Props in Network Theory

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Abstract

Long before the invention of Feynman diagrams, engineers were using similar diagrams to reason about electrical circuits and more general networks containing mechanical, hydraulic, thermodynamic and chemical components. We can formalize this reasoning using props: that is, strict symmetric monoidal categories where the objects are natural numbers, with the tensor product of objects given by addition. In this approach, each kind of network corresponds to a prop, and each network of this kind is a morphism in that prop. A network with m inputs and n outputs is a morphism from m to n, putting networks together in series is composition, and setting them side by side is tensoring. Here we work out the details of this approach for various kinds of electrical circuits, starting with circuits made solely of ideal perfectly conductive wires, then circuits with passive linear components, and then circuits that also have voltage and current sources. Each kind of circuit corresponds to a mathematically natural prop. We describe the 'behavior' of these circuits using morphisms between props. In particular, we give a new construction of the black-boxing functor of Fong and the first author; unlike the original construction, this new one easily generalizes to circuits with nonlinear components. We also use a morphism of props to clarify the relation between circuit diagrams and the signal-flow diagrams in control theory. Technically, the key tools are the Rosebrugh–Sabadini–Walters result relating circuits to special commutative Frobenius monoids, the monadic adjunction between props and signatures, and a result saying which symmetric monoidal categories are equivalent to props.

1 Introduction

In his 1963 thesis, Lawvere [29] introduced functorial semantics: the use of categories with specified extra structure as 'theories' whose 'models' are structure-preserving functors into other such categories. In particular, a 'Lawvere theory' is a category with finite cartesian products and a distinguished object X such that each object is a power X^n for some unique n. These can serve as theories of mathematical structures that are sets X equipped with n-ary operations $f\colon X^n \to X$ obeying equational laws. However, structures of a more linear-algebraic nature are often vector spaces equipped with operations of the form $f\colon X^{\otimes m} \to X^{\otimes n}$. To extend functorial semantics to these, Mac Lane [31] introduced props—or as he called them, 'PROPs', where the acronym stands for 'products and permutations'. A prop is a symmetric monoidal category equipped with a distinguished object X such that every object is a tensor power $X^{\otimes n}$ for some unique n. Working with tensor products rather than cartesian products puts operations having multiple outputs on an equal footing with operations having multiple inputs.

Already in 1949 Feynman had introduced his famous diagrams, which he used to describe theories of elementary particles [17]. For a theory with just one type of particle, Feynman's method amounts

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to specifying a prop where an operation $f\colon X^{\otimes m} \to X^{\otimes n}$ describes a process with m particles coming in and n going out. Although Feynman diagrams quickly caught on in physics [26], only in the 1980s did it become clear that they were a method of depicting morphisms in symmetric monoidal categories. A key step was the work of Joyal and Street [25], which rigorously justified reasoning in any symmetric monoidal category using 'string diagrams'—a generalization of Feynman diagrams.

By now, many mathematical physicists are aware of props and the general idea of functorial semantics. In constrast, props seem to be virtually unknown in engineering. However, long before physicists began using Feynman diagrams, engineers were using similar diagrams to describe electrical circuits. In the 1940's Olson [35] explained how to apply circuit diagrams to networks of mechanical, hydraulic, thermodynamic and chemical components. By 1961, Paynter [36] had made the analogies between these various systems mathematically precise. By 1963 Forrester [19] was using circuit diagrams in economics, and in 1984 Odum [34] published an influential book on their use in biology and ecology.

We can use props to study circuit diagrams of all these kinds. The underlying mathematics is similar in each case, so we focus on just one example: electrical circuits. The reader interested in applying props to other examples can do so with the help of various textbooks that explain the analogies between electrical circuits and other networks [11, 27].

We illustrate the usefulness of props by giving a new, shorter construction of the 'black-boxing functor' introduced by Fong and the first author [4, Thm. 1.1]. A 'black box' is a system with inputs and outputs whose internal mechanisms are unknown or ignored. We can treat an electrical circuit as a black box by forgetting its inner workings and recording only the relation it imposes between its inputs and outputs. Circuits of a given kind with inputs and outputs can be seen as morphisms in a category, where composition uses the outputs of the one circuit as the inputs of another. Black-boxing is a functor from this category to a suitable category of relations.

In an electrical circuit, associated to each wire there is a pair of variables called the potential ϕ and current I. When we black-box such a circuit, we record only the relation it imposes between these variables on its input and output wires. Since these variables come in pairs, this is a relation between even-dimensional vector spaces. But these vector spaces turn out to be equipped with extra structure: they are 'symplectic' vector spaces, meaning they are equipped with a nondegenerate antisymmetric bilinear form. Black-boxing gives a relation that respects this extra structure: it is a 'Lagrangian' relation.

Why does symplectic geometry show up when we black-box an electrical circuit? A circuit made of linear resistors acts to minimize the total power dissipated in the form of heat. More generally, any circuit made of linear resistors, inductors and capacitors obeys a generalization of this 'principle of minimum power'. Whenever a system obeys a minimum principle, it establishes a Lagrangian relation between input and output variables. This fact was first noticed in classical mechanics, where systems obey the 'principle of least action'. Indeed, symplectic geometry has its origins in classical mechanics [24, 44, 45]. But it applies more generally: for any sort of system governed by a minimum principle, black-boxing should give a functor to some category where the morphisms are Lagrangian relations [5, Sec. 13].

The first step toward proving this for electrical circuits is to treat circuits as morphisms in a suitable category. We start with circuits made only of ideal perfectly conductive wires. These are morphisms in a prop we call Circ, defined in Section 3. In Section 8 we construct a black-boxing functor

$$\blacksquare\colon \mathrm{Circ} \to \mathrm{LagRel}_k$$

sending each such circuit to the relation it defines between its input and output potentials and currents. Here LagRel_k is a prop with symplectic vector spaces of the form k^{2n} as objects and linear Lagrangian relations as morphisms, and $\blacksquare$ is a morphism of props. We work in a purely algebraic fashion, so k here can be any field.

In Section 9 we extend black-boxing to a larger class of circuits that include linear resistors, inductors and capacitors. This gives a new proof of a result of Fong and the first author: namely, there is a morphism of props

$$\blacksquare \colon \mathrm{Circ}_k \to \mathrm{LagRel}_k$$

sending each such linear circuit to the Lagrangian relation it defines between its input and output potentials and currents. The ease with which we can extend the black-boxing functor is due to the fact that all our categories with circuits as morphisms are props. We can describe these props using generators and relations, so that constructing a black-boxing functor simply requires that we choose where it sends each generator and check that all the relations hold. In Section 10 we explain how electric circuits are related to signal-flow diagrams, used in control theory. Finally, in Section 11, we illustrate how props can be used to study nonlinear circuits.

Plan of the paper

In Section 2 we explain a general notion of '$\mathcal{L}$-circuit' first introduced by Rosebrugh, Sabadini and Walters [40]. This is a cospan of finite sets where the apex is the set of nodes of a graph whose edges are labelled by elements of some set $\mathcal{L}$. In applications to electrical engineering, the elements of $\mathcal{L}$ describe different 'circuit elements' such as resistors, inductors and capacitors. We discuss a symmetric monoidal category $\mathrm{Circ}_{\mathcal{L}}$ whose objects are finite sets and whose morphisms are (isomorphism classes of) $\mathcal{L}$-circuits.

In Section 3 we consider $\mathrm{Circ}_{\mathcal{L}}$ when $\mathcal{L}$ is a 1-element set. We call this category simply Circ. In applications to engineering, a morphism in Circ describes circuit made solely of ideal conductive wires. We show how such a circuit can be simplified in two successive stages, described by symmetric monoidal functors:

$$\mathrm{Circ} \xrightarrow{G} \mathrm{FinCospan} \xrightarrow{H} \mathrm{FinCorel}.$$

Here FinCospan is the category of cospans of finite sets, while FinCorel is the category of 'corelations' between finite sets. Corelations, categorically dual to relations, are already known to play an important role in network theory [4, 14, 19, 20]. Just as a relation can be seen as a jointly monic span, a corelation can be seen as a jointly epic cospan. The functor G crushes any graph down to its set of components, while H reduces any cospan to a jointly epic one.

In Section 4 we turn to props. Propositions 11 and 12, proved in Appendix A.1 with the help of Steve Lack, characterize which symmetric monoidal categories are equivalent to props and which symmetric monoidal functors are isomorphic to morphisms of props. We use these to find props equivalent to $\mathrm{Circ}_{\mathcal{L}}$, Circ, FinCospan and FinCorel, and to reinterpret G and H as morphisms of props. In Section 5 we discuss presentations of props. Proposition 19, proved in Appendix A.2 using a result of Todd Trimble, shows that the category of props is monadic over the category of 'signatures', $\mathrm{Set}^{\mathbb{N}\times\mathbb{N}}$. This lets us work with props using generators and relations. We conclude by recalling a presentation of FinCospan due to Lack [28] and a presentation of FinCorel due to Fong and the second author [14].

In Section 6 we introduce the prop FinRel_k. This prop is equivalent to the symmetric monoidal category with finite-dimensional vector spaces over the field k as objects and linear relations as morphisms, with *direct sum* as its tensor product. A presentation of this prop was given by Erbele and the first author [3, 16], and independently by Bonchi, Sobociński and Zanasi [9, 10, 48]. In Section 7 we state a fundamental result of Rosebrugh, Sabadini and Walters [40]. This result can be seen as giving a presentation of $\mathrm{Circ}_{\mathcal{L}}$. Equivalently, it says that $\mathrm{Circ}_{\mathcal{L}}$ is the coproduct, in the category of props, of FinCospan and the free prop on a set of unary operations, one for each element of $\mathcal{L}$. This result makes it easy to construct morphisms from $\mathrm{Circ}_{\mathcal{L}}$ to other props.

In Section 8 we introduce the prop LagRel_k where morphisms are Lagrangian linear relations between symplectic vector spaces, and construct the black-boxing functor $\blacksquare \colon \mathrm{Circ} \to \mathrm{LagRel}_k$.

Mathematically, this functor is the composite

$$\mathrm{Circ} \xrightarrow{G} \mathrm{FinCospan} \xrightarrow{H} \mathrm{FinCorel} \xrightarrow{K} \mathrm{LagRel}_k$$

where K is a symmetric monoidal functor defined by its action on the generators of FinCorel. In applications to electrical engineering, the black-boxing functor maps any circuit of ideal conductive wires to its 'behavior': that is, to the relation that it imposes on the potentials and currents at its inputs and outputs.

In Section 9 we extend the black-boxing functor to circuits that include resistors, inductors, capacitors and certain other linear circuit elements. The most elegant prop having such circuits as morphisms is Circ_k, meaning $\mathrm{Circ}_{\mathcal{L}}$ with the label set $\mathcal{L}$ taken to be the field k. We characterize this black-boxing functor $\blacksquare\colon \mathrm{Circ}_k \to \mathrm{LagRel}_k$ in Theorem 41.

In Section 10 we expand the scope of inquiry to include 'signal-flow diagrams', a type of diagram used in control theory. We recall the work of Erbele and others showing that signal-flow diagrams are a syntax for linear relations [3, 16, 9, 10, 48]. Concretely, this means that signal-flow diagrams are morphisms in a free prop $\mathrm{SigFlow}_k$ with the same generators as FinRel_k, but no relations. There is thus a morphism of props

$$\square\colon \mathrm{SigFlow}_k \to \mathrm{FinRel}_k$$

mapping any signal-flow diagrams to the linear relation that it denotes. It is natural to wonder how this is related to the black-boxing functor

$$\blacksquare\colon \mathrm{Circ}_k \to \mathrm{LagRel}_k.$$

The answer involves the free prop $\widetilde{\mathrm{Circ}}_k$ which arises when we take the simplest presentation of Circ_k and omit the relations. This comes with a map $P\colon \widetilde{\mathrm{Circ}}_k \to \mathrm{Circ}_k$ which reinstates those relations, and in Theorem 44 we show there is a map of props $T\colon \widetilde{\mathrm{Circ}}_k \to \mathrm{SigFlow}_k$ making this diagram commute:

$$\begin{tikzcd}
\widetilde{\mathrm{Circ}}_k \arrow[r, "P"] \arrow[d, "T"'] & \mathrm{Circ}_k \arrow[r, "\blacksquare"] & \mathrm{LagRel}_k \arrow[d, hook] \\
\mathrm{SigFlow}_k \arrow[rr, "\square"] & & \mathrm{FinRel}_k.
\end{tikzcd}$$

Finally, in Section 11 we illustrate how props can also be used to study nonlinear circuits. Namely, we show how to include voltage and current sources. Black-boxing these gives Lagrangian *affine* relations between symplectic vector spaces.

Acknowledgements

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2 Circuits

Rosebrugh, Sabadini and Walters [40] explained how to construct a category where the morphisms are circuits made of wires with 'circuit elements' on them, such as resistors, inductors and capacitors. We use their work to construct a symmetric monoidal category $\mathrm{Circ}_{\mathcal{L}}$ for any set $\mathcal{L}$ of circuit elements.

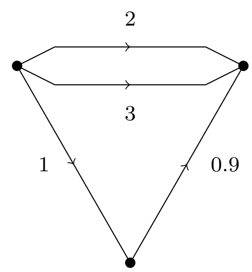
To begin with, a circuit consists of a finite graph with 'wires' as edges:

Definition 1. A **graph** is a finite set E of **edges** and a finite set N of **nodes** equipped with a pair of functions $s, t\colon E \to N$ assigning to each edge its **source** and **target**. We say that $e \in E$ is an edge **from** $s(e)$ **to** $t(e)$.

We then label the edges with elements of some set $\mathcal{L}$:

Definition 2. Given a set $\mathcal{L}$ of **labels**, an **$\mathcal{L}$-graph** is a graph $s, t\colon E \to N$ equipped with a function $\ell\colon E \to \mathcal{L}$ assigning a label to each edge.

For example, we can describe a circuit made from resistors by labeling each edge with a 'resistance' chosen from the set $\mathcal{L} = (0, \infty)$. Here is a typical $\mathcal{L}$-graph in this case:



However, to make circuits into the morphisms of a category, we need to specify some 'input' and 'output' nodes. We could do this by picking out two subsets of the set of nodes, but it turns out to be better to use two maps into this set.

Definition 3. Given a set $\mathcal{L}$ and finite sets X and Y, an **$\mathcal{L}$-circuit from X to Y** is a cospan of finite sets

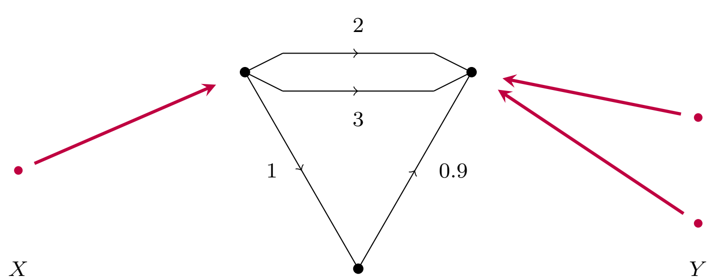
$$\begin{tikzcd} & N & \\ X \arrow[ur, "i"] & & Y \arrow[ul, "o"'] \end{tikzcd}$$

together with an $\mathcal{L}$-graph

$$\mathcal{L} \xleftarrow{\ell} E \overset{s}{\underset{t}{\rightrightarrows}} N.$$

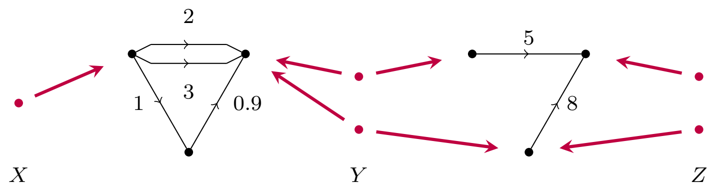
We call the sets $i(X)$, $o(Y)$, and $\partial N := i(X) \cup o(Y)$ the **inputs**, **outputs**, and **terminals** of the $\mathcal{L}$-circuit, respectively.

Here is an example of an $\mathcal{L}$-circuit:

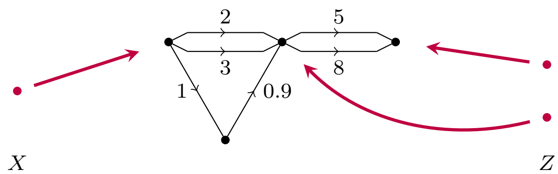


Note that in this example, two points of Y map to the same node.

