end of the day nothing depends on this choice of background structure. It also turns out that the final theory has no local degrees of freedom: all the measurable quantities are global in character. For example, there is no operator on $Z(S)$ corresponding to the 'length of a triangle's edge', but there is an operator corresponding to the length of the shortest geodesic wrapping around the space S in a particular way. These miracles are among the main reasons for interest in quantum topology. They only happen because of the carefully chosen recipe for computing amplitudes for spacetime geometries. This recipe is the real core of the whole construction. Sadly, it is a bit too technical to describe here, so the reader will have to turn elsewhere for details [19, 30]. I can say this, though: the reason this recipe works so well is that it neatly combines ideas from general relativity, quantum field theory, and a third subject that might at first seem unrelated — higher-dimensional algebra.

5 Higher-Dimensional Algebra

One of the most remarkable accomplishments of the early 20th century was to formalize all of mathematics in terms of a language with a deliberately impoverished vocabulary: the language of set theory. In Zermelo-Fraenkel set theory, everything is a set, the only fundamental relationships between sets are membership and equality, and two sets are equal if and only if they have the same elements. If in Zermelo-Fraenkel set theory you ask what sort of thing is the number π, the relationship 'less than', or the exponential function, the answer is always the same: a set! Of course one must bend over backwards to think of such varied entities as sets, so this formalization may seem almost deliberately perverse. However, it represents the culmination of a worldview in which things are regarded as more fundamental than processes or relationships.

More recently, mathematicians have developed a somewhat more flexible language, the language of category theory. Category theory is an attempt to put processes and relationships on an equal status with things. A category consists of a collection of 'objects', and for each pair of objects x and y, a collection of 'morphisms' from x to y. We write a morphism from x to y as $f: x \to y$. We demand that for any morphisms $f: x \to y$ and $g: y \to z$, we can 'compose' them to obtain a morphism $gf: x \to z$. We also demand that composition be associative. Finally, we demand that for any object x there be a morphism 1_x, called the 'identity' of x, such that $f 1_x = f$ for any morphism $f: x \to y$ and $1_x g = g$ for any morphism $g: y \to x$.

Perhaps the most familiar example of a category is Set. Here the objects are sets and the morphisms are functions between sets. However, there are many other examples. Fundamental to quantum theory is the category Hilb. Here the objects are complex Hilbert spaces and the morphisms are linear operators between Hilbert spaces. In Section 3 we also met a category important in differential topology, the category nCob. Here the objects are $(n-1)$-dimensional manifolds and the morphisms are cobordisms between such manifolds. Note that in this example, the morphisms are not functions! Nonetheless we can still think of them as 'processes' going from one object to another.

An important part of learning category theory is breaking certain habits one may have acquired from set theory. For example, in category theory one must resist the temptation to 'peek into the objects'. Traditionally, the first thing one asks about a set is: what are its elements? A set is like a container, and the contents of this container are the most interesting thing about it. But in category theory, an object need not have 'elements' or any sort of internal structure. Even if it does, this is not what really matters! What really matters about an object is its morphisms to and from other objects. Thus category theory encourages a relational worldview in which things are described, not in terms of their constituents, but by their relationships to other things.

Category theory also downplays the importance of equality between objects. Given two elements of a set, the first thing one asks about them is: are they equal? But for objects in a category, we should ask instead whether they are isomorphic. Technically, the objects x and y are said to be 'isomorphic' if there is a morphism $f: x \to y$ that has an 'inverse': a morphism $f^{-1}: y \to x$ for which $f^{-1} f = 1_x$ and $f f^{-1} = 1_y$. A morphism with an inverse is called an 'isomorphism'. An isomorphism between two objects lets us turn any morphism to or from one of them into a morphism to or from the other in a reversible sort of way. Since what matters about objects are their morphisms to and from other objects, specifying an isomorphism between two objects lets us treat them as 'the same' for all practical purposes.

Categories can be regarded as higher-dimensional analogs of sets. As shown in Fig.

5, we may visualize a set as a bunch of points, namely its elements. Similarly, we may visualize a category as a bunch of points corresponding to its objects, together with a bunch of 1-dimensional arrows corresponding to its morphisms. (For simplicity, I have not drawn the identity morphisms in Fig. 5.)

