Quantum Gravity Seminar

"Quantization

&

Categorification"

Fall 2003

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(Notes by Derek K. Wise)

QUANTUM GRAVITY SEMINAR
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Quantization & Categorification

Neither of these are completely systematic processes in general yet, but they're "attempted inverses" of some more systematic process.

Quantization: going from classical mechanics to quantum mechanics.

Mathematically, one way to formalize this is as follows: In classical mechanics, observables are described using a commutative algebra of real-valued functions on the "phase space" – the space of states of the system in question. In quantization, we replace this algebra by a noncommutative algebra, where the failure of commutativity is usually on the order of Planck's constant $\hbar \cong 6.6 \times 10^{-34} \, m^2 \frac{kg}{s}$.

$m^2 \frac{kg}{s}$: unit of ACTION = length × momentum
= time × energy, where energy = $\text{mass} \times \frac{\text{length}^2}{\text{time}^2}$
= angle × angular momentum, where angular momentum = $\text{mass} \times \frac{\text{length}^2}{\text{time}}$

ACTION is the product of:

- Length & Momentum
- or Time & Energy
- or Angle & Angular Momentum

(Angle & Angular Momentum: "canonically" conjugate quantities)

Canonically conjugate quantities fail to commute.

Simplest example:

$q =$ position of particle (length)

$p =$ momentum of particle $\left(\text{mass} \cdot \frac{\text{length}}{\text{time}}\right)$

For a particle on a line, the phase space is $\mathbb{R}^2$ w. q, p as coordinate fns – in classical mechanics.

Heisenberg asserted that in quantum mechanics, we instead have

$$pq - qp = -i\hbar$$

or

$$[p, q] = -i\hbar \quad \text{for short.}$$

In Heisenberg's MATRIX MECHANICS (invented ~1925) he found certain $\infty \times \infty$ matrices p & q satisfying the canonical commutation relations $[p, q] = -i\hbar$.

This is an amazing idea, which we've been struggling to understand ever since. We'll study examples of this:

① the harmonic oscillator 

Classically the position q of the oscillator is a fn

$$q : \mathbb{R} \longrightarrow \mathbb{R}$$
$$t \longmapsto q(t)$$

(t = time)

satisfying

$$\frac{d^2q}{dt^2} = -\alpha q$$

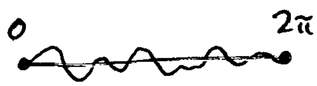
Quantizing the harmonic oscillator forces you to (re)invent:

1) Gaussians
2) Hermite Polynomials
3) Fourier Transform
4) The symplectic group

This is a quantum mechanics problem (as opposed to QFT problem - only finitely many degrees of freedom.)

We can also do quantum field theory - ∞ly many degrees of freedom. Eg:

(2) the Wave equation



(e.g. a violin string)

$$\varphi : \mathbb{R} \times [0, 2\pi] \longrightarrow \mathbb{R}$$
$$(t, x) \longmapsto \varphi(t,x)$$

↖ height of string at time t, position x.

$$\boxed{\frac{\partial^2 \varphi}{\partial t^2} = \frac{\partial^2 \varphi}{\partial x^2}}$$

Or, consider

$$\varphi : \mathbb{R} \times S^1 \longrightarrow \mathbb{R}$$

↖ identify endpts of $[0, 2\pi]$ and use periodic boundary conditions.

or :

$$\varphi : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$$

or :

$$\varphi : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R} \qquad \text{"scalar field"}$$

$$(t, \vec{x}) \longmapsto \varphi(t, \vec{x})$$

$$\frac{\partial^2 \varphi}{\partial t^2} = \nabla^2 \varphi \qquad \text{wave equation}$$

or :

$$\frac{\partial^2 \varphi}{\partial t^2} = (\nabla^2 - m^2)\varphi \qquad \text{"Klein-Gordon equation"}$$

($m \in \mathbb{R}$) — mass-m spin-0 particle

Quantizing these forces you to generalize all the previous stuff to fns of ∞ly many variables.

in particular:

i) integration on ∞-dim vector spaces (esp. Hilbert spaces)

(Problem: $\nexists$ any "decent" translation invariant countably additive measure on an infinite dimensional space)

Categorification –

the replacement of sets by categories.

In 1947, Eilenberg & MacLane invented categories, functors, & natural transformations. (in one paper! Their goal was to understand natural transformations / natural isomorphisms not to invent categories)

Def: A category $\mathcal{C}$ consists of

1. A collection of objects (if x is an object in $\mathcal{C}$, we write $x \in \mathcal{C}$)
2. given $x, y \in \mathcal{C}$ a collection of morphisms from x to y, $\hom(x,y)$.
   (If we have $f \in \hom(x,y)$ we write $f: x \to y$)
3. given $f: x \to y$ & $g: y \to z$, a morphism $fg: x \to z$ called the composite.
4. given $x \in \mathcal{C}$, a morphism $1_x : x \to x$ called the identity of x

s.t.

1. $(fg)h = f(gh)$
2. $1_x f = f = f 1_x$

Examples: Everything! e.g.

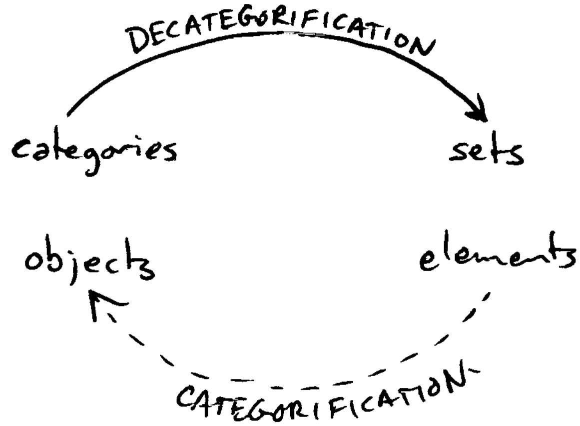
1) $\mathcal{C} = \mathrm{Set}$, where objects are sets, morphisms are functions
2) $\mathcal{C} = \mathrm{FinSet}$, where " finite sets, " " "

Decategorification:

A morphism $f: x \to y$ in some category is an <u>isomorphism</u> if $\exists\ g: y \to x$ s.t. $fg = 1_x$ & $gf = 1_y$.

<u>Prop</u>: "being isomorphic" is an equivalence relation
(We say $x \cong y$ if $\exists$ iso. $f: x \to y$).

Given a category $\mathcal{C}$, its decategorification $\underline{\mathcal{C}}$ is the collection of isomorphism classes of objects. We call the isomorphism class of $x \in \mathcal{C}$ "$\underline{x}$", so $\underline{x} \in \underline{\mathcal{C}}$.



<u>Puzzle</u>: FinSet has what famous set as its decategorification?

$x, y \in$ FinSet are isomorphic iff there's a bijection $f: x \to y$ iff x & y have the same cardinality.
So we get

$$\underline{\text{FinSet}} \xrightarrow{\ \cong\ } \mathbb{N}$$
$$\underline{x} \longmapsto |x|$$

↖ cardinality of x.

$n \times m$ in $\mathbb{N}$ corresponds to $n \times m$, the Cartesian product in FinSet:

$$\underline{n} \times \underline{m} := \underline{n \times m}$$

Similarly $\underline{n} + \underline{m}$ corresponds to the disjoint union $n+m$:

$$\underline{n} + \underline{m} := \underline{n+m}$$

This quarter we'll study "categorified quantum mechanics" in which the equations of quantum mechanics (usually viewed as numerical equations) are viewed instead as morphisms between objects in some category. Part of the weirdness of QM is actually because of decategorification!!