We can now compose $\mathcal{L}$-circuits by gluing the outputs of one to the inputs of another. For example, we can compose these two $\mathcal{L}$-circuits:



and obtain this one:



We formalize this using composition of cospans. Given cospans $X \xrightarrow{i} N \xleftarrow{o} Y$ and $Y \xrightarrow{i'} N' \xleftarrow{o'} Z$, their composite is $X \xrightarrow{fi} N +_Y N' \xleftarrow{f'o'} Z$, where the finite set $N +_Y N'$ and the functions f and f' are defined by this pushout:

$$\begin{tikzcd}
& & N +_Y N' & & \\
& N \arrow[ur, "f"] & & N' \arrow[ul, "f'"'] & \\
X \arrow[ur, "i"] & & Y \arrow[ul, "o"'] \arrow[ur, "i'"] & & Z. \arrow[ul, "o'"']
\end{tikzcd}$$

To make the composite cospan into an $\mathcal{L}$-circuit, we must choose an $\mathcal{L}$-graph whose set of nodes is $N +_Y N'$. We use this $\mathcal{L}$-graph:

$$\begin{tikzcd}
\mathcal{L} & E + E' \arrow[l, "\langle \ell, \ell' \rangle"'] \arrow[r, shift left, "j(s+s')"] \arrow[r, shift right, "j(t+t')"'] & N +_Y N'
\end{tikzcd}$$

where

$$\langle \ell, \ell' \rangle \colon E + E' \to \mathcal{L}$$

is the 'copairing' of ℓ and ℓ, i.e., the function that equals ℓ on E and ℓ' on E', while the functions

$$s + s', t + t' \colon E + E' \to N + N'$$

are coproducts, and

$$j_{N,N'} \colon N + N' \to N +_Y N'$$

is the natural map from the coproduct to the pushout.

However, the pushout is only unique up to isomorphism, so to make composition associative we must use isomorphism classes of $\mathcal{L}$-circuits. Since an $\mathcal{L}$-circuit can be seen as a graph with an extra structure, an isomorphism between $\mathcal{L}$-circuits is an isomorphism of graphs that preserves this extra structure:

Definition 4. Two $\mathcal{L}$-circuits

$$\begin{tikzcd}
& N & \\
X \arrow[ur, "i"] & & Y \arrow[ul, "o"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{L} & E \arrow[l, "\ell"'] \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & N
\end{tikzcd}$$

and

$$\begin{tikzcd} & N' & \\ X \arrow[ur, "i'"] & & Y \arrow[ul, "o'"'] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{L} & E' \arrow[l, "\ell'"'] \arrow[r, shift left, "s'"] \arrow[r, shift right, "t'"'] & N' \end{tikzcd}$$

are **isomorphic** if there are bijections

$$f_E \colon E \to E', \quad f_N \colon N \to N'$$

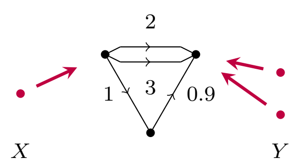
such that these diagrams commute:

$$\begin{tikzcd} & N \arrow[dd, "f_N"] & \\ X \arrow[ur, "i"] \arrow[dr, "i'"'] & & Y \arrow[ul, "o"'] \arrow[dl, "o'"] \\ & N' & \end{tikzcd} \qquad \begin{tikzcd} & E \arrow[dl, "\ell"'] \arrow[dd, "f_E"] \\ \mathcal{L} & \\ & E' \arrow[ul, "\ell'"] \end{tikzcd}$$

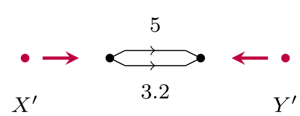
$$\begin{tikzcd} E \arrow[r, "s"] \arrow[d, "f_E"'] & N \arrow[d, "f_N"] \\ E' \arrow[r, "s'"'] & N' \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r, "t"] \arrow[d, "f_E"'] & N \arrow[d, "f_N"] \\ E' \arrow[r, "t'"'] & N' \end{tikzcd}$$

It is easy to check that composition of $\mathcal{L}$-circuits is well-defined and associative at the level of isomorphism classes.

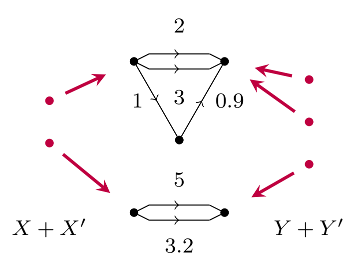
We can also 'tensor' two $\mathcal{L}$-circuits by setting them side by side. This uses coproducts, or disjoint unions, both of sets and of $\mathcal{L}$-graphs. For example, tensoring this $\mathcal{L}$-circuit:



with this one:



give this one:



In general, given $\mathcal{L}$-circuits

$$\begin{tikzcd} & N & \\ X \arrow[ur, "i"] & & Y \arrow[ul, "o"'] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{L} & E \arrow[l, "\ell"'] \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & N \end{tikzcd}$$

and

$$\begin{tikzcd} & N' & \\ X' \arrow[ur, "i'"] & & Y' \arrow[ul, "o'"'] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{L} & E' \arrow[l, "\ell'"'] \arrow[r, shift left, "s'"] \arrow[r, shift right, "t'"'] & N' \end{tikzcd}$$

their tensor product is

$$\begin{tikzcd} & N+N' & \\ X+X' \arrow[ur, "i+i'"] & & Y+Y' \arrow[ul, "o+o'"'] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{L} & E+E' \arrow[l, "\langle \ell,\ell' \rangle"'] \arrow[r, shift left, "s+s'"] \arrow[r, shift right, "t+t'"'] & N+N'. \end{tikzcd}$$

This operation is well-defined at the level of isomorphism classes. Indeed, we obtain a symmetric monoidal category:

Proposition 5. *For any set $\mathcal{L}$, there is a symmetric monoidal category $\mathbf{Circ}_{\mathcal{L}}$ where the objects are finite sets, the morphisms are isomorphism classes of $\mathcal{L}$-circuits, and composition and the tensor product are defined as above.*

Proof. This was proved by Rosebrugh, Sabadini and Walters [40], who call this category $\mathrm{Csp}(\mathrm{Graph}_{\mathrm{fin}}/\Sigma)$, where Σ is their name for the label set $\mathcal{L}$. Another style of proof uses Fong's theory of decorated cospans, which implies that $\mathrm{Circ}_{\mathcal{L}}$ is a special sort of symmetric monoidal category called a 'hypergraph category'. Fong used this method in the special case where $\mathcal{L}$ is the set of positive real numbers [21, Sec. 5.1], but the argument does not depend on the nature of the set $\mathcal{L}$. $\square$

In fact, the work of Courser [13] and Stay [42] implies that $\mathrm{Circ}_{\mathcal{L}}$ comes from a compact closed symmetric monoidal bicategory. The objects in this bicategory are still finite sets, but the morphisms in this bicategory are actual $\mathcal{L}$-circuits, not isomorphism classes. We expect this bicategorical approach to become important, but for now we are content to work with mere categories.

3 Circuits of ideal conductive wires

The simplest circuits are those made solely of ideal perfectly conductive wires. Electrical engineers would consider these circuits trivial. Nonetheless, they provide the foundation on which all our results rest. In this case the underlying graph of the circuit has unlabeled edges—or equivalently, the set of labels has a single element. So, we make the following definition:

Definition 6. Let Circ be symmetric monoidal category $\mathrm{Circ}_{\mathcal{L}}$ where $\mathcal{L} = \{\ell\}$ is a 1-element set.

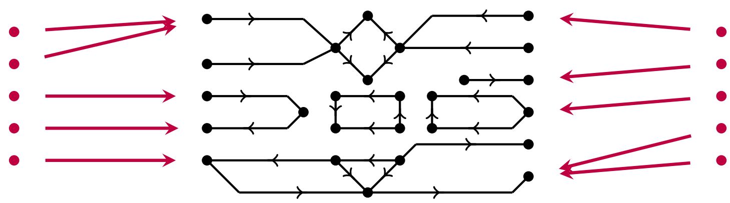
We can treat a morphism in Circ as an isomorphism class of cospans of finite sets

$$\begin{tikzcd} & N & \\ X \arrow[ur, "i"] & & Y \arrow[ul, "o"'] \end{tikzcd}$$

together with a graph Γ having N as its set of vertices:

$$\Gamma = \{\ E \overset{s}{\underset{t}{\rightrightarrows}} N\ \}$$

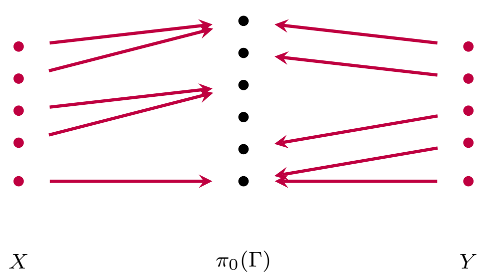
For example, a morphism in Circ might look like this:



As we shall see in detail later, for the behavior of a circuit made of ideal wires, all that matters about the underlying graph is whether any given pair of nodes lie in the same connected component or not: that is, whether or not there exists a path of edges and their reverses from one node to another. If they lie in the same component, current can flow between them; otherwise not, and this is all there is to say. We may thus replace the graph Γ by its set of connected components, $\pi_0(\Gamma)$. There is map $p_\Gamma \colon N \to \pi_0(\Gamma)$ sending each node to the connected component it lies in. We thus obtain a cospan of finite sets:

$$\begin{tikzcd}
 & \pi_0(\Gamma) & \\
X \arrow[ur, "p_\Gamma i"] & & Y \arrow[ul, "p_\Gamma o"']
\end{tikzcd}$$

In particular, the above example gives this cospan of finite sets:



This way of simplifying a circuit made of ideal wires defines a functor

$$G \colon \mathrm{Circ} \to \mathrm{FinCospan}$$

where FinCospan is the category where an object is a finite set and a morphism is an isomorphism classes of cospans. We study this functor in Example 29.

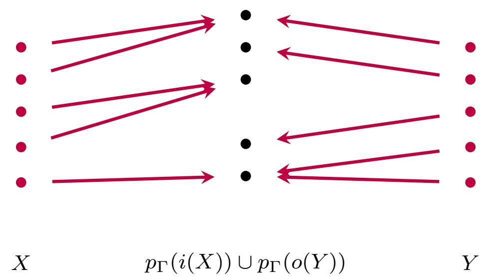
We can simplify a circuit made of ideal wires even further, because connected components of the graph G that contain no terminals—that is, no points in $i(X) \cup o(Y)$—can be discarded without affecting the behavior of the circuit. In other words, given a cospan of finite sets:

$$\begin{tikzcd}
 & S & \\
X \arrow[ur, "f"] & & Y \arrow[ul, "g"']
\end{tikzcd}$$

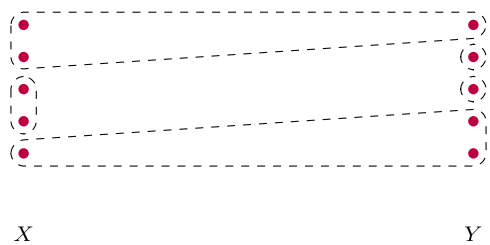
we can replace S by the subset set $f(X) \cup g(Y)$. The resulting cospan is 'jointly epic':

Definition 7. A cospan $X \xrightarrow{f} S \xleftarrow{g} Y$ in the category of finite sets is **jointly epic** if $f(X) \cup g(Y) = S$.

When we apply this second simplification process to our example, we obtain this jointly epic cospan:



A jointly epic cospan $X \xrightarrow{f} S \xleftarrow{g} Y$ determines a partition of $X+Y$, where two points $p, q \in X+Y$ are in the same block of the partition if and only if they map to the same point of S via the function $\langle f, g\rangle \colon X+Y \to S$. Moreover, two jointly epic cospans from X to Y are isomorphic, in the usual sense of isomorphism of cospans, if and only if they determine the same partition of $X+Y$. For example, the above jointly epic cospan gives this partition:



Thus, one makes the following definition:

Definition 8. Given sets X and Y, a **corelation** from X to Y is a partition of $X+Y$, or equivalently, an isomorphism class of jointly epic cospans $X \xrightarrow{f} S \xleftarrow{g} Y$.

The reason for the word 'corelation' is that a relation from X to Y corresponds, in a similar way, to an isomorphism class of jointly monic spans $X \xleftarrow{f} S \xrightarrow{g} Y$. Corelations have been studied by Lawvere and Rosebrugh [30], and Ellerman [15] used them in an approach to logic and set theory, dual to the usual approach, in which propositions correspond to partitions rather than subsets. Fong and the second author [14, 19] have continued the study of corelations, and we summarize some of their results in Example 15.

To begin with, there is a category FinCorel where the objects are finite sets and morphisms are corelations. We compose two corelations by treating them as jointly epic cospans:

$$\begin{tikzcd}
 & S & & S' & \\
X \arrow[ur, "f"] & & Y \arrow[ul, "g"'] \arrow[ur, "f'"] & & Z, \arrow[ul, "g'"']
\end{tikzcd}$$

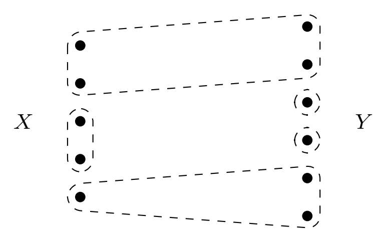
forming the pushout:

$$\begin{tikzcd}
 & & S +_Y S' & & \\
 & S \arrow[ur, "h"] & & S' \arrow[ul, "h'"'] & \\
X \arrow[ur, "f"] & & Y \arrow[ul, "g"'] \arrow[ur, "f'"] & & Z, \arrow[ul, "g'"']
\end{tikzcd}$$

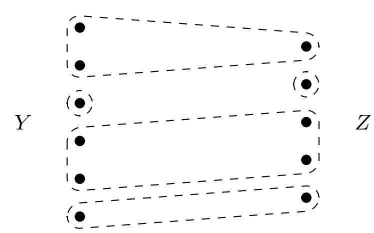
and then forcing the composite cospan to be jointly epic:

$$\begin{tikzcd}
 & h(S) \cup h'(S') & \\
X \arrow[ur, "hf"] & & Z. \arrow[ul, "h'g'"']
\end{tikzcd}$$

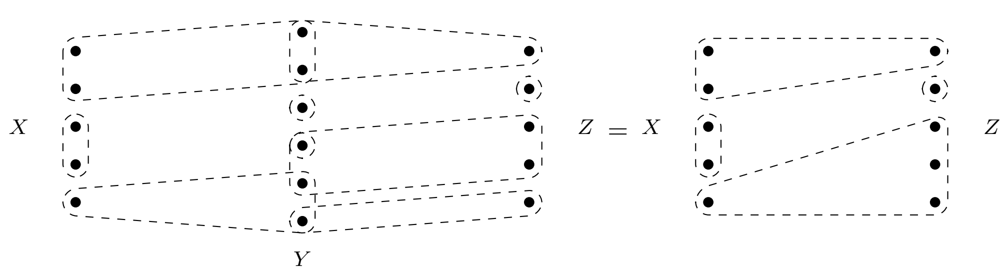
Equivalently, if we treat a corelation $R\colon X \to Y$ as an equivalence relation on $X + Y$, we compose corelations $R\colon X \to Y$ and $S\colon Y \to Z$ by forming the union of relations $R \cup S$ on $X + Y + Z$, taking its transitive closure to get an equivalence relation, and then restricting this equivalence relation to $X + Z$. For example, given this corelation from X to Y:



and this corelation from Y to Z:



we compose them as follows:



Fong [22, Corollary 3.18] showed that the process of taking a cospan of finite sets:

$$\begin{tikzcd} & S & \\ X \arrow[ur, "f"] & & Y \arrow[ul, "g"'] \end{tikzcd}$$

and making it jointly epic as follows:

$$\begin{tikzcd} & f(X) \cup g(Y) & \\ X \arrow[ur, "f"] & & Y \arrow[ul, "g"'] \end{tikzcd}$$

defines a functor

$$H \colon \mathrm{FinCospan} \to \mathrm{FinCorel}.$$

We describe this functor in more detail in Example 23 below.

In short, we can simplify a circuit made of ideal conductive wires in two successive stages:

$$\mathrm{Circ} \xrightarrow{G} \mathrm{FinCospan} \xrightarrow{H} \mathrm{FinCorel}.$$

In Section 8 we define a 'black-boxing functor'

$$\blacksquare \colon \mathrm{Circ} \to \mathrm{FinRel}_k,$$

which maps any such circuit to its 'behavior': that is, the linear relation $R \subseteq k^{2n}$ that it imposes between potentials and currents at its inputs and outputs. We construct this by composing $HG \colon \mathrm{Circ} \to \mathrm{FinCorel}$ with a functor

$$K \colon \mathrm{FinCorel} \to \mathrm{FinRel}_k.$$

This makes precise the sense in which the behavior of such a circuit depends only on its underlying corelation.

4 Props

We now introduce the machinery of 'props' in order to use generators and relations to describe the symmetric monoidal categories we are studying. Mac Lane [31] introduced these structures in 1965 to generalize Lawvere's algebraic theories [29] to contexts where the tensor product is not cartesian. He called them 'PROPs', which is an acronym for 'products and permutations'. We feel it is finally time to drop the rather ungainly capitalization here and treat props as ordinary mathematical citizens like groups and rings:

Definition 9. A **prop** is a strict symmetric monoidal category having the natural numbers as objects, with the tensor product of objects given by addition. We define a morphism of props to be a strict symmetric monoidal functor that is the identity on objects. Let **PROP** be the category of props.