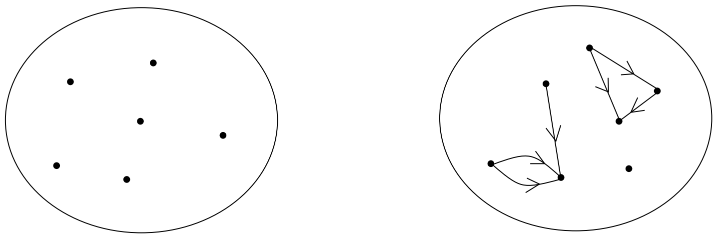

Figure 5: A set and a category

We may use the analogy between sets and categories to 'categorify' almost any set-theoretic concept, obtaining a category-theoretic counterpart [8]. For example, just as there are functions between sets, there are 'functors' between categories. A function from one set to another sends each element of the first to an element of the second. Similarly, a functor F from one category to another sends each object x of the first to an object $F(x)$ of the second, and also sends each morphism $f\colon x \to y$ of the first to a morphism $F(f)\colon F(x) \to F(y)$ of the second. In addition, functors are required to preserve composition and identities:

$$F(f'f) = F(f')F(f)$$

and

$$F(1_x) = 1_{F(x)}.$$

Functors are important because they allow us to apply the relational worldview discussed above, not just to objects in a given category, but to categories themselves. Ultimately what matters about a category is not its 'contents' — its objects and morphisms — but its functors to and from other categories!

SET THEORY	CATEGORY THEORY
elements	objects
equations between elements	isomorphisms between objects
sets	categories
functions between sets	functors between categories
equations between functions	natural isomorphisms between functors

Table 2. Analogy between set theory and category theory

We summarize the analogy between set theory and category theory in Table 2. In addition to the terms already discussed there is a concept of 'natural isomorphism' between functors. This is the correct analog of an equation between functions, but we will not need it here — I include it just for the sake of completeness.

The full impact of category-theoretic thinking has taken a while to be felt. Categories were invented in the 1940s by Eilenberg and Mac Lane for the purpose of clarifying relationships between algebra and topology. As time passed they became increasingly recognized as a powerful tool for exploiting analogies throughout mathematics [21]. In the early 1960s they led to revolutionary — and still controversial — developments in mathematical logic [17]. It gradually became clear that category theory was a part of a deeper subject, 'higher-dimensional algebra', in which the concept of a category is generalized to that of an 'n-category'. But only by the 1990s did the real importance of categories for physics become evident, with the discovery that higher-dimensional algebra is the perfect language for topological quantum field theory [14, 20].

Why are categories important in topological quantum field theory? The most obvious answer is that a TQFT is a functor. Recall from Section 3 that a TQFT maps each manifold S representing space to a Hilbert space $Z(S)$ and each cobordism $M: S \to S'$ representing spacetime to an operator $Z(M): Z(S) \to Z(S')$, in such a way that composition and identities are preserved. We may summarize all this by saying that a TQFT is a functor

$$Z: n\mathrm{Cob} \to \mathrm{Hilb}.$$

In short, category theory makes the analogy in Table 1 completely precise. In terms of this analogy, many somewhat mysterious aspects of quantum theory correspond to easily understood facts about spacetime! For example, the noncommutativity of operators in quantum theory corresponds to the noncommutativity of composing cobordisms. Similarly, the all-important 'adjoint' operation in quantum theory, which turns an operator $A: H \to H'$ into an operator $A^*: H' \to H$, corresponds to the operation of reversing the roles of past and future in a cobordism $M: S \to S'$, obtaining a cobordism $M^*: S' \to S$.

But the role of category theory goes far beyond this. The real surprise comes when one examines the details of specific TQFTs. In Section 4 I sketched the construction of 3-dimensional quantum gravity, but I left out the recipe for computing amplitudes for spacetime geometries. Thus the most interesting features of the whole business were left as unexplained 'miracles': the background-independence of the Hilbert spaces $Z(S)$ and operators $Z(M)$, and the fact that they satisfy Atiyah's axioms for a TQFT. In fact, the recipe for amplitudes and the verification of these facts make heavy use of category theory. The same is true for all other theories for which Atiyah's axioms have been verified. For some strange reason, it seems that category theory is precisely suited to explaining what makes a TQFT tick.

For the last 10 years or so, various researchers have been trying to understand this more deeply. Much remains mysterious, but it now seems that TQFTs are intimately related to category theory because of special properties of the category nCob. While nCob is defined using concepts from differential topology, a great deal of evidence suggests that it admits a simple description in terms of 'n-categories'.

I have already alluded to the concept of 'categorification' — the process of replacing sets by categories, functions by functors and so on, as indicated in Table 2. The concept of 'n-category' is obtained from the concept of 'set' by categorifying it n times! An n-category has objects, morphisms between objects, 2-morphisms between morphisms, and so on up to n-morphisms, together with various composition operations satisfying various reasonable laws [5]. Increasing the value of n allows an ever more nuanced treatment of the notion of 'sameness'. A 0-category is just a set, and in a set the elements are simply equal or unequal. A 1-category is a category, and in this context we may speak not only of equal but also of isomorphic objects. Unfortunately, this careful distinction between equality and isomorphism breaks down when we study the morphisms. Morphisms in a category are either the same or different; there is no concept of isomorphic morphisms. In a 2-category this is remedied by introducing 2-morphisms between morphisms. Unfortunately, in a 2-category we cannot speak of isomorphic 2-morphisms. To remedy this we must introduce the notion of 3-category, and so on.