A prop T has, for any natural numbers m and n, a homset $T(m,n)$. In circuit theory we take this to be the set of circuits with m inputs and n outputs. In other contexts $T(m,n)$ can serve as a set of 'operations' with m inputs and n outputs. In either case, we often study props by studying their algebras:

Definition 10. If T is a prop and C is a strict symmetric monoidal category, an **algebra of T in C** is a strict symmetric monoidal functor $F\colon T \to C$. We define a morphism of algebras of T in C to be a monoidal natural transformation between such functors.

For example, if T is a prop for which morphisms $f \in T(m,n)$ are circuits of some sort, 'black-boxing' should be a strict symmetric monoidal functor $F\colon T \to C$ that describes the 'behavior' of each circuit as a morphism in C. We work out many examples of this in the sections to come.

It has long been interesting to take familiar symmetric monoidal categories, treat them as props, and study their algebras. The most important feature of a prop is that every object is isomorphic to a tensor power of some chosen object x. However, not every symmetric monoidal category of this sort is a prop, or even isomorphic to one. There are two reasons: first, it may not be strict, and second, not every object may be *equal* to some tensor power of x. Luckily these two problems are 'purely technical', and can be fixed as follows:

Proposition 11. *A symmetric monoidal category C is equivalent to a prop if and only if there is an object $x \in C$ such that every object of C is isomorphic to $x^{\otimes n} = x \otimes (x \otimes (x \otimes \cdots))$ for some $n \in \mathbb{N}$.*

Proof. See Section A.1 for a precise statement and proof. The proof gives a recipe for actually constructing a prop equivalent to C when this is possible. □

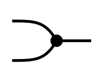
Proposition 12. *Suppose T and C are props and $F\colon T \to C$ is a symmetric monoidal functor. Then F is isomorphic to a strict symmetric monoidal functor $G\colon T \to C$. If $F(1) = 1$, then G is a morphism of props.*

Proof. See Section A.1 for a precise statement and proof. □

We can use Proposition 11 to turn some categories we have been discussing into props:

Example 13. Consider the category of finite sets and functions, made into a symmetric monoidal category where the tensor product of sets is their coproduct, or disjoint union. By Proposition 11 this symmetric monoidal category is equivalent to a prop. From now on, we use **FinSet** to stand for this prop. We identify this prop with a skeleton of the category of finite sets and functions, having finite ordinals $0, 1, 2, \ldots$ as objects.

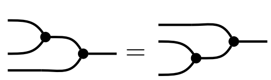
It is well known that the algebras of FinSet are commutative monoids [37]. Recall that a **commutative monoid** (x, μ, ι) in a strict symmetric monoidal category C is an object $x \in C$ together with a **multiplication** $\mu\colon x \otimes x \to x$ and **unit** $\iota\colon I \to x$ obeying the associative, unit and commutative laws. If we draw the multiplication and unit using string diagrams:



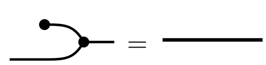
$\mu\colon x \otimes x \to x$

$\iota\colon I \to x$

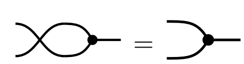
these laws are as follows:



(associativity)



(unitality)



(commutativity)

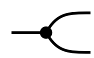
where $\times$ is the braiding on $x \otimes x$. In addition to the 'upper' or 'right' unit law shown above, the mirror image 'lower' or 'left' unit law also holds, due to commutativity and the naturality of the braiding.

To get a sense for why the algebras of FinSet are commutative monoids, write $m\colon 2 \to 1$ and $i\colon 0 \to 1$ for the unique functions of their type. Then given a strict symmetric monoidal functor $F\colon \mathrm{FinSet} \to C$, the object $F(1)$ becomes a commutative monoid in C with multiplication $F(m)\colon F(1) \otimes F(1) \to F(1)$ and unit $F(i)$. The associative, unit and commutative laws are easy to check. Conversely—and this requires more work—any commutative monoid in C arises in this way from a unique choice of F.

Similarly, morphisms between algebras of FinSet in C correspond to morphisms of commutative monoids in C. We thus say that FinSet is **the prop for commutative monoids**.

Example 14. Consider the category of finite sets and isomorphism classes of cospans, made into a symmetric monoidal category where the tensor product is given by disjoint union. By Proposition 11 this symmetric monoidal category is equivalent to a prop. We henceforth call this prop **FinCospan**. As with FinSet, we can identify the objects of FinCospan with finite ordinals.

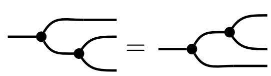
Lack [28] has shown that FinCospan is the prop for special commutative Frobenius monoids. To understand this, recall that a **cocommutative comonoid** (x, δ, ϵ) in C is an object $x \in C$ together with morphisms



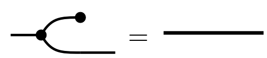
$\delta\colon x \to x \otimes x$

$\epsilon\colon x \to I$

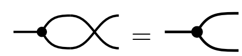
obeying these equations:



(coassociativity)

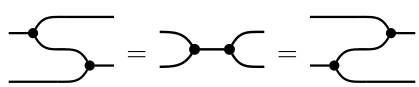


(counitality)



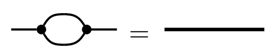
(cocommutativity)

A **commutative Frobenius monoid** in C is a commutative monoid (x, μ, ι) and a cocommutative comonoid (x, δ, ϵ) in C which together obey the **Frobenius law**:



In fact if any two of these expressions are equal so are all three. Furthermore, a monoid and comonoid obeying the Frobenius law is commutative if and only if it is cocommutative. A **morphism** of commutative Frobenius monoids in C is a morphism between the underlying objects that preserves the multiplication, unit, comultiplication and counit. In fact, any morphism of commutative Frobenius monoids is an isomorphism.

A commutative Frobenius monoid is **special** if comultiplication followed by multiplication is the identity:



To get a sense for why the algebras of FinCospan are special commutative Frobenius monoids, note that any function $f\colon X \to Y$ between finite sets gives rise to a cospan $X \xrightarrow{f} Y \xleftarrow{1} Y$ but also a cospan $Y \xrightarrow{1} Y \xleftarrow{f} X$. The aforementioned functions $m\colon 2 \to 1$ and $i\colon 0 \to 1$ thus give cospans $\mu\colon 2 \to 1$, $\iota\colon 0 \to 1$ but also cospans $\delta\colon 1 \to 2$, $\epsilon\colon 1 \to 0$. These four cospans make the object $1 \in \mathrm{FinCospan}$ into a special commutative Frobenius monoid.

This much is easy; Lack's accomplishment was to find an elegant proof that the category of algebras of FinCospan in any strict symmetric monoidal category C is equivalent to the category of special commutative Frobenius monoids in C. We thus say that FinCospan is the **prop for special commutative Frobenius monoids**.

Example 15. Consider the category of finite sets and corelations, again made into a symmetric monoidal category where the tensor product is given by disjoint union. By Proposition 11, this symmetric monoidal category is equivalent to a prop. We call this prop **FinCorel**, and we identify this prop with a skeleton of the category of finite sets and corelations having finite ordinals as objects.

A special commutative Frobenius monoid is **extraspecial** if the unit followed by the counit is the identity:

 =

where the blank at right denotes the identity on the unit object for the tensor product. The Frobenius law and the special law go back at least to Carboni and Walters [12], but this 'extra' law is newer, appearing under this name in the work of Baez and Erbele [3], as the 'bone law' in [9, 23], and as the 'irredundancy law' in [48]. In terms of circuits, it says that a loose wire not connected to anything else can be discarded without affecting the behavior of the circuit.

Fong and the second author [14] showed that the category of algebras of FinCorel in a strict symmetric monoidal category C is equivalent to the category of extraspecial commutative Frobenius monoids in C. We thus say that FinCorel is the **prop for extraspecial commutative Frobenius monoids**.

We conclude with two examples that are in some sense 'dual' to the previous two. These are important in their own right, but also useful in understanding the prop of linear relations, discussed in Section 6.

Example 16. Consider the category of finite sets and isomorphism classes of spans, made into a symmetric monoidal category where the tensor product is given by disjoint union. By Proposition 11 this symmetric monoidal category is equivalent to a prop. We call this prop **FinSpan**.

Lack [28] has shown that FinSpan is the prop for bicommutative bimonoids. Recall that a **bimonoid** is a monoid and comonoid where all the monoid operations are comonoid homomorphisms—or equivalently, all the comonoid operations are monoid homomorphisms. A bimonoid is **bicommutative** if its underlying monoid is commutative and its underlying comonoid is cocommutative.

Example 17. Consider the category of finite sets and relations, made into a symmetric monoidal category where the tensor product is given by disjoint union. Yet again this is equivalent to a prop, which we call **FinRel**. Fong and the second author [14] have shown that FinRel is the prop for special bicommutative bimonoids. Here a bimonoid is **special** if its comultiplication followed by its multiplication is the identity. The 'extra' law, saying that the unit followed by the counit is the identity, holds in any bimonoid.

In short, we have this picture [14]:

spans	**cospans**
extra bicommutative bimonoids	special bicommutative Frobenius monoids
relations	**corelations**
extraspecial bicommutative bimonoids	extraspecial bicommutative Frobenius monoids

Here we are making the pattern clearer by speaking of 'extra' and 'extraspecial' bicommutative bimonoids, even though the 'extra' law holds for any bimonoid. We also mention 'bicommutative' Frobenius monoids, even though any commutative Frobenius monoid is automatically bicommutative.

5 Presenting props

Just as we can present groups using generators and relations, we can do the same for props. Such presentations play a key role in our work. We can handle them using the tools of universal algebra, which in its modern form involves monads arising from algebraic theories. Just as we can talk about the free group on a set, we can talk about the free prop on a 'signature'. The underlying signature of a prop T is roughly the collection of all its homsets $T(m, n)$ where $m, n \in \mathbb{N}$. It is good to organize this collection into a functor from $\mathbb{N} \times \mathbb{N}$ to Set, where we consider $\mathbb{N} \times \mathbb{N}$ as a discrete category. Thus:

Definition 18. We define the category of **signatures** to be the functor category $\mathrm{Set}^{\mathbb{N}\times\mathbb{N}}$.

A signature Σ has **elements**, which are the elements of the sets $\Sigma(m, n)$ for $m, n \in \mathbb{N}$. Given an element $f \in \Sigma(m, n)$ we write $f : m \to n$.

Proposition 19. *There is a forgetful functor*

$$U : \mathrm{PROP} \to \mathrm{Set}^{\mathbb{N}\times\mathbb{N}}$$

sending any prop to its underlying signature and any morphism of props to its underlying morphism of signatures. This functor is monadic: that is, it has a left adjoint

$$F : \mathrm{Set}^{\mathbb{N}\times\mathbb{N}} \to \mathrm{PROP}$$

and the category of algebras of the monad UF is equivalent, via the canonical comparison functor, to the category PROP*.*

Proof. This follows from the fact that props are algebras of a multi-sorted or 'typed' Lawvere theory, together with a generalization of Lawvere's fundamental result [29] to this context. We prove this in Appendix A. □

For any signature Σ, we call $F\Sigma$ the **free prop** on Σ. The first benefit of the previous theorem is that it lets us describe any prop using a presentation. In other words, any prop can be obtained from a free prop by taking a coequalizer:

Corollary 20. *The category* PROP *is cocomplete, and any prop T is the coequalizer of some diagram*

$$F(E) \underset{\rho}{\overset{\lambda}{\rightrightarrows}} F(\Sigma).$$

Proof. Cocompleteness follows from a result of Trimble [43, Prop. 3.1]: the category of algebras of a multi-sorted Lawvere theory in a category C is cocomplete if C is cocomplete and has finite products, with finite products distributing over colimits. The latter fact holds simply because PROP is equivalent to the category of algebras of a monad: as noted by Barr and Wells [6, Sec. 3.2, Prop. 4], we can take $\Sigma = U(T)$ and $E = UFU(T)$, with $\lambda = FU\epsilon_T$ and $\rho = \epsilon_{FU(T)}$, where $\epsilon : FU \Rightarrow 1$ is the counit of the adjunction in Proposition 19. □

Here elements of the signature Σ serve as generators for T, while elements of E give relations—though as we will often be discussing relations of another sort, we prefer to call elements of E 'equations'. The idea is that given $e \in E(m, n)$, the morphisms $\lambda(e)$ and $\rho(e)$ in the free prop on Σ are set equal to each other in T. To illustrate this idea, we give presentations for the props FinSet, FinCospan and FinCorel:

Example 21. To present FinSet we can take Σ_0 to be the signature with $\mu : 2 \to 1$ and $\iota : 0 \to 1$ as its only elements, and let the equations $\lambda, \rho : F(E_0) \to F(\Sigma)$ be those governing a commutative monoid with multiplication μ and unit ι, namely the associative law:

$$\mu(\mu \otimes 1) = \mu(1 \otimes \mu),$$

the left and right unit laws:

$$\mu(\iota \otimes 1) = 1 = \mu(1 \otimes \iota),$$

and the commutative law:

$$\mu = \mu B,$$

where 1 above denotes the identity morphism on the object $1 \in F(\Sigma)$, while B is the braiding on two copies of this object. We obtain a coequalizer diagram

$$F(E_0) \overset{\lambda}{\underset{\rho}{\rightrightarrows}} F(\Sigma_0) \longrightarrow \mathrm{FinSet}.$$

Example 22. To present FinCospan we can take Σ to be the signature with elements $\mu\colon 2 \to 1, \iota\colon 0 \to 1, \delta\colon 1 \to 2, \epsilon\colon 1 \to 0$, and let the equations be those governing a special commutative Frobenius monoid with multiplication μ, unit ι, comultiplication δ and counit ϵ. We obtain a coequalizer diagram

$$F(E) \overset{\lambda}{\underset{\rho}{\rightrightarrows}} F(\Sigma) \longrightarrow \mathrm{FinCospan}.$$

Example 23. To present FinCorel we can use the same signature Σ as for FinCospan, but the equations E' include one additional equation, the so-called **extra** law:

$$\epsilon\iota = 1$$

in the definition of extraspecial commutative Frobenius monoids. Fong and the second author [14] show we obtain a coequalizer diagram

$$F(E') \overset{\lambda'}{\underset{\rho'}{\rightrightarrows}} F(\Sigma) \longrightarrow \mathrm{FinCorel}.$$

They also show that the inclusion $i\colon E \to E'$ gives rise to a diagram of this form:

$$\begin{tikzcd}
F(E) \arrow[r, shift left, "\lambda"] \arrow[r, shift right, "\rho"'] \arrow[d, "Fi"'] & F(\Sigma) \arrow[r] \arrow[d, "F1"] & \mathrm{FinCospan} \\
F(E') \arrow[r, shift left, "\lambda'"] \arrow[r, shift right, "\rho'"'] & F(\Sigma) \arrow[r] & \mathrm{FinCorel}
\end{tikzcd}$$

where the two squares at left, one involving λ and λ' and the other involving ρ and ρ', each commute. Thus, by the universal property of the coequalizer, we obtain a morphism $H\colon \mathrm{FinCospan} \to \mathrm{FinCorel}$. This expresses FinCorel as a 'quotient' of FinCospan.

The morphism of props $H\colon \mathrm{FinCospan} \to \mathrm{FinCorel}$ is, in fact, unique:

Proposition 24. *There exists a unique morphism of props* $H\colon \mathrm{FinCospan} \to \mathrm{FinCorel}$.

Proof. It suffices to show that H is uniquely determined on the generators $\mu, \iota, \delta, \epsilon$. By Example 22, $H(\mu)\colon 2 \to 1, H(\iota)\colon 0 \to 1, H(\delta)\colon 1 \to 2$ and $H(\epsilon)\colon 1 \to 0$ must make $1 \in \mathrm{FinCorel}$ into a special commutative Frobenius monoid.

There is a unique corelation from 0 to 1, so $H(\iota)$ is uniquely determined. Similarly, $H(\epsilon)$ is uniquely determined. Each partition of a 3-element set gives a possible choice for $H(\mu)$. If we write $H(\mu)$ as a partition of the set $2 + 1$ with $2 = \{a, b\}$ and $1 = \{c\}$ these choices are:

$$\{\{a,b,c\}\}, \;\; \{\{a\},\{b\},\{c\}\}, \;\; \{\{a,b\},\{c\}\}, \;\; \{\{a\},\{b,c\}\}, \;\; \{\{a,c\},\{b\}\}.$$

The commutative law rules out those that are not invariant under switching a and b, leaving us with these:

$$\{\{a,b,c\}\}, \ \ \{\{a\},\{b\},\{c\}\}, \ \ \{\{a,b\},\{c\}\}.$$

The associative law rules out the last of these. Either of the remaining two makes $1 \in \mathrm{FinCorel}$ into a commutative monoid. Dually, there are two choices for $H(\delta)$ making 1 into a commutative comonoid. However, the 'special' law demands that $H(\mu)H(\delta)$ be the identity corelation. This forces $H(\mu)$ to be the partition with just one block, namely $\{\{a,b,c\}\}$, and similarly for $H(\delta)$. $\square$

6 The prop of linear relations

Since the simplest circuits impose *linear* relations between potentials and currents at their terminals, the study of circuits forces us to think carefully about linear relations. As we shall see, for any field k there is a prop FinRel_k where a morphism $f\colon m \to n$ is a linear relation from k^m to k^n. A presentation for this prop has been worked out by Erbele and the first author [3, 16] and independently by Bonchi, Sobociński and Zanasi [9, 10, 48]. Since we need some facts about this presentation to describe the black-boxing of circuits, we recall it here.

For any field k, there is a category where an object is a finite-dimensional vector space over k, while a morphism from U to V is a **linear relation**, meaning a linear subspace $L \subseteq U \oplus V$. We write a linear relation from U to V as $L\colon U \nrightarrow V$ to distinguish it from a linear map. Since the direct sum $U \oplus V$ is also the cartesian product of U and V, a linear relation is a relation in the usual sense, and we can compose linear relations $L\colon U \nrightarrow V$ and $L'\colon V \nrightarrow W$ in the usual way:

$$L'L = \{(u,w)\colon\ \exists\, v \in V \ \ (u,v) \in L \text{ and } (v,w) \in L'\}$$

the result being a linear relation $L'L\colon U \nrightarrow W$. Given linear relations $L\colon U \nrightarrow V$ and $L'\colon U' \nrightarrow V'$, the direct sum of subspaces gives a linear relation $L \oplus L'\colon U \oplus U' \nrightarrow V \oplus V'$, and this gives our category a symmetric monoidal structure. By Proposition 11, this symmetric monoidal category is equivalent to a prop. Concretely, we may describe this prop as follows:

Definition 25. Let $\mathbf{FinRel}_{\boldsymbol{k}}$ be the prop where a morphism $f\colon m \to n$ is a linear relation from k^m to k^n, composition is the usual composition of relations, and the symmetric monoidal structure is given by direct sum.

To give a presentation of FinRel_k, we use a simple but nice fact: the object $1 \in \mathrm{FinRel}_k$, or in less pedantic terms the 1-dimensional vector space k, is an extraspecial Frobenius monoid in two fundamentally different ways. To understand these, first note that for any linear relation $L\colon U \to V$ there is a linear relation $L^\dagger\colon V \to U$ given by

$$L^\dagger = \{(v,u) :\ (u,v) \in L\}.$$

This makes FinRel_k into a dagger-compact category [2, 3, 41]. Also recall that a linear map is a special case of a linear relation.

The first way of making k into an extraspecial commutative Frobenius monoid in FinRel_k uses these morphisms:

- as comultiplication, the linear map called **duplication**:

$$\Delta\colon k \nrightarrow k^2$$

$$\Delta = \{(x,x,x) :\ x \in k\} \subseteq k \oplus k^2$$

- as counit, the linear map called **deletion**:

$$!\colon k \nrightarrow \{0\}$$

$$! = \{(x,0) :\ x \in k\} \subseteq k \oplus \{0\}$$

- as multiplication, the linear relation called **coduplication**:

$$\Delta^\dagger \colon k^2 \nrightarrow k$$

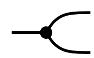
$$\Delta^\dagger = \{(x, x, x) : \; x \in k\} \subseteq k^2 \oplus k$$

- as unit, the linear relation called **codeletion**:

$$!^\dagger \colon \{0\} \nrightarrow k$$

$$!^\dagger = \{(0, x) : \; x \in k\} \subseteq \{0\} \oplus k.$$

We call this the **duplicative Frobenius structure** on k. In circuit theory this is important for working with the electric potential. The reason is that in a circuit of ideal conductive wires the potential is constant on each connected component, so wires like this:



have the effect of duplicating the potential.

The second way of making k into an extraspecial commutative Frobenius monoid in FinRel_k uses these morphisms:

- as multiplication, the linear map called **addition**:

$$+ \colon k^2 \nrightarrow k$$

$$+ = \{(x, y, x + y) : \; x, y \in k\} \subseteq k^2 \oplus k$$

- as unit, the linear map called **zero**:

$$0 \colon \{0\} \nrightarrow k$$

$$0 = \{(0, x) : \; x \in k\} \subseteq \{0\} \oplus k$$

- as comultiplication, the linear relation called **coaddition**:

$$+^\dagger \colon k \nrightarrow k \oplus k$$

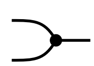
$$+^\dagger = \{(x + y, x, y) : \; x, y \in k\} \subseteq k \oplus k^2$$

- as counit, the linear relation called **cozero**:

$$0^\dagger \colon k \nrightarrow \{0\}$$

$$0^\dagger = \{(x, 0)\} \subseteq k \oplus \{0\}.$$

We call this the **additive Frobenius structure** on k. In circuit theory this structure is important for working with electric current. The reason is that Kirchhoff's current law says that the sum of input currents must equal the sum of the output currents, so wires like this:



have the effect of adding currents.

The prop FinRel_k is generated by the eight morphisms listed above together with a morphism for each element $c \in k$, namely the map from k to itself given by multiplication by c. We denote this simply as c:

$$\begin{array}{cccc} c \colon & k & \to & k \\ & x & \mapsto & cx. \end{array}$$

From the generators we can build two other important morphisms:

- the **cup** $\cup\colon k^2 \to \{0\}$: this is the composite of coduplication $\Delta^\dagger\colon k^2 \to k$ and deletion $!\colon k \to \{0\}$.
- the **cap** $\cap\colon \{0\} \to k^2$: this is the composite of codeletion $!^\dagger\colon \{0\} \to k$ and duplication $\Delta\colon k \to k^2$;

These are the unit and counit for an adjunction making k into its own dual. Since every object in FinRel_k is a tensor product of copies of k, every object becomes self-dual. Thus, FinRel_k becomes a dagger-compact category. This explains the use of the dagger notation for half of the eight morphisms listed above.

Thanks to the work of Baez and Erbele [3, 16] and also Bonchi, Sobociński and Zanasi [9, 10, 48], the prop FinRel_k has a presentation of this form:

$$F(E_k) \underset{\rho_k}{\overset{\lambda_k}{\rightrightarrows}} F(\Sigma) + F(\Sigma) + F(k) \xrightarrow{\square} \mathrm{FinRel}_k$$

Here the signature Σ has elements $\mu\colon 2 \to 1, \iota\colon 0 \to 1, \delta\colon 1 \to 2, \epsilon\colon 1 \to 0$. In this presentation for FinRel_k, the first copy of $F(\Sigma)$ is responsible for the duplicative Frobenius structure on k, so we call its generators

- **coduplication**, $\Delta^\dagger\colon 2 \to 1$,
- **codeletion**, $!^\dagger\colon 0 \to 1$,
- **duplication**, $\Delta\colon 1 \to 2$,
- **deletion**, $!\colon 1 \to 0$.

The second copy of $F(\Sigma)$ is responsible for the additive Frobenius structure, so we call its generators

- **addition**, $+\colon 2 \to 1$,
- **zero**, $0\colon 0 \to 1$,
- **coaddition**, $+^\dagger\colon 1 \to 2$,
- **cozero**, $0^\dagger\colon 2 \to 1$.

Finally, we have a copy of $F(k)$, consisting of elements we call

- **scalar multiplication**, $c\colon 1 \to 1$,

one for each $c \in k$. All these generators are mapped by $\square$ to the previously described morphisms with the same names in FinRel_k.

We do not need a complete list of the equations in this presentation of FinRel_k, but among them are equations saying that in FinRel_k

1. $(k, \Delta^\dagger, !, \Delta, !^\dagger)$ is an extraspecial commutative Frobenius monoid;
2. $(k, +, 0, +^\dagger, 0^\dagger)$ is an extraspecial commutative Frobenius monoid;
3. $(k, +, 0, \Delta, !)$ is a bicommutative bimonoid;
4. $(k, \Delta^\dagger, !^\dagger, +^\dagger, 0^\dagger)$ is a bicommutative bimonoid.

In Example 15 we saw that FinCorel is the prop for extraspecial commutative Frobenius monoids. Thus, items 1 and 2 give two prop morphisms from FinCorel to FinRel_k. We use these to define the black-boxing functor in Section 8. Similarly, in Example 16 we saw that FinSpan is the prop for bicommutative bimonoids. Thus, items 3 and 4 give two prop morphisms from FinSpan to FinRel_k, but these play no role in this paper.

7 Props of circuits

We now introduce the most important props in this paper, give a presentation for them, and describe their algebras. All this is a rephrasing of the fundamental work of Rosebrugh, Sabadini and Walters [40].

Fix a set $\mathcal{L}$. Recall the symmetric monoidal category $\mathrm{Circ}_{\mathcal{L}}$ described in Proposition 5, where an object is a finite set and a morphism from X to Y is an isomorphism class of $\mathcal{L}$-circuits from X to Y. By Proposition 11, $\mathrm{Circ}_{\mathcal{L}}$ is equivalent to a prop. Henceforth, by a slight abuse of language, we use $\mathbf{Circ}_{\mathcal{L}}$ to denote this prop.

To describe the algebras of $\mathrm{Circ}_{\mathcal{L}}$, we make a somewhat nonstandard definition. We say a set $\mathcal{L}$ acts on an object x if for each element of $\mathcal{L}$ we have a morphism from x to itself:

Definition 26. An **action** of a set $\mathcal{L}$ on an object x in a category C is a function $\alpha\colon \mathcal{L} \to \hom(x,x)$. We also call this an **$\mathcal{L}$-action**. Given $\mathcal{L}$-actions $\alpha\colon \mathcal{L} \to \hom(x,x)$ and $\beta\colon \mathcal{L} \to \hom(y,y)$, a **morphism of $\mathcal{L}$-actions** is a morphism $f\colon x \to y$ in C such that $f\alpha(\ell) = \beta(\ell)f$ for all $\ell \in \mathcal{L}$.

Proposition 27. *An algebra of* $\mathrm{Circ}_{\mathcal{L}}$ *in a strict symmetric monoidal category C is a special commutative Frobenius monoid in C whose underlying object is equipped with an action of $\mathcal{L}$. A morphism of algebras of* $\mathrm{Circ}_{\mathcal{L}}$ *in C is a morphism of special commutative Frobenius monoids that is also a morphism of $\mathcal{L}$-actions.*

Proof. This was proved by Rosebrugh, Sabadini and Walters [40], though stated in quite different language. □

We may thus say that $\mathrm{Circ}_{\mathcal{L}}$ is the prop for special commutative Frobenius monoids whose underlying object is equipped with an action of $\mathcal{L}$.

Unsurprisingly, $\mathrm{Circ}_{\mathcal{L}}$ is coproduct of two props: the prop for special commutative Frobenius monoids and the prop for $\mathcal{L}$-actions. To describe the latter, consider a signature with one unary operation for each element of $\mathcal{L}$, and no other operations. For simplicity we call this signature simply $\mathcal{L}$. The free prop $F(\mathcal{L})$ has a morphism $\ell\colon 1 \to 1$ for each $\ell \in \mathcal{L}$. For any strict symmetric monoidal category C, the category of algebras of $F(\mathcal{L})$ in C is the category of $\mathcal{L}$-actions and morphisms of $\mathcal{L}$-actions. We thus call $F(\mathcal{L})$ **the prop for $\mathcal{L}$-actions**.

Proposition 28. $\mathrm{Circ}_{\mathcal{L}}$ *is the coproduct of* $\mathrm{FinCospan}$ *and the prop for $\mathcal{L}$-actions.*

Proof. Let

$$F(E) \underset{\rho}{\overset{\lambda}{\rightrightarrows}} F(\Sigma)$$

be the presentation of FinCospan given in Example 22. Here Σ is the signature with elements $\mu\colon 2 \to 1, \iota\colon 0 \to 1, \delta\colon 1 \to 2$ and $\epsilon\colon 1 \to 0$, and the equations are the laws for a special commutative Frobenius monoid.

Since left adjoints preserve colimits, we have a natural isomorphism $F(\Sigma)+F(\mathcal{L}) \cong F(\Sigma+\mathcal{L})$. Let $\iota\colon F(\Sigma) \to F(\Sigma+\mathcal{L})$ be the resulting monomorphism. By Corollary 20 we can form the coequalizer X here:

$$F(E) \underset{\iota\rho}{\overset{\iota\lambda}{\rightrightarrows}} F(\Sigma+\mathcal{L}) \longrightarrow X.$$

We claim that $X \cong \mathrm{Circ}_{\mathcal{L}}$.

On the one hand, there is a morphism $f\colon F(\Sigma) \to \mathrm{Circ}_{\mathcal{L}}$ sending μ, ι, δ and ϵ in $F(\Sigma)$ to the corresponding morphisms in $\mathrm{Circ}_{\mathcal{L}}$, and $f\lambda = f\rho$ because these morphisms make $1 \in \mathrm{Circ}_{\mathcal{L}}$ into a special commutative Frobenius monoid. We thus have a commutative diagram

$$\begin{tikzcd} F(E) \arrow[r, shift left, "\iota\lambda"] \arrow[r, shift right, "\iota\rho"'] & F(\Sigma+\mathcal{L}) \arrow[r, "j"] \arrow[dr, "f"'] & X \\ & & \mathrm{Circ}_{\mathcal{L}}. \end{tikzcd}$$

By the universal property of the coequalizer, there is a unique morphism $g\colon X \to \mathrm{Circ}_{\mathcal{L}}$ with $gj = f$. On the other hand, the object $1 \in X$ is, by construction, a special commutative Frobenius monoid with $\mathcal{L}$-action. By Proposition 27 we thus obtain an algebra of $\mathrm{Circ}_{\mathcal{L}}$ in X, that is, a morphism h as follows:

$$\begin{tikzcd} F(E) \arrow[r, shift left, "\iota\lambda"] \arrow[r, shift right, "\iota\rho"'] & F(\Sigma+\mathcal{L}) \arrow[r, "j"] \arrow[dr, "f"'] & X \arrow[d, shift left, "g"] \\ & & \mathrm{Circ}_{\mathcal{L}}. \arrow[u, shift left, "h"] \end{tikzcd}$$

It is easy to check that $hf = j$ by seeing how both sides act on the elements $\mu, \iota, \delta, \epsilon, \ell$ of $F(\Sigma+\mathcal{L})$. By the universal property of X we have $hg = 1$, and because the Frobenius monoid in $\mathrm{Circ}_{\mathcal{L}}$ has no nontrivial automorphisms we also have $gh = 1$. Thus $X \cong \mathrm{Circ}_{\mathcal{L}}$ and

$$\begin{tikzcd} F(E) \arrow[r, shift left, "\lambda"] \arrow[r, shift right, "\rho"'] & F(\Sigma+\mathcal{L}) \arrow[r, "f"] & \mathrm{Circ}_{\mathcal{L}} \end{tikzcd} \tag{1}$$

is a coequalizer.

In general, given presentations of props X_1 and X_2, we have coequalizers:

$$\begin{tikzcd} F(E_i) \arrow[r, shift left, "\lambda_i"] \arrow[r, shift right, "\rho_i"'] & F(\Sigma_i) \arrow[r, "g_i"] & X_i, \end{tikzcd} \qquad i = 1, 2$$

whose coproduct is another coequalizer:

$$\begin{tikzcd} F(E_1)+F(E_2) \arrow[r, shift left, "\lambda_1+\lambda_2"] \arrow[r, shift right, "\rho_1+\rho_2"'] & F(\Sigma_1)+F(\Sigma_2) \arrow[r, "g_1+g_2"] & X_1+X_2. \end{tikzcd}$$

Since left adjoints preserve coproducts, we obtain a coequalizer

$$\begin{tikzcd} F(E_1+E_2) \arrow[r, shift left] \arrow[r, shift right] & F(\Sigma_1+\Sigma_2) \arrow[r] & X_1+X_2 \end{tikzcd}$$

which gives a presentation for $X_1 + X_2$.

In our situation $E \cong E + 0$ where 0 is the initial or 'empty' signature, so we have a coequalizer

$$\begin{tikzcd} F(E)+F(0) \arrow[r, shift left] \arrow[r, shift right] & F(\Sigma)+F(\mathcal{L}) \arrow[r] & \mathrm{FinCorel}+F(\mathcal{L}) \end{tikzcd}$$

Combining this with Equation (1), we see $\mathrm{Circ}_{\mathcal{L}} \cong \mathrm{FinCospan} + F(\mathcal{L})$. □

Proposition 28 plays a large part in the rest of the paper. For example, we can use it to get a more algebraic description of the functor $G\colon \mathrm{Circ} \to \mathrm{FinCospan}$ discussed in Section 3.

Example 29. We can take $\mathcal{L}$ to be a one-element set, say $\{\ell\}$. In this case we abbreviate the prop $\mathrm{Circ}_{\mathcal{L}}$ as Circ, abusing language slightly since in Definition 6 we used the same name for an equivalent symmetric monoidal category.

By Proposition 28 we can identify Circ with the coproduct FinCospan + $F(\{\ell\})$. There thus exists a unique morphism of props

$$G\colon \mathrm{Circ} \to \mathrm{FinCospan}$$

such that

$$G(f) = f$$

for any morphism f in FinCospan and

$$G(\ell) = 1_1.$$

In other words, G does nothing to morphisms in the sub-prop FinCospan, while it sends the morphism ℓ to the identity. It thus a retraction for the inclusion of FinCospan in Circ.