We may visualize the objects of an n-category as points, the morphisms as arrows going between these points, the 2-morphisms as 2-dimensional surfaces going between these arrows, and so on. There is thus a natural link between n-categories and n-dimensional topology. Indeed, one reason why n-categories are a bit formidable is that calculations with them are most naturally done using n-dimensional diagrams. But this link between n-categories and n-dimensional topology is precisely why there may be a nice description of nCob in the language of n-categories.

Dolan and I have proposed such a description, which we call the 'cobordism hypothesis' [7]. Much work remains to be done to make this hypothesis precise and prove or disprove it. Proving it would lay the groundwork for understanding topological quantum field theories in a systematic way. But beyond this, it would help us towards a purely algebraic understanding of 'space' and 'spacetime' — which is precisely what we need to marry them to the quantum-mechanical notions of 'state' and 'process'.

6 4-Dimensional Quantum Gravity

How important are the lessons of topological quantum field theory for 4-dimensional quantum gravity? This is still an open question. Since TQFTs lack local degrees of freedom, they are at best a warmup for the problem we really want to tackle: constructing a background-free quantum theory with local degrees of freedom propagating causally. Thus, even though work on TQFTs has suggested new ideas linking quantum theory and general relativity, these ideas may be too simplistic to be useful in real-world physics.

However, physics is not done by sitting on ones hands and pessimistically pondering the immense magnitude of the problems. For decades our only insights into quantum gravity came from general relativity and quantum field theory on spacetime with a fixed background metric. Now we can view it from a third angle, that of topological quantum field theory. Surely it makes sense to invest some effort in trying to combine the best aspects of all three theories!

And indeed, in the last few years various people have begun to do just this, largely motivated by tantalizing connections between topological quantum field theory and loop quantum gravity. In loop quantum gravity, the preliminary Hilbert space has a basis given by 'spin networks' — roughly speaking, graphs with edges labelled by spins [4, 29]. We now understand quite well how a spin network describes a quantum state of the geometry of space. But spin networks are also used to describe states in TQFTs, where they arise naturally from considerations of higher-dimensional algebra. For example, in 3-dimensional quantum gravity the state shown in Fig. 4 can also be described using the spin network shown in Fig. 6.

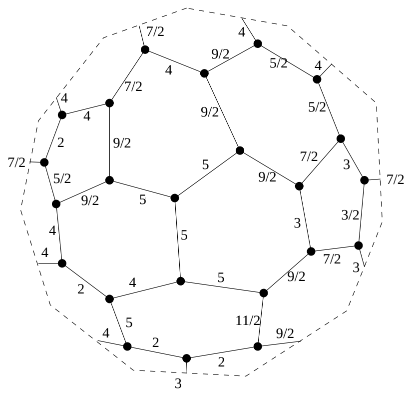

Figure 6: A spin network

Using the relationships between 4-dimensional quantum gravity and topological quantum field theory, researchers have begun to formulate theories in which the quantum geometry of spacetime is described using 'spin foams' — roughly speaking, 2-dimensional structures made of polygons joined at their edges, with all the polygons being labelled by spins [6, 11, 16, 23, 24]. The most important part of a spin foam model is a recipe assigning an amplitude to each spin foam. Much as Feynman diagrams in ordinary quantum field theory describe processes by which one collection of particles evolves into another, spin foams describe processes by which one spin network evolves into another. Indeed, there is a category whose objects are spin networks and whose morphisms are spin foams! And like nCob, this category appears to arise very naturally from purely n-categorical considerations.

In the most radical approaches, the concepts of 'space' and 'state' are completely merged in the notion of 'spin network', and similarly the concepts of 'spacetime' and 'process' are merged in the notion of 'spin foam', eliminating the scaffolding of a spacetime manifold entirely. To me, at least, this is a very appealing vision. However, there are a great many obstacles to overcome before we have a full-fledged theory of quantum gravity along these lines. Let me mention just a few of the most pressing. First there is the problem of developing quantum theories of Lorentzian rather than Riemannian metrics. Second, and closely related, we need to better understand the concept of 'causal structure' in the context of spin foam models. Only the work of Markopoulou and Smolin [22] has addressed this point so far. Third, there is the problem of formulating physical questions in these theories in such a way that divergent sums are eliminated. And fourth, there is the problem of developing computational techniques to the point where we can check whether these theories approximate general relativity in the limit of large distance scales — i.e., distances much greater than the Planck length. Starting from familiar territory we have sailed into strange new waters, but only if we circle back to the physics we know will the journey be complete.

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