In Section 3 we explained how a morphism in Circ can be seen as a cospan of finite sets

$$\begin{tikzcd} & N & \\ X \arrow[ur, "i"] & & Y \arrow[ul, "o"'] \end{tikzcd}$$

together with a graph Γ having N as its set of vertices. In our new description of Circ, each edge of Γ corresponds to a copy of the morphism ℓ. The functor G has the effect of collapsing each edge of Γ to a point, since it sends ℓ to the identity. The result is cospan of finite sets where the apex is the set of connected components $\pi_0(\Gamma)$.

8 Black-boxing circuits of ideal conductive wires

In Section 3 we looked at circuits made of ideal perfectly conductive wires and described symmetric monoidal functors

$$\mathrm{Circ} \xrightarrow{G} \mathrm{FinCospan} \xrightarrow{H} \mathrm{FinCorel}.$$

In Example 23 we described H as a morphism of props, and in Example 29 we did the same for G. However, so far we have only briefly touched on the 'behavior' of such circuits: that is, what they actually *do*. A circuit provides a relation between potentials and currents at its inputs and outputs. For a circuit with m inputs and n outputs, this is a linear relation on $2m + 2n$ variables. We now describe a functor called 'black-boxing', which takes any circuit of ideal conductive wires and extracts this linear relation.

In Section 6 we saw that the object $k \in \mathrm{FinRel}_k$ has two extraspecial commutative Frobenius monoids: the 'duplicative' structure and the 'additive' structure. The first is relevant to potentials, while the second is relevant to currents [4]. In any symmetric monoidal category, the tensor product of two monoids is a monoid in standard way, and dually for comonoids. In the same way, the tensor product of extraspecial commutative Frobenius monoids becomes another extraspecial commutative Frobenius monoid. Thus, we can make $k \oplus k$ into an extraspecial commutative Frobenius monoid in FinRel_k where the first copy of k has the duplicative Frobenius structure and the second copy has the additive structure. Thanks to Proposition 27, this determines a strict symmetric monoidal functor

$$K\colon \mathrm{FinCorel} \to \mathrm{FinRel}_k.$$

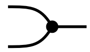
Composing this with

$$HG\colon \mathrm{Circ} \to \mathrm{FinCorel}$$

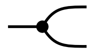
we get the **black-boxing** functor

$$\blacksquare = KHG\colon \mathrm{Circ} \to \mathrm{FinRel}_k.$$

Here is what black-boxing does to the generators of Circ:



$$\xmapsto{\blacksquare} \{(\phi_1, I_1, \phi_2, I_2, \phi_3, I_3) :\ \phi_1 = \phi_2 = \phi_3, I_1 + I_2 = I_3\}$$



$$\xmapsto{\blacksquare} \{(\phi_1, I_1, \phi_2, I_2, \phi_3, I_3) :\ \phi_1 = \phi_2 = \phi_3, I_1 = I_2 + I_3\}$$

$$\xmapsto{\blacksquare} \{(\phi_2, I_2) :\ I_2 = 0\}$$

$$\xmapsto{\blacksquare} \{(\phi_1, I_1) :\ I_1 = 0\}$$

$$\xmapsto{\blacksquare} \{(\phi_1, I_1, \phi_2, I_2) :\ \phi_1 = \phi_2, I_1 = I_2\}$$

Here ℓ is the generator corresponding to an ideal conductive wire; black-boxing maps it to the identity morphism on k^2. Since black-boxing is a symmetric monoidal functor, we can decompose a large circuit made of ideal conductive wires into simple building blocks in order to determine the relation it imposes between the potentials and currents at its inputs and outputs.

The black-boxing functor as described so far is not a morphism of props, since it sends the object $1 \in \mathrm{Circ}$ to the object $2 \in \mathrm{FinRel}_k$, that is, the vector space k^2. However, it can reinterpreted as a morphism of props with the help of some symplectic geometry. Instead of linear relations between finite-dimensional vector spaces, we should use Lagrangian relations between symplectic vector spaces. For a detailed explanation of this idea, see our work with Fong [4]; here we simply state the key facts.

Definition 30. A **symplectic** vector space V over a field k is a finite-dimensional vector space equipped with a map $\omega\colon V \times V \to k$ that is:

- bilinear,
- alternating: $\omega(v, v) = 0$ for all $v \in V$,
- nondegenerate: if $\omega(u, v) = 0$ for all $u \in V$ then $v = 0$.

Such a map ω is called a **symplectic structure**.

There is a standard way to make $k \oplus k$ into a symplectic vector space, namely

$$\omega((\phi, I), (\phi', I')) = \phi I' - \phi' I.$$

Given two symplectic vector spaces (V_1, ω_1) and (V_2, ω_2), we give their direct sum the symplectic structure

$$(\omega_1 \oplus \omega_2)((u_1, u_2), (v_1, v_2)) = \omega_1(u_1, v_1) + \omega_2(u_2, v_2).$$

In what follows, whenever we treat as $(k \oplus k)^n$ as a symplectic vector space, we give it the symplectic structure obtained by taking a direct sum of copies of $k \oplus k$ with the symplectic structure described above. Every symplectic vector space is isomorphic to $(k \oplus k)^n$ for some n, so every symplectic vector space is even dimensional [24, Thm. 21.2].

The concept of a 'Lagrangian relation' looks subtle at first, but it has become clear in mathematical physics that for many purposes this is the right notion of morphism between symplectic vector spaces [44, 45]. Lagrangian relations are also known as 'canonical relations'. The definition has a few prerequisites:

Definition 31. Given a symplectic structure ω on a vector space V, we define its **conjugate** to be the symplectic structure $\overline{\omega} = -\omega$, and write the conjugate symplectic vector space $(V, \overline{\omega})$ as $\overline{V}$.

Definition 32. A subspace L of a symplectic vector space (V, ω) is **isotropic** if $\omega(v, w) = 0$ for all $v, w \in V$. It is **Lagrangian** if is isotropic and not properly contained in any other isotropic subspace.

One can show that a subspace $L \subseteq V$ is Lagrangian if and only it is isotropic and $\dim(L) = \frac{1}{2}\dim(V)$. This condition is often easier to check.

Definition 33. Given symplectic vector spaces (V, ω) and (V', ω'), a **linear Lagrangian relation** $L\colon V \nrightarrow V'$ is a Lagrangian subspace $L \subseteq \overline{V} \oplus V'$.

We need the conjugate symplectic structure on V to show that the identity relation is a linear Lagrangian relation. With this twist, linear Lagrangian relations are also closed under composition: for a self-contained proof of this well-known fact, see [4, Prop. 6.8]. There is thus a category with symplectic vector spaces as objects and linear Lagrangian relations as morphisms. This becomes symmetric monoidal using direct sums: in particular, if the linear relations $L\colon U \nrightarrow V$ and $L'\colon U' \nrightarrow V'$ are Lagrangian, so is $L \oplus L'\colon U \oplus U' \nrightarrow V \oplus V'$. One can show using Proposition 11 that this symmetric monoidal category is equivalent to the following prop.

Definition 34. Let $\mathbf{LagRel}_{\boldsymbol{k}}$ be the prop where a morphism from m to n is a linear Lagrangian relation from $(k \oplus k)^m$ to $(k \oplus k)^n$, composition is the usual composition of relations, and the symmetric monoidal structure is given by direct sum.

We can now redefine the functor $K\colon \mathrm{FinCorel} \to \mathrm{FinRel}_k$, and the black-boxing functor $\blacksquare\colon \mathrm{Circ} \to \mathrm{FinRel}_k$, to be morphisms of props taking values in LagRel_k. This is the view we take henceforth.

Proposition 35. *The strict symmetric monoidal functor* $K\colon \mathrm{FinCorel} \to \mathrm{FinRel}_k$ *maps any morphism* $f\colon m \to n$ *is any morphism in* FinCospan *to a Lagrangian linear relation* $K(f)\colon (k \oplus k)^m \nrightarrow (k \oplus k)^n$. *It thus defines a morphism of props, which we call*

$$K\colon \mathrm{Circ} \to \mathrm{LagRel}_k.$$

Proof. First note that LagRel_k is a symmetric monoidal subcategory of FinRel_k: composition and direct sum for linear Lagrangian relations is a special case of composition and direct sum for linear relations. Second, note that while K applied to the object $n \in \mathrm{FinCospan}$ gives the vector space $(k \oplus k)^n$, which is the object $2n$ in FinRel_k, this is the object n in LagRel_k. Thus, to check that strict symmetric monoidal functor $K\colon \mathrm{FinCospan} \to \mathrm{FinRel}_k$ defines a morphism of props from FinCospan to LagRel_k, we just need to check that $K(f)$ is Lagrangian for each generator f of FinCospan. We have

$$\begin{array}{rcl} K(m) & = & \{(\phi_1, I_1, \phi_2, I_2, \phi_3, I_3) :\ \phi_1 = \phi_2 = \phi_3,\ I_1 + I_2 = I_3\} \\ K(\Delta) & = & \{(\phi_1, I_1, \phi_2, I_2, \phi_3, I_3) :\ \phi_1 = \phi_2 = \phi_3,\ I_1 = I_2 + I_3\} \\ K(\iota) & = & \{(\phi_2, I_2) :\ I_2 = 0\} \\ K(\epsilon) & = & \{(\phi_1, I_1) :\ I_1 = 0\} \end{array}$$

In each case the relation is an isotropic subspace of half the total dimension, so it is Lagrangian. □

We can characterize this new improved K as follows:

Proposition 36. *There exists a unique morphism of props*

$$K\colon \mathrm{FinCorel} \to \mathrm{LagRel}_k$$

sending the extraspecial commutative Frobenius monoid $1 \in \mathrm{FinCorel}$ *to the extraspecial commutative Frobenius monoid* $k \oplus k \in \mathrm{LagRel}_k$, *where the first copy of* k *is equipped with its additive Frobenius structure and the second is equipped with its duplicative Frobenius structure.*

Proof. Existence follows from Proposition 35; uniqueness follows from the fact that FinCorel is the prop for extraspecial commutative Frobenius monoids. $\square$

We can now reinterpret black-boxing of circuits of ideal conductive wires as a morphism of props:

Definition 37. We define **black-boxing** to be the morphism of props $\blacksquare\colon \mathrm{Circ} \to \mathrm{LagRel}_k$ given by the composite

$$\mathrm{Circ} \xrightarrow{G} \mathrm{FinCospan} \xrightarrow{H} \mathrm{FinCorel} \xrightarrow{K} \mathrm{LagRel}_k.$$

9 Black-boxing linear circuits

Black-boxing circuits of ideal conductive wires is just the first step: one can extend black-boxing to circuits made of wires labeled by elements of any set $\mathcal{L}$. The elements of $\mathcal{L}$ play the role of 'circuit elements' such as resistors, inductors and capacitors. The extended black-boxing functor can be chosen so that these circuit elements are mapped to arbitrary Lagrangian linear relations from $k \oplus k$ to itself.

The key to doing this is Proposition 28, which says that

$$\mathrm{Circ}_{\mathcal{L}} \cong \mathrm{FinCospan} + F(\mathcal{L}).$$

We also need Propositions 24 and 36, which give prop morphisms $H\colon \mathrm{FinCospan} \to \mathrm{FinCorel}$ and $K\colon \mathrm{FinCorel} \to \mathrm{LagRel}_k$, respectively.

Theorem 38. *For any field k and label set $\mathcal{L}$, there exists a unique morphism of props*

$$\blacksquare\colon \mathrm{Circ}_{\mathcal{L}} \cong \mathrm{FinCospan} + F(\mathcal{L}) \to \mathrm{LagRel}_k$$

such that $\blacksquare|_{\mathrm{FinCospan}}$ is the composite

$$\mathrm{FinCospan} \xrightarrow{H} \mathrm{FinCorel} \xrightarrow{K} \mathrm{LagRel}_k$$

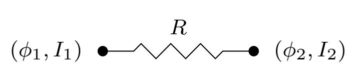
and $\blacksquare|_{F(\mathcal{L})}$ maps each $\ell \in \mathcal{L}$ to an arbitrarily chosen Lagrangian linear relation from $k \oplus k$ to itself.

Proof. By the universal property of the coproduct and the fact that $F(\mathcal{L})$ is the prop for $\mathcal{L}$-actions, there exists a unique morphism of props

$$\blacksquare\colon \mathrm{FinCospan} + F(\mathcal{L}) \to \mathrm{LagRel}_k$$

such that $\blacksquare|_{\mathrm{FinCospan}} = K \circ H$ and $\phi|_{F(\mathcal{L})}$ maps each $\ell \in \mathcal{L}$ to an arbitrarily chosen Lagrangian linear relation from $k \oplus k$ to itself. $\square$

We can apply this theorem to circuits made of resistors. Any resistor has a **resistance** R, which is a positive real number. Thus, if we take the label set $\mathcal{L}$ to be $\mathbb{R}^+$, we obtain a prop $\mathrm{Circ}_{\mathcal{L}}$ that models circuits made of resistors. Electrical engineers typically draw a resistor as a wiggly line:



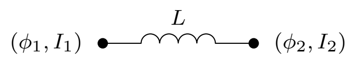
Here $(\phi_1, I_1) \in \mathbb{R} \oplus \mathbb{R}$ are the **potential** and **current** at the resistor's input and $(\phi_2, I_2) \in \mathbb{R} \oplus \mathbb{R}$ are the potential and current at its output. To define a black-boxing functor

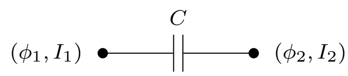
$$\blacksquare\colon \mathrm{Circ}_{\mathcal{L}} \to \mathrm{FinRel}_k$$

we need to choose a linear relation between these four quantities for each choice of the resistance R. To do this, recall first that Kirchhoff's current law requires that the current flowing in equals the current flowing out: $I_1 = I_2$. Second, Ohm's Law says that $V = RI$ where $I = I_1 = I_2$ is called the **current through** the resistor and $V = \phi_2 - \phi_1$ is called the **voltage across** the resistor. Thus, for each $R \in \mathbb{R}^+$ we choose

$$\blacksquare(R) = \{(\phi_1, I_1, \phi_2, I_2) : \ \phi_2 - \phi_1 = RI_1, I_1 = I_2\}.$$

We could stop here, but suppose we also want to include inductors and capacitors. An inductor comes with an inductance $L \in \mathbb{R}^+$ (not to be confused with our notation for a label set), while a capacitor comes with a capacitance $C \in \mathbb{R}^+$. These circuit elements are drawn as follows:





These circuit elements apply to time-dependent currents and voltages, and they impose the relations $V = L\dot{I}$ and $I = C\dot{V}$, where the dot stands for the time derivative. Engineers deal with this using the Laplace transform. As explained in detail elsewhere [3, 4], this comes down to adjoining a variable s to the field $\mathbb{R}$ and letting $k = \mathbb{R}(s)$ be the field of rational functions in one real variable. The variable s has the meaning of a time derivative. We henceforth use I and V to denote the Laplace transforms of current and voltage, respectively, and obtain the relations $V = sLI$ for the inductor, $I = sCV$ for the capacitor, and $V = RI$ for the resistor. Thus if we extend our label set to the disjoint union of three copies of $\mathbb{R}^+$, defining

$$RLC = \mathbb{R}^+ + \mathbb{R}^+ + \mathbb{R}^+,$$

we obtain a prop Circ_{RLC} that describes circuits of resistors, inductors and capacitors. The name 'Circ_{RLC}' is a bit of a pun, since electrical engineers call a circuit made of one resistor, one inductor and one capacitor an 'RLC circuit'.

To construct the black-boxing functor

$$\blacksquare \colon \mathrm{Circ}_{RLC} \to \mathrm{FinRel}_k$$

we specify it separately on each kind of circuit element. Thus, on the first copy of $\mathbb{R}^+$, corresponding to resistors, we set

$$\blacksquare(R) = \{(\phi_1, I_1, \phi_2, I_2) : \ \phi_2 - \phi_1 = RI_1, \ I_1 = I_2\}$$

as before. On the second copy we set

$$\blacksquare(L) = \{(\phi_1, I_1, \phi_2, I_2) : \ \phi_2 - \phi_1 = sLI_1, \ I_1 = I_2\}$$

and on the third we set

$$\blacksquare(C) = \{(\phi_1, I_1, \phi_2, I_2) : \ sC(\phi_2 - \phi_1) = I_1, \ \ I_1 = I_2\}.$$

We have:

Proposition 39. *If $f \colon m \to n$ is any morphism in Circ_{RLC}, the linear relation $\blacksquare(f) : (k \oplus k)^m \nrightarrow (k \oplus k)^n$ is Lagrangian. We thus obtain a morphism of props*

$$\blacksquare \colon \mathrm{Circ}_{RLC} \to \mathrm{LagRel}_k$$

where $k = \mathbb{R}(s)$.

Proof. By Theorem 38 it suffices to check that the linear relations $\blacksquare(R), \blacksquare(L)$ and $\blacksquare(C)$ are Lagrangian for any $R, L, C \in \mathbb{R}^+$. To do this, one can check that these relations are 2-dimensional isotropic subspaces of $\overline{(k \oplus k)} \oplus (k \oplus k)$. $\square$

A similar result was proved by Baez and Fong [4, 22] using different methods: decorated cospan categories rather than props. In their work, resistors, inductors and capacitors were subsumed in a mathematically more natural class of circuit elements. We can do something similar here. At the same time, we might as well generalize to an arbitrary field k and work with the prop Circ_k, meaning $\mathrm{Circ}_{\mathcal{L}}$ where the label set $\mathcal{L}$ is taken to be k.

Definition 40. We call a morphism in Circ_k a **linear circuit**.

Engineers might instead call such a morphism a 'passive' linear circuit [4], but we will never need any other kind.

Theorem 41. *For any field k there exists a unique morphism of props*

$$\blacksquare\colon \mathrm{Circ}_k \cong \mathrm{FinCospan} + F(k) \to \mathrm{LagRel}_k$$

such that $\blacksquare|_{\mathrm{FinCospan}}$ is the composite

$$\mathrm{FinCospan} \xrightarrow{H} \mathrm{FinCorel} \xrightarrow{K} \mathrm{LagRel}_k$$

and for each $Z \in k$, the linear Lagrangian relation

$$\blacksquare(Z)\colon k \oplus k \nrightarrow k \oplus k$$

is given by

$$\blacksquare(Z) = \{(\phi_1, I_1, \phi_2, I_2) :\ \phi_2 - \phi_1 = Z I_1,\ I_1 = I_2\}.$$

Proof. By Theorem 38 it suffices to check that $\blacksquare(Z)$ is a linear Lagrangian relation for any $Z \in k$. This follows from the fact that $\blacksquare(Z)$ is a 2-dimensional isotropic subspace of $\overline{(k \oplus k)} \oplus (k \oplus k)$. $\square$

In electrical engineering, Z is called the 'impedance': a circuit element with one input and one output has **impedance** Z if the voltage across it is Z times the current through it. Resistance is a special case of impedance. In particular, an ideal conductive wire has impedance zero. Mathematically, this fact is reflected in an inclusion of props

$$\mathrm{Circ} \hookrightarrow \mathrm{Circ}_k$$

that sends the generator ℓ in $\mathrm{Circ} \cong \mathrm{FinCospan} + F(\{\ell\})$ to the generator $0 \in k$ in $\mathrm{Circ} \cong \mathrm{FinCospan} + F(k)$, while it is the identity on FinCospan. Black-boxing for linear circuits then extends black-boxing as previously defined for circuits of ideal conductive wires. That is, we have a commutative triangle:

$$\begin{tikzcd}
\mathrm{Circ} \arrow[d, hook] \arrow[dr, "\blacksquare"] & \\
\mathrm{Circ}_k \arrow[r, "\blacksquare"'] & \mathrm{LagRel}_k.
\end{tikzcd} \tag{2}$$

10 Signal-flow diagrams and circuit diagrams

Control theory is the branch of engineering that studies the behavior of 'open' dynamical systems: that is, systems with inputs and outputs. Control theorists have intensively studied *linear* open dynamical systems, and they specify these using 'signal-flow diagrams'. We now know that signal-flow diagrams are a syntax for linear relations. In other words, we can see signal-flow diagrams as morphisms in a free prop that maps onto the prop of linear relations, FinRel_k. This is a nice example of functorial semantics drawn from engineering.

The machinery of props lets us map circuit diagrams to signal-flow diagrams in a manner compatible with composition, addressing a problem raised by Willems [46]. To do this, because circuits as we have defined them obey some nontrivial equations, while the prop of signal-flow diagrams is free, we need to introduce free props that map onto Circ and Circ_k.

In Section 6 we discussed this presentation of FinRel_k:

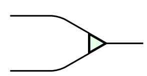
$$F(E_k) \overset{\lambda_k}{\underset{\rho_k}{\rightrightarrows}} F(\Sigma) + F(\Sigma) + F(k) \xrightarrow{\square} \mathrm{FinRel}_k.$$

The morphisms in the free prop $F(\Sigma) + F(\Sigma) + F(k)$ can be drawn as string diagrams, and these roughly match what control theorists call signal flow diagrams. So, we make the following definition:

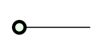
Definition 42. Define $\mathbf{SigFlow}_{\boldsymbol{k}}$ to be $F(\Sigma) + F(\Sigma) + F(k)$. We call a morphism in $\mathrm{SigFlow}_k$ a **signal-flow diagram**.

The prop $\mathrm{SigFlow}_k$ is free on eight generators together with one generator for each element of k. The meaning of these generators is best understood in terms of the linear relations they are mapped to under $\square$. We discussed those linear relations in Section 8. So, we give the generators of $\mathrm{SigFlow}_k$ the same names. Erbele also drew pictures of them, loosely modeled after the notation in signal-flow diagrams [3, 16]. The generators of the first copy of $F(\Sigma)$ are then:

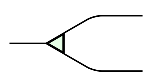
- **coduplication**, $\Delta^\dagger \colon 2 \to 1$



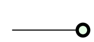
- **codeletion**, $!^\dagger \colon 0 \to 1$



- **duplication**, $\Delta \colon 1 \to 2$

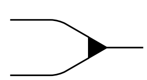


- **deletion**, $! \colon 1 \to 1$

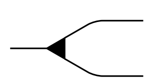


The generators of the second copy of $F(\Sigma)$ are:

- **addition**, $+ \colon 2 \to 1$

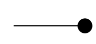


- **coaddition**, $+^\dagger \colon 1 \to 2$



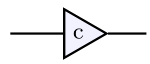
- **zero**, $0\colon 0 \to 1$

- **cozero**, $0^\dagger\colon 1 \to 0$



The generators of $F(k)$ are:

- for each $c \in k$, **scalar multiplication**, $c\colon 1 \to 1$



Since $\mathrm{SigFlow}_k$ is a free prop, while Circ_k is not, there is no useful morphism of props from Circ_k to $\mathrm{SigFlow}_k$. However, there is a free prop $\widetilde{\mathrm{Circ}}_k$ having Circ_k as a quotient, and a morphism from this free prop to $\mathrm{SigFlow}_k$. This morphism lifts the black-boxing functor described in Theorem 41 to a morphism between free props.

Definition 43. For any set $\mathcal{L}$ define the prop $\widetilde{\mathbf{Circ}}_{\mathcal{L}}$ by

$$\widetilde{\mathrm{Circ}}_{\mathcal{L}} = F(\Sigma) + F(\mathcal{L}).$$

Here $\mathcal{L}$ stands for the signature with one unary operation for each element of $\mathcal{L}$ while Σ is the signature with elements $\mu\colon 2 \to 1, \iota\colon 0 \to 1, \delta\colon 1 \to 2$ and $\epsilon\colon 1 \to 0$.

In Example 22 we saw this presentation for FinCospan:

$$F(E) \overset{\lambda}{\underset{\rho}{\rightrightarrows}} F(\Sigma) \longrightarrow \mathrm{FinCospan}$$

where the equations in E are the laws for a special commutative Frobenius monoid. In the proof of Proposition 28 we derived this presentation for $\mathrm{Circ}_{\mathcal{L}}$:

$$F(E) \overset{\iota\lambda}{\underset{\iota\rho}{\rightrightarrows}} F(\Sigma) + F(\mathcal{L}) \longrightarrow \mathrm{FinCospan} + F(\mathcal{L}) \cong \mathrm{Circ}_{\mathcal{L}}$$

where ι is the inclusion of $F(\Sigma)$ in $F(\Sigma) + F(\mathcal{L})$. Since $F(\Sigma) + F(k) = \widetilde{\mathrm{Circ}}_k$, we can rewrite this as

$$F(E) \overset{\iota\lambda}{\underset{\iota\rho}{\rightrightarrows}} \widetilde{\mathrm{Circ}}_{\mathcal{L}} \longrightarrow \mathrm{Circ}_{\mathcal{L}}$$

The last arrow here, which we call $P\colon \widetilde{\mathrm{Circ}}_{\mathcal{L}} \to \mathrm{Circ}_{\mathcal{L}}$, imposes the laws of a special commutative Frobenius monoid on the object 1.

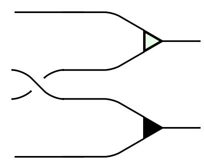
The most important case of this construction is when $\mathcal{L}$ is some field k:

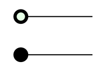
Theorem 44. *For any field k, there is a strict symmetric monoidal functor $T\colon \widetilde{\mathrm{Circ}}_k \to \mathrm{SigFlow}_k$ giving a commutative square of strict symmetric monoidal functors*

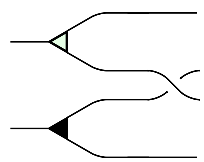
$$\begin{tikzcd}
\widetilde{\mathrm{Circ}}_k \arrow[r, "P"] \arrow[d, "T"'] & \mathrm{Circ}_k \arrow[r, "\blacksquare"] & \mathrm{LagRel}_k \arrow[d, hook] \\
\mathrm{SigFlow}_k \arrow[rr, "\square"] & & \mathrm{FinRel}_k.
\end{tikzcd}$$

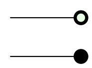
The horizontal arrows in this diagram are morphisms of props. The vertical ones are not, because they send the object 1 to the object 2.

Proof. We define the strict symmetric monoidal functor $T\colon \widetilde{\mathrm{Circ}}_k \to \mathrm{SigFlow}_k$ as follows. It sends the object 1 to 2 and has the following action on the generating morphisms of $\widetilde{\mathrm{Circ}}_k = F(\Sigma) + F(k)$:

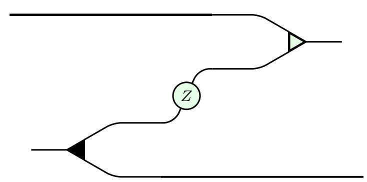
$T\colon \mu \longmapsto$ 

$T\colon \iota \longmapsto$ 

$T\colon \delta \longmapsto$ 

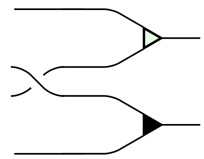
$T\colon \epsilon \longmapsto$ 

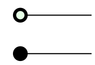
and for each element $Z \in k$,

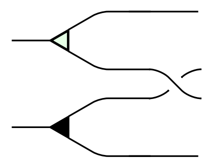
$T\colon Z \longmapsto$ 

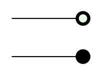
where we use string diagram notation for morphisms in $\mathrm{SigFlow}_k$. To check that T with these properties exists and is unique, let $\mathrm{SigFlow}_k^{\mathrm{ev}}$ be the full subcategory of $\mathrm{SigFlow}_k$ whose objects are even natural numbers. This becomes a prop if we rename each object $2n$, calling it n. Then, since $\widetilde{\mathrm{Circ}}_k$ is free, there exists a unique morphism of props $T\colon \widetilde{\mathrm{Circ}}_k \to \mathrm{SigFlow}_k^{\mathrm{ev}}$ defined on generators as above. Since $\mathrm{SigFlow}_k^{\mathrm{ev}}$ is a symmetric monoidal subcategory of $\mathrm{SigFlow}_k$, we can reinterpret T as a strict symmetric monoidal functor $T\colon \widetilde{\mathrm{Circ}}_k \to \mathrm{SigFlow}_k$, and this too is uniquely determined by its action on the generators.

To prove that the square in the statement of the theorem commutes, it suffices to check it on the generators of $\widetilde{\mathrm{Circ}}_k$. For this we use the properties of black-boxing stated in Theorem 41. First, note that these morphisms in $\mathrm{SigFlow}_k$:

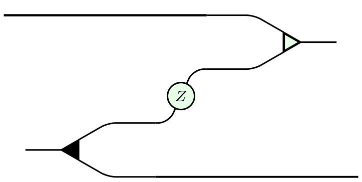
$T(\mu) \quad = $ 

$T(\iota) \quad = $ 

$T(\delta) \quad = $ 

$T(\epsilon) \quad = $ 

are mapped by $\square$ to the same multiplication, unit, comultiplication and counit on $k \oplus k$ as given by $\blacksquare(\mu), \blacksquare(\iota), \blacksquare(\delta)$ and $\blacksquare(\epsilon)$. Namely, these four linear relations make $k \oplus k$ into Frobenius monoid where the first copy of k has the duplicative Frobenius structure and the second copy has the additive Frobenius structure. Second, note that the morphism

$T(Z) = $ 

in $\mathrm{SigFlow}_k$ is mapped by $\square$ to the linear relation

$$\{(\phi_1, I_1, \phi_2, I_2): \; \phi_2 - \phi_1 = ZI_1, \; I_1 = I_2\},$$

while Theorem 41 states that $\blacksquare(Z)$ is the same relation, viewed as a Lagrangian linear relation. $\square$

Recall that when $\mathcal{L} = \{\ell\}$, we call $\mathrm{Circ}_{\mathcal{L}}$ simply Circ. We have seen that the map $\{\ell\} \to k$ sending ℓ to 0 induces a morphism of props $\mathrm{Circ} \hookrightarrow \mathrm{Circ}_k$, which expresses how circuits of ideal conductive wires are a special case of linear circuits. We can define a morphism of props $\widetilde{\mathrm{Circ}} \hookrightarrow \widetilde{\mathrm{Circ}}_k$ in an analogous way. Due to the naturality of the above construction, we obtain a commutative square

$$\begin{tikzcd}
\widetilde{\mathrm{Circ}} \arrow[r, "P"] \arrow[d, hook] & \mathrm{Circ} \arrow[d, hook] \\
\widetilde{\mathrm{Circ}}_k \arrow[r, "P"] & \mathrm{Circ}_k
\end{tikzcd}$$

We can combine this with the commutative square in Theorem 44 and the commutative triangle in Equation 2, which expands to a square when we use the definition of black-boxing for circuits of ideal conductive wires. The resulting diagram summarizes the relationship between linear circuits, cospans, corelations, and signal-flow diagrams:

$$\begin{tikzcd}
\widetilde{\mathrm{Circ}} \arrow[r, "P"] \arrow[d, hook] & \mathrm{Circ} \arrow[r, "G"] \arrow[d, hook] & \mathrm{FinCospan} \arrow[r, "H"] & \mathrm{FinCorel} \arrow[d, "K"] \\
\widetilde{\mathrm{Circ}}_k \arrow[r, "P"] \arrow[d, "T"'] & \mathrm{Circ}_k \arrow[rr, "\blacksquare"] & & \mathrm{LagRel}_k \arrow[d, hook] \\
\mathrm{SigFlow}_k \arrow[rrr, "\square"] & & & \mathrm{FinRel}_k.
\end{tikzcd}$$

In conclusion, we warn the reader that Erbele [16, Defn. 21] uses a different definition of $\mathrm{SigFlow}_k$. His prop with this name is free on the following generators:

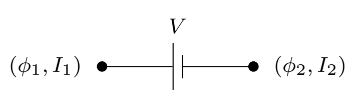
- addition, $+\colon 2 \to 1$
- zero, $0\colon 0 \to 1$
- duplication, $\Delta\colon 1 \to 2$
- deletion, $!\colon 1 \to 0$
- for each $c \in k$, scalar multiplication $c\colon 1 \to 1$
- the cup, $\cup\colon 2 \to 0$
- the cap, $\cap\colon 0 \to 2$

The main advantage is that string diagrams for morphisms in his prop more closely resemble the signal-flow diagrams actually drawn by control theorists; however, see his discussion of some subtleties. All the results above can easily be adapted to Erbele's definition.

11 Voltage and current sources

In our previous work on electrical circuits [4], batteries were not included. Resistors, capacitors, and inductors define linear relations between potential and current. Batteries, also known as 'voltage sources', define *affine* relations between these quantities. The same is true of current sources. Thus, to handle these additional circuit elements, we need a black-boxing functor that takes values in a different prop. The ease with which we can do this illustrates the flexibility of working with props. In what follows, we continue to work over an arbitrary field k, which in electrical engineering is either $\mathbb{R}$ or $\mathbb{R}(s)$.

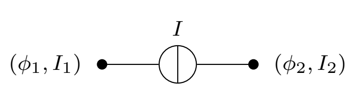
A voltage source is typically drawn as follows:



It sets the difference between the output and input potentials to a constant value $V \in k$. Thus, to define a black-boxing functor for voltage sources, we want to set

$$\blacksquare(V) = \{(\phi_1, I_1, \phi_2, I_2) : \ \phi_2 - \phi_1 = V, \ I_1 = I_2\}.$$

Similarly, a current source is drawn as



and it fixes the current at both input and output to a constant value I, giving this relation:

$$\blacksquare(I) = \{(\phi_1, I_1, \phi_2, I_2) : \ I_1 = I_2 = I\}.$$

We could define a black-boxing functor suitable for voltage and current sources by using a prop where the morphisms $f\colon m \to n$ are arbitrary relations from $(k \oplus k)^m$ to $(k \oplus k)^n$. However, the relations shown above are better than average. First, they are 'affine relations': that is, translates of linear subspaces of $(k \oplus k)^m \oplus (k \oplus k)^n$. For voltage sources we have

$$\blacksquare(V) = (0, 0, V, 0) + \{(\phi_1, I_1, \phi_1, I_1)\}$$

and for current sources we have

$$\blacksquare(I) = (I, 0, I, 0) + \{(\phi_1, 0, \phi_2, 0)\}.$$

Second, these affine relations are 'Lagrangian': that is, they are translates of Lagrangian linear relations.

Thus, we proceed as follows:

Definition 45. Given symplectic vector spaces (V, ω) and (V', ω'), an **Lagrangian affine relation** $R\colon V \nrightarrow V'$ is an affine subspace $R \subseteq \overline{V} \oplus V'$ that is also a Lagrangian subvariety of $\overline{V} \oplus V'$.

Here recall that a subset A of a vector space is said to be an **affine subspace** if it is closed under affine linear combinations: if $a, a' \in A$ then so is $ta + (1-t)a'$ for all $t \in \mathbb{R}$. A subvariety $R \subseteq \overline{V} \oplus V'$ is said to be **Lagrangian** if each of its tangent spaces, when identified with a linear subspace of $\overline{V} \oplus V'$, is Lagrangian. If R is an affine subspace of $\overline{V} \oplus V'$ it is automatically a subvariety, and it is either empty or a translate $L + (v, v')$ of some linear subspace $L \subseteq \overline{V} \oplus V'$. In the latter case all its tangent spaces become the same when identified with linear subspaces of $\overline{V} \oplus V'$: they are all simply L. We thus have:

Proposition 46. *Given symplectic vector spaces (V, ω) and (V', ω'), any Lagrangian affine relation $R\colon V \nrightarrow V'$ is either empty or a translate of a Lagrangian linear relation.*

This allows us to construct the following category:

Proposition 47. *There is a category where the objects are symplectic vector spaces, the morphisms are Lagrangian affine relations, and composition is the usual composition of relations. This is a symmetric monoidal subcategory of the category of sets and relations with the symmetric monoidal structure coming from the cartesian product of sets.*

Proof. It suffices to check that morphisms are closed under composition and tensor product, and that the braiding is a Lagrangian affine relation. Let $R\colon U \nrightarrow V$ and $S\colon V \nrightarrow W$ be two Lagrangian affine relations. If the composite $S \circ R$ is empty then we are done, so suppose it is not. In this case there exist $u \in U, v \in V, w \in W$ such that $(u, v) \in R, (v, w) \in S$, and $(u, w) \in S \circ R$. Consider the subspaces $-(u, v) + R = L$ and $-(v, w) + S = M$. These are both affine subspaces containing the origin, so they are linear subspaces. Since R and S are Lagrangian affine subspaces, their translates L and M are Lagrangian linear relations from U to W. It follows that $M \circ L\colon V \nrightarrow W$ is a Lagrangian linear relation. We claim that $S \circ R = (u, w) + M \circ L$, so that morphisms in our proposed category are closed under composition. First write $R = L + (u, v)$ and $S = M + (v, w)$, so that

$$\begin{aligned}
S \circ R &= \{(x, z) | \exists y \in V \text{ s.t. } (x, y) \in R \text{ and } (y, z) \in S\} \\
&= \{(x, z) | \exists y \in V \text{ s.t. } (x, y) \in L + (u, v) \text{ and } (y, z) \in M + (v, w)\} \\
&= \{(x, z) | \exists y \in V \text{ s.t. } \begin{array}{ll} x = l_1 + u & y = m_1 + v \\ y = l_2 + v & z = m_2 + w \\ (l_1, l_2) \in L & (m_1, m_2) \in M\} \end{array}
\end{aligned}$$

This gives $l_2 = m_1$, so finally we have

$$\begin{aligned} S \circ R &= \{(l_1 + u, m_2 + w) | \exists l_2 \in V \text{ s.t. } (l_1, l_2) \in L \text{ and } (l_2, m_2) \in M\} \\ &= (u, w) + M \circ L \end{aligned}$$

as desired.

The tensor product of Lagrangian affine relations $R\colon U \nrightarrow V$ and $R'\colon U' \nrightarrow V'$ is given by

$$R \oplus R' = \{(u, u', v, v') : \ (u, v) \in R, \ (u', v') \in R'\} \colon U \oplus U' \nrightarrow V \oplus V',$$

and this is a Lagrangian affine relation because it is a translate of a Lagrangian linear relation. Finally, note that the braiding morphism $B_{U,V}\colon U \oplus V \nrightarrow V \oplus U$ defined by $B_{U,V} = \{(u, v, v, u) | u \in U, v \in V\}$ is a Lagrangian linear relation and thus a Lagrangian affine relation. □

By Proposition 11, the above symmetric monoidal category is equivalent to the following prop:

Definition 48. Let $\mathbf{AffLagRel}_{\boldsymbol{k}}$ be the prop where a morphism from m to n is a Lagrangian affine relation from $(k \oplus k)^m$ to $(k \oplus k)^n$, composition is the usual composition of relations, and the symmetric monoidal structure is given as above.

We can extend the black-boxing functor from linear circuits to circuits that include voltage and/or current sources. The target of this extended black-boxing functor will be, not LagRel_k, but $\mathrm{AffLagRel}_k$.

Theorem 49. *For any field k and label set $\mathcal{L}$, there exists a unique morphism of props*

$$\blacksquare\colon \mathrm{Circ}_{\mathcal{L}} \cong \mathrm{FinCospan} + F(\mathcal{L}) \to \mathrm{AffLagRel}_k$$

such that $\blacksquare|_{\mathrm{FinCospan}}$ is the composite

$$\mathrm{FinCospan} \xrightarrow{H} \mathrm{FinCorel} \xrightarrow{K} \mathrm{LagRel}_k \hookrightarrow \mathrm{AffLagRel}_k$$

and $\blacksquare|_{F(\mathcal{L})}$ maps each $\ell \in \mathcal{L}$ to an arbitrarily chosen Lagrangian affine relation from $k \oplus k$ to itself.

Proof. The proof mimics that of Theorem 38. □

Using the formulas given above for the relations between potentials and currents for voltage sources and current sources, we can use this theorem to define black-boxing functors for circuits that include these additional circuit elements. A similar strategy can be used to define black-boxing for circuits containing other nonlinear circuit elements, such as transistors. We merely need to expand the target of this black-boxing functor to include all the relations between potentials and currents that arise.

A The mathematics of props

A.1 Props from symmetric monoidal categories

There is a 2-category SymmMonCat where:

- objects are symmetric monoidal categories,
- morphisms are symmetric monoidal functors, and
- 2-morphisms are monoidal natural transformations.

Here our default notions are the 'weak' ones (which Mac Lane [32] calls 'strong'), where all laws hold up to coherent natural isomorphism. Props, on the other hand, are *strict* symmetric monoidal categories where every object is *equal* to a natural number, and morphisms between them are *strict* symmetric monoidal functors sending each object n to itself. Categorical structures found in nature are often weak. Thus, to study them using props, one needs to 'strictify' them. Thanks to conversations with Steve Lack we can state the following results, which accomplish this strictification.

The first question is when a symmetric monoidal category C is equivalent, as an object of SymmMonCat, to a prop. In other words: when is does there exist a prop T and symmetric monoidal functors $j\colon T \to C$, $k\colon C \to T$ together with monoidal natural isomorphisms $jk \cong 1_C$ and $kj \cong 1_T$? This is answered by Proposition 11:

Proposition 11. *A symmetric monoidal category C is equivalent, as an object of* SymmMonCat*, to a prop if and only if there is an object $x \in C$ such that every object of C is isomorphic to the nth tensor power of x for some $n \in \mathbb{N}$.*

Proof. The 'only if' condition is obvious, so suppose that C is any symmetric monoidal category with $x \in C$ such that every object of C is isomorphic to a tensor power of x. We use a method due to A. J. Power, based on this lemma:

Lemma 50 (Lemma 3.3, [38])**.** *1. Any functor $f\colon A \to B$ can be factored as je where e is bijective on objects and j is fully faithful.*

2. Given a square that commutes up to a natural isomorphism α:

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "u"'] \arrow[dr, phantom, "\swarrow\alpha"] & B \arrow[d, "v"] \\
C \arrow[r, "g"'] & D
\end{tikzcd}$$

where h is bijective on objects and g is fully faithful, there exists a unique functor $w\colon B \to C$ and natural transformation $\beta\colon v \Rightarrow gw$ such that $wh = u$ and $\beta h = \alpha$. Moreover β is a natural isomorphism.

Let N be the strict symmetric monoidal category with one object for each natural number and only identity morphisms, with the tensor product of objects being given by addition. There exists a symmetric monoidal functor $f\colon N \to C$ that sends the nth object of N to $x^{\otimes n} = x \otimes (x \otimes (x \otimes \cdots))$. By Part 1 of Lemma 50 we can factor f as a composite

$$N \xrightarrow{e} T \xrightarrow{j} C$$

where e is bijective on objects and j is fully faithful. By our condition on C, f is essentially surjective. It follows that j is also essentially surjective, and thus an equivalence of categories. We claim that T can be given the structure of a strict symmetric monoidal category making j symmetric monoidal. It will follow that T is a prop and $j\colon T \to C$ can be promoted to an equivalence in SymmMonCat.

To prove the claim, we use the 2-monad P on Cat whose strict algebras are strict symmetric monoidal categories. For any category A, $P(A)$ is the 'free strict symmetric monoidal category' on A. Explicitly, an object of $P(A)$ consists of a finite list $(a_1 \ldots, a_n)$ of objects of A. A morphism from $(a_1, \ldots, a_n)$ to $(b_1, \ldots b_m)$ exists only if $n = m$, in which case it consists of a permutation $\sigma \in S_n$ and a morphism from a_i to $b_{\sigma(i)}$ in A for each i. For the rest of the 2-monad structure see for example [18, Sec. 4.1]. Any symmetric monoidal category can be made into a pseudoalgebra of P, and then the pseudomorphisms between such pseudoalgebras are the symmetric monoidal functors.

Power's method [38] applies to 2-monads that preserve the class of functors that are bijective on objects. The 2-monad P has this property.

We can take N above to be $P(1)$, since they are isomorphic. Any object $x \in C$ determines a functor $1 \to C$ and so a pseudomorphism $F\colon P(1) \to C$. Replacing F by an equivalent pseudomorphism if necessary, we can assume $F(n) = x^{\otimes n}$, so the situation in the first paragraph holds with this choice of F, and we can factor F as $P(1) \xrightarrow{e} T \xrightarrow{j} C$ as before. Since F is a pseudomorphism, this square commutes up to a natural isomorphism α:

$$\begin{tikzcd}
P(P(1)) \arrow[r, "P(e)"] \arrow[d, "m_1"'] & P(T) \arrow[d, "P(j)"] \\
P(1) \arrow[d, "e"'] \arrow[r, phantom, "\swarrow_{\alpha}"] & P(C) \arrow[d, "a"] \\
T \arrow[r, "j"'] & C
\end{tikzcd}$$

where a comes from the pseudoalgebra structure and m_1 comes the multiplication in the 2-monad. The functor $P(e)$ is bijective on objects because P is, and $j\colon T \to C$ is fully faithful. Thus, by Part 2 of Lemma 50, there exists a unique functor $w\colon P(T) \to T$ and natural isomorphism $\beta\colon aP(j) \to jw$ such that $wP(e) = em_1$ and $\beta P(e) = \alpha$. Thus, w makes T into a strict algebra of P and β makes j into a pseudomorphism from T to C. This proves the claim: T has been given the structure of a symmetric monoidal category for which $j\colon T \to C$ is a symmmetric monoidal functor. $\square$

The second question is when a symmetric monoidal functor $f\colon T \to C$ between props is isomorphic, in SymmMonCat, to a morphism of props. In other words: when is there a morphism of props $g\colon T \to C$ and a monoidal natural isomorphism $f \cong g$? This is answered by Proposition 12:

Proposition 12. *Suppose T and C are props and $f\colon T \to C$ is a symmetric monoidal functor. Then f is isomorphic to a strict symmetric monoidal functor $g\colon T \to C$. If $f(1) = 1$, then g is a morphism of props.*

Proof. As in the previous proof, let P be the 2-monad on Cat whose strict algebras are strict monoidal categories. The objects of $P(1)$ correspond to natural numbers, with tensor product being given addition, so we can write the nth object as n. There is a unique strict monoidal functor $e\colon P(1) \to T$ with $e(n) = n$ for all n. By a result of Blackwell, Kelly and Power [8, Cor. 5.6], any free algebra of a 2-monad is 'flexible', meaning that pseudomorphisms out of this algebra are isomorphic to strict morphisms. Thus, the symmetric monoidal functor $fe\colon P(1) \to C$ is isomorphic, in SymmMonCat, to a strict symmetric monoidal functor $h\colon P(1) \to C$. Let $\alpha\colon fe \Rightarrow h$ be the isomorphism.

We can define a strict symmetric monoidal functor $g\colon T \to C$ as follows. On objects, define $g(n) = h(n)$. For any morphism $\phi\colon m \to n$, there is a unique morphism $g(\phi)$ making this square commute:

$$\begin{tikzcd}
f(m) \arrow[r, "f(\phi)"] \arrow[d, "\alpha_m"'] & f(n) \arrow[d, "\alpha_n"] \\
g(m) \arrow[r, "g(\phi)"'] & g(n)
\end{tikzcd}$$

One can check that g is a strict symmetric monoidal functor. The above square gives a natural isomorphism between f and g, which by abuse of language we could call $\alpha\colon f \Rightarrow g$. It is easy to check that this is a monoidal natural isomorphism. $\square$

It is worth noting that Propositions 11 and 12, and the proofs just given, generalize straightforwardly from props to 'C-colored' props, with $\mathbb{N}$ replaced everywhere by the free commutative monoid on the set of colors, C.

A.2 The adjunction between props and signatures

Our goal in this section is to prove Proposition 19:

Proposition 19. *There is a forgetful functor*

$$U\colon \mathrm{PROP} \to \mathrm{Set}^{\mathbb{N}\times\mathbb{N}}$$

sending any prop to its underlying signature and any morphism of props to its underlying morphism of signatures. This functor is monadic.

Proof. The plan of the proof is as follows. We show that props are models of a typed Lawvere theory Θ_{PROP} whose set of types is $\mathbb{N}\times\mathbb{N}$. We write this as follows:

$$\mathrm{PROP} \simeq \mathrm{Mod}(\Theta_{\mathrm{PROP}}).$$

This lets us apply the following theorem, which says that for any typed Lawvere theory Θ with T as its set of types, the forgetful functor

$$U\colon \mathrm{Mod}(\Theta) \to \mathrm{Set}^T$$

is monadic. The desired result follows.

Theorem 51. *If Θ is a typed Lawvere theory with T as its set of types, then the forgetful functor $U\colon \mathrm{Mod}(\Theta) \to \mathrm{Set}^T$ is monadic.*

Proof. The origin of this theorem may be lost in the mists of time, though the case $T = 1$ is famous, and was proved in Lawvere's thesis [29]. The general theorem is Theorem A.41 in Adámek, Rosický and Vitale's book on algebraic theories [1]. Trimble has proved a further generalization where Set is replaced by any category C that is cocomplete and has the property that finite products distribute over colimits [43]. An even more general result appears in the work of Nishisawa and Power [33]. □

Now let us define the terms here and see how this result applies to our situation. First, for any set T, let $\mathbb{N}[T]$ be the set of finite linear combinations of elements of T with natural number coefficients. This becomes a commutative monoid under addition, in fact the free commutative monoid on T. Define a ***T*-typed Lawvere theory** to be a category Θ with finite products whose set of objects is $\mathbb{N}[T]$, with the product of objects given by addition in $\mathbb{N}[T]$. We call the elements of T **types**.

Suppose Θ is an T-typed Lawvere theory. Let **Mod(Θ)** be the category whose objects are functors $F\colon \Theta \to \mathrm{Set}$ preserving finite products, and whose morphisms are natural transformations between such functors. We call an object of $\mathrm{Mod}(\Theta)$ a **model** of Θ, and call a morphism in $\mathrm{Mod}(\Theta)$ a **morphism of models**.

There is an inclusion $T \hookrightarrow \mathbb{N}[T]$, since $\mathbb{N}[T]$ is the free commutative monoid on T. Thus, any model M of Θ gives, for each type $t \in T$, a set $M(t)$. Similarly, any morphism of models $\alpha\colon M \to M'$ gives, for each type $t \in T$, a function $\alpha_t\colon M(t) \to M'(t)$. Indeed, there is a functor

$$U\colon \mathrm{Mod}(\Theta) \to \mathrm{Set}^T$$

with $U(M)(t) = M(t)$ for each model M and $U(\alpha)(t) = \alpha_t$ for each morphism of models $\alpha\colon M \to M'$. If we call Set^T the category of **signatures** for T-typed Lawvere theories, then U sends models to their underlying signatures and morphisms of models to morphisms of their underlying signatures. Theorem 51 says that U is monadic.

To complete the proof of Proposition 19 we need to give a typed Lawvere theory Θ_{PROP} whose models are props. We can do this by giving a 'sketch'. Since this idea has been described very carefully by Barr and Wells [6] and others, we content ourselves with a quick intuitive explanation of the special case we really need which could be called a 'products sketch'. This is a way of presenting a T-typed Lawvere theory by specifying a set T of generating objects (or types), a set of generating morphisms between formal products of these generating objects, and a set of relations given as commutative diagrams. These commutative diagrams can involve the generating morphisms and also morphisms built from these using the machinery available in a category with finite products.

To present the typed Lawvere theory Θ_{PROP} we start by taking $T = \mathbb{N} \times \mathbb{N}$. For the purposes of easy comprehension, we call the corresponding generating objects $\hom(m, n)$ for $m, n \in \mathbb{N}$, since these will be mapped by any model of Θ_{PROP} to the homsets that a prop must have. We then include the following generating morphisms:

- For any m we include a morphism $\iota_m \colon 1 \to \hom(m, m)$. These give rise to the identity morphisms in any model of Θ_{PROP}.
- For any ℓ, m, n, we include a morphism $\circ_{\ell,m,n} \colon \hom(m, n) \times \hom(\ell, m) \to \hom(\ell, n)$. These give us the ability to compose morphisms in any model of Θ_{PROP}.
- For any m, n, m', n', we include a morphism $\otimes_{m,n,m',n'} \colon \hom(m', n') \times \hom(m, n) \to \hom(m + m', n + n')$. These allow us to take the tensor product of morphisms in any model of Θ_{PROP}.
- For any m, m', we include a morphism $b_{m,m'} \colon 1 \to \hom(m + m', m' + m)$. These give the braidings in any model of Θ_{PROP}.

Finally, we impose relations via the following commutative diagrams. In these diagrams, unlabeled arrows are morphisms provided by the structure of a category with finite products. We omit the subscripts on morphisms since they can be inferred from context. We begin with a set of diagrams, one for each $m, n \in \mathbb{N}$, that ensure associativity of composition in any model of Θ_{PROP}:

$$\begin{tikzcd}
& \hom(m,n) \times \hom(l,m) \times \hom(k,l) \arrow[dl, "\circ \times 1"'] \arrow[dr, "1 \times \circ"] & \\
\hom(l,n) \times \hom(k,l) \arrow[dr, "\circ"'] & & \hom(m,n) \times \hom(k,m) \arrow[dl, "\circ"] \\
& \hom(k,n) &
\end{tikzcd}$$

Next, a diagram that ensures each ι_m picks out an identity morphism:

$$\begin{tikzcd}
1 \times \hom(m,n) \arrow[r] \arrow[d, "\iota \times 1"] & \hom(m,n) \arrow[d, "1"] & \hom(m,n) \times 1 \arrow[l] \arrow[d, "1 \times \iota"] \\
\hom(n,n) \times \hom(m,n) \arrow[r, "\circ"] & \hom(m,n) & \hom(m,n) \times \hom(m,m) \arrow[l, "\circ"']
\end{tikzcd}$$

Next, a diagram that ensures associativity of the tensor product of morphisms:

$$
\begin{tikzcd}
& \hom(m,n) \times \hom(m',n') \times \hom(m'',n'') \arrow[dl, "\otimes \times 1"'] \arrow[dr, "1 \times \otimes"] & \\
\hom(m+m', n+n') \times \hom(m''+n'') \arrow[dr, "\otimes"'] & & \hom(m,n) \times \hom(m'+m'', n'+n'') \arrow[dl, "\otimes"] \\
& \hom(m+m'+m'', n+n'+n'') &
\end{tikzcd}
$$

Next, a diagram that ensures that tensoring with the identity morphism on 0 acts trivially on morphisms:

$$
\begin{tikzcd}
1 \times \hom(m,n) \arrow[d, "\iota \times 1"'] \arrow[dr] & & \hom(m,n) \times 1 \arrow[dl] \arrow[d, "1 \times \iota"] \\
\hom(0,0) \times \hom(m,n) \arrow[r, "\otimes"] & \hom(m,n) & \hom(m,n) \times \hom(0,0) \arrow[l, "\otimes"']
\end{tikzcd}
$$

Next, a diagram that ensures that the tensor product preserves composition:

$$
\begin{tikzcd}
\hom(m',n') \times \hom(l',m') \times \hom(m,n) \times \hom(l,m) \arrow[rr] \arrow[d, "\circ \times \circ"'] & & \hom(m',n') \times \hom(m,n) \times \hom(l',m') \times \hom(l,m) \arrow[d, "\otimes \times \otimes"] \\
\hom(l',n') \times \hom(l,n) \arrow[dr, "\otimes"'] & & \hom(m+m', n+n') \times \hom(l+l', m+m') \arrow[dl, "\circ"] \\
& \hom(l+l', n+n') &
\end{tikzcd}
$$

Next, a diagram that ensures the naturality of the braiding:

$$
\begin{tikzcd}
1 \times \hom(m',n') \times \hom(m,n) \arrow[d, "b \times \otimes"'] & \hom(m',n') \times 1 \times \hom(m,n) \arrow[l] \arrow[r] & \hom(m',n') \times \hom(m,n) \times 1 \arrow[d, "\otimes \times b"] \\
\hom(n+n', n'+n) \times \hom(m+m', n+n') \arrow[dr, "\circ"'] & & \hom(m+m', n+n') \times \hom(m+m', m'+m) \arrow[dl, "\circ"] \\
& \hom(m+m', n'+n) &
\end{tikzcd}
$$

Next, a diagram that ensures that the braiding is a symmetry:

$$
\begin{tikzcd}
 & 1 \times 1 \arrow[dl, "b_{m',m} \times b_{m,m'}"'] \arrow[dr] & \\
\hom(m'+m, m+m') \times \hom(m+m', m'+m) \arrow[dr, "\circ"'] & & 1 \arrow[dl, "\iota_{m+m'}"] \\
 & \hom(m+m', m+m') &
\end{tikzcd}
$$

Next, a diagram that ensures that the braidings $b_{0,n}$ are identity morphisms:

$$
\begin{tikzcd}
1 \arrow[r, bend left, "\iota_n"] \arrow[r, bend right, "b_{0,n}"'] & \hom(n, n)
\end{tikzcd}
$$

Finally, we need a diagram to ensure the braiding obeys the hexagon identities. However, since the associators are trivial and the braiding is a symmetry, the two hexagons reduce to a single triangle. To provide for this, we use the following diagram:

$$
\begin{tikzcd}
1 \arrow[ddd, "b"'] & 1\times 1\times 1\times 1 \arrow[l] \arrow[d, "b\times\iota\times\iota\times b"] \\
& \hom(m+m'', m''+m)\times\hom(m',m')\times\hom(m'',m'')\times\hom(m+m',m'+m) \arrow[d, "\otimes\times\otimes"] \\
& \hom(m'+m+m'', m'+m''+m)\times\hom(m+m'+m'', m'+m+m'') \arrow[dl, "\circ"] \\
\hom(m+m'+m'', m'+m''+m) &
\end{tikzcd}
$$

This completes the list of commutative diagrams in the sketch for Θ_{PROP}. These diagrams simply state the definition of a PROP, so there is a 1-1 correspondence between models of Θ_{PROP} in Set and props. Similarly, morphisms of models of Θ_{PROP} in Set correspond to morphisms of props. This gives an isomorphism of categories $\mathrm{PROP} \cong \mathrm{Mod}(\Theta_{\mathrm{PROP}})$ as desired. This concludes the proof. $\square$

It is worth noting that Proposition 19, and the proof just given, generalize straightforwardly from props to 'C-colored' props, with $\mathbb{N}$ replaced everywhere by the free commutative monoid on a set C, called the set of 'colors'. There is also a version for operads, a version for C-colored operads, and a version for T-typed Lawvere theories: there is a typed Lawvere theory whose models in Set are T-typed Lawvere theories! In each case we simply need to write down a sketch that describes the structure under consideration.

Proposition 19 has a wealth of consequences; we conclude with two that Erbele needed in his work on control theory [16, Prop. 6]. In rough terms, these results say that adding generators to a presentation of a prop P gives a new prop P' having P as a sub-prop, while adding equations gives a new prop P' that is a quotient of P. We actually prove more general statements.

In all that follows, let Θ be a T-typed Lawvere theory. Let $U\colon \mathrm{Mod}(\Theta)\to \mathrm{Set}^T$ be the forgetful functor and $F\colon \mathrm{Set}^T \to \mathrm{Mod}(\Theta)$ its left adjoint. Further, suppose we have two coequalizer diagrams in $\mathrm{Mod}(\Theta)$:

$$
\begin{tikzcd}
F(E) \arrow[r, shift left, "\lambda"] \arrow[r, shift right, "\rho"'] & F(\Sigma) \arrow[r, "\pi"] & P
\end{tikzcd}
$$

$$
\begin{tikzcd}
F(E') \arrow[r, shift left, "\lambda'"] \arrow[r, shift right, "\rho'"'] & F(\Sigma') \arrow[r, "\pi'"] & P'
\end{tikzcd}
$$

together with morphisms $f\colon E\to E'$, $g\colon \Sigma\to\Sigma'$ such that these squares commute:

$$
\begin{tikzcd}
F(E) \arrow[r, "\lambda"] \arrow[d, "F(f)"'] & F(\Sigma) \arrow[d, "F(g)"] \\
F(E') \arrow[r, "\lambda'"] & F(\Sigma')
\end{tikzcd}
\qquad
\begin{tikzcd}
F(E) \arrow[r, "\rho"] \arrow[d, "F(f)"'] & F(\Sigma) \arrow[d, "F(g)"] \\
F(E') \arrow[r, "\rho'"] & F(\Sigma')
\end{tikzcd}
$$

Then, thanks to the universal property of P, there exists a unique morphism $h\colon P \to P'$ making the square at right commute:

$$\begin{tikzcd}
F(E) \arrow[r, shift left, "\lambda"] \arrow[r, shift right, "\rho"'] \arrow[d, "F(f)"'] & F(\Sigma) \arrow[r, "\pi"] \arrow[d, "F(g)"'] & P \arrow[d, "h"] \\
F(E') \arrow[r, shift left, "\lambda'"] \arrow[r, shift right, "\rho'"'] & F(\Sigma') \arrow[r, "\pi'"] & P'
\end{tikzcd}$$

In this situation, adding extra equations makes P' into a quotient object of P. More precisely, and also more generally:

Corollary 52. *If g is an epimorphism, then h is a regular epimorphism.*

Proof. Given that g is an epimorphism in Set^T, it is a regular epimorphism. So is $F(g)$, since left adjoints preserve regular epimorphisms, and so is π', by definition. It follows that $\pi' \circ F(g) = h \circ \pi$ is a regular epimorphism, and thus so is h. $\square$

One might hope that in the same situation, adding extra generators makes P into a subobject of P'. More precisely, one might hope that if f is an isomorphism and g is a monomorphism, h is a monomorphism. This is not true in general, but it is when the typed Lawvere theory Θ is Θ_{PROP}.

To see why some extra conditions are needed, consider a counterexample provided by Charles Rezk [39]. There is a typed Lawvere theory with two types whose models consist of:

- a ring R,
- a set S,
- a function $f\colon S \to R$ with $f(s) = 0$ and $f(s) = 1$ for all $s \in S$.

Thanks to the peculiar laws imposed on f, the only models are pairs (R, S) where R is an arbitrary ring and S is empty, and pairs (R, S) where R is a terminal ring (one with $0 = 1$) and S is an arbitrary set. The free model on $(\emptyset, \emptyset) \in \mathrm{Set}^2$ is $(\mathbb{Z}, \emptyset)$, while the free model on $(\emptyset, 1)$ is $(\{0\}, 1)$. Thus, the monomorphism $(\emptyset, \emptyset) \to (\emptyset, 1)$ in Set^2 does not induce a monomorphism between the corresponding free models: the extra generator in the set part of (R, S) causes the ring part to 'collapse'.

This problem does not occur for Lawvere theories with just one type, nor does it happen for typed Lawvere theories that arise from typed operads, more commonly known as 'colored' operads [7, 47]. A typed Lawvere theory arises from a typed operad when it can be presented in terms of operations obeying purely equational laws for which each variable appearing in an equation shows up exactly once on each side. The laws governing props are of this form: for example, the operations for composition of morphisms obey the associative law

$$(f \circ g) \circ h = f \circ (g \circ h).$$

It follows that Θ_{PROP} arises from a typed operad, so the following corollary applies to this example:

Corollary 53. *Suppose that either Θ is a T-typed Lawvere theory with $T = 1$ or Θ arises from a T-typed operad. If f is an isomorphism and g is a monomorphism, then h is a monomorphism.*

Proof. We may assume without loss of generality that $f\colon E \to E'$ is the identity and $g\colon \Sigma \to \Sigma'$ is monic. Since monomorphisms in Set^T are just T-tuples of injections, we can write $\Sigma' \cong \Sigma + \Delta$ for some signature Δ in such a way that $g\colon \Sigma \to \Delta$ is isomorphic to the coprojection $\Sigma \to \Sigma + \Delta$. It follows that $F(g)$ is isomorphic to the coprojection $i\colon F(\Sigma) \to F(\Sigma) + F(\Delta)$, and the diagram

$$\begin{tikzcd}
F(E) \arrow[r, shift left, "\lambda"] \arrow[r, shift right, "\rho"'] \arrow[d, "F(f)"'] & F(\Sigma) \arrow[r, "\pi"] \arrow[d, "F(g)"'] & P \arrow[d, "h"] \\
F(E') \arrow[r, shift left, "\lambda'"] \arrow[r, shift right, "\rho'"'] & F(\Sigma') \arrow[r, "\pi'"] & P'
\end{tikzcd}$$

is isomorphic to this diagram:

$$\begin{tikzcd}
F(E) \arrow[r, shift left, "\lambda"] \arrow[r, shift right, "\rho"'] \arrow[d, "1"'] & F(\Sigma) \arrow[r, "\pi"] \arrow[d, "i"'] & P \arrow[d, "j"] \\
F(E') \arrow[r, shift left, "\lambda"] \arrow[r, shift right, "\rho"'] & F(\Sigma) + F(\Delta) \arrow[r, "\pi+1"] & P + F(\Delta)
\end{tikzcd}$$

where j is the coprojection from P to $P + F(\Delta)$. Thus it suffices to prove the following:

Lemma 54. *Suppose that either Θ is a T-typed Lawvere theory with $T = 1$ or Θ arises from a T-typed operad. If $P \in \mathrm{Mod}(\Theta)$ and $\Delta \in \mathrm{Set}^T$ then the coprojection $j\colon P \to P + F(\Delta)$ is a monomorphism.*

Proof. We thank Todd Trimble for this proof. First suppose $T = 1$. To show that j is monic it suffices to show that $U(j)$ is injective, since $U\colon \mathrm{Mod}(\Theta) \to \mathrm{Set}$ is faithful and thus it reflects monomorphisms [1, Prop. 11.8]. Either $U(P)$ is empty and the injectivity is trivial, or $U(P)$ is nonempty, in which case we can split the coprojection $j\colon P \to P + F(\Delta)$, since all we need for this is a morphism $F(\Delta) \to P$, or equivalently, a function $\Delta \to U(P)$.

Next suppose that Θ is a T-typed Lawvere theory that comes from a T-typed operad O. Here we can use the following construction: given $P \in \mathrm{Mod}(\Theta)$, we can form a model $P^* \in \mathrm{Mod}(\Theta)$ that has an extra element for each type $t \in T$. To do this, we first set

$$M^*(t) = M(t) \sqcup \{x_t\}$$

for all $t \in T$, where x_t is an arbitrary extra element. Then, we make P^* into an algebra of O as follows. Suppose $f \in O(t_1, \ldots, t_n; t)$ is any operation of O with inputs of type $t_1, \ldots, t_n \in T$ and output of type $t \in T$. Since P is an algebra of O, f acts on P as some function

$$P(f)\colon P(t_1) \times \cdots \times P(t_n) \to P(t).$$

Then we let f act on P^* as the function

$$P^*(f)\colon P^*(t_1) \times \cdots \times P^*(t_n) \to P^*(t).$$

that equals $P(f)$ on n-tuples $(p_1, \ldots, p_n)$ with $p_i \in P(t_i)$ for all i, and otherwise gives x_t. One can readily check that this really defines an algebra of O and thus a model of Θ. The evident morphism of models $k\colon P \to P^*$ is monic because again U is faithful [1, Prop. 14.8] and the underlying morphism of signatures $U(k)\colon U(P) \to U(P^*)$ is monic.

With this construction in hand, we can show that the coprojection $j\colon P \to P + F(\Delta)$ is monic. We have just constructed a monomorphism $k\colon P \to P^*$. Now extend this to a morphism $\ell\colon P + F(\Delta) \to P^*$: to do this, we just need a morphism $F(\Delta) \to P^*$, which we can take to be the one corresponding to the map $X \to U(P^*)$ whose component $X(t) \to U(P^*)(t)$ is the function mapping every element of $X(t)$ to x_t. Then, we have $k = \ell \circ j$, and since k is monic, j must be as well. $\square$

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