30 Sept 2003

"I'm going to quantize today more than categorify"

Classical Harmonic Oscillator

$q : \mathbb{R} \to \mathbb{R}$ satisfies $F = ma$

$t \longmapsto q(t)$

time position

where $m > 0$ is the mass

$a(t) = \ddot{q}(t)$ is the acceleration

$F = F(q(t))$ is the force when particle is at position $q(t)$

For the oscillator:

$$F(x) = -kx$$

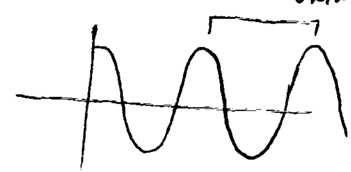
where k is the spring constant. So $F = ma$ gives

$$\ddot{q}(t) = -\frac{k}{m} q(t)$$

This has solutions

$$q(t) = \alpha \sin \sqrt{\frac{k}{m}}\, t + \beta \cos \sqrt{\frac{k}{m}}\, t$$

$\tau = \frac{2\pi}{\sqrt{k/m}}$



The period is $\frac{2\pi}{\sqrt{k/m}} = \frac{2\pi}{\omega}$ where $\omega = \sqrt{\frac{k}{m}}$ is the (angular) frequency.

Soon we will choose units of time s.t. $\omega = 1$ & units of mass s.t. $m = 1$, & thus $k = 1$.

Hamiltonian approach:

$$E(q, \dot{q}) = \underbrace{K(\dot{q})}_{\text{kinetic}} + \underbrace{V(q)}_{\text{potential}}$$

$$K(\dot{q}) = \frac{1}{2} m \dot{q}^2$$

$V(x)$ has: $F(x) = -V'(x)$ so for oscillator

$$-kx = -V'(x)$$

$$V(x) = \frac{1}{2} k x^2 \quad (+\text{ constant})$$

So, for the harmonic oscillator:

$$E(q,\dot q) = \frac{1}{2} m\dot q^2 + \frac{1}{2} k q^2$$

- so E is a quadratic form on the vector space ~~the~~ with basis $\{q, \dot q\}$. The point of energy is that it's conserved:

$$\frac{d}{dt} E(q(t),\dot q(t)) = \frac{\partial E}{\partial q}\frac{dq}{dt} + \frac{\partial E}{\partial \dot q}\frac{d\dot q}{dt}$$

$$= \underbrace{V'(q(t))}_{-\text{force}}\frac{dq}{dt} + m\frac{dq}{dt}\ddot q(t) = 0$$

Energy Conservation

+force by $F=ma$

We can introduce momentum

$$p = m\frac{dq}{dt} = \frac{\partial E}{\partial \dot q}$$

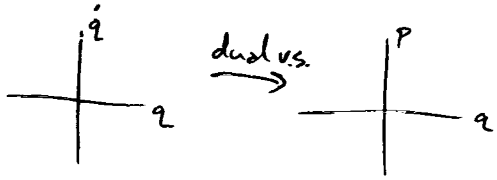
This is important because $F = ma = m\ddot q$ says $F = \dot p$

In terms of momentum & position (as opposed to velocity and position) energy gets called the "Hamiltonian":

$$H(q,p) = \frac{p^2}{2m} + V(q)$$

or specifically for our oscillator:

$$H(q,p) = \frac{p^2}{2m} + \frac{k}{2}q^2$$



a quadratic form (nondegenerate) lets us switch between these

Now let $m, k, w = 1$! $(p = \dot{q})$

$$E(q, \dot{q}) = \frac{1}{2}(q^2 + \dot{q}^2)$$

$$H(q, p) = \frac{1}{2}(q^2 + p^2)$$

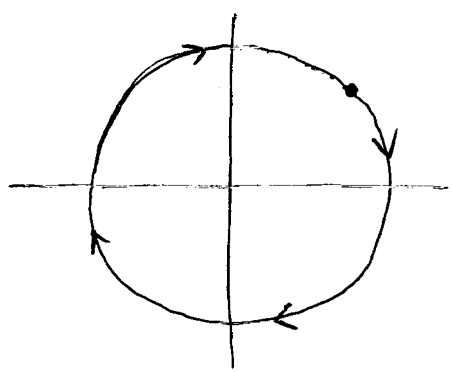
$\frac{dp}{dt} = F = -kq = -q$

$$\frac{dq}{dt} = p \qquad \frac{dp}{dt} = -q$$

This has general solution

$$q = q_0 \cos t + p_0 \sin t \qquad q(0) = q_0$$
$$p = -q_0 \sin t + p_0 \cos t \qquad p(0) = p_0$$

$(q(t), p(t))$ traces out a circle in the phase space $\mathbb{R}^2 \ni (q, p)$

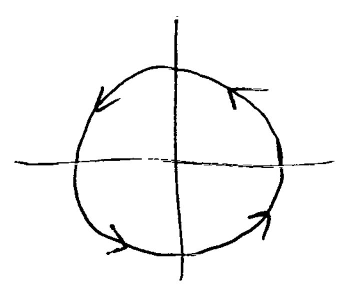


(note the direction! (Due to the bad arbitrary choice of putting i here $+$ instead of here $+$))

<u>OR</u> use the opposite order $(p(t), q(t))$ so time evolution goes counterclockwise, so that if we let

$$z = p + iq$$

then



$$\frac{dz}{dt} = -q + ip = iz$$

so

$$z(t) = e^{it} z_0$$
$$\& \ H = \frac{1}{2}(p^2 + q^2) = \frac{1}{2}|z|^2$$

(this may seem like a small point, but if we use (q, p) and have cw circles then we are forced to talk about <u>anti</u>holomorphic fns. on phase space)

Recall that if $F = ma$ & $F = -V'$ we have:

$$\frac{dq}{dt} = \frac{1}{m}p = \frac{\partial H}{\partial p} \qquad \leftarrow \boxed{H(q,p) = \frac{1}{2m}p^2 + V(q)}$$

$$\frac{dp}{dt} = F(q) = -\frac{\partial H}{\partial q}$$

Hamilton's equations:

$$\boxed{\frac{dq}{dt} = \frac{\partial H}{\partial p} \quad \Big| \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}}$$

reminiscent of $i(a+ib) = (-b+ia)$

An <u>observable</u> for classical harmonic oscillator is any function $O \in C^\infty(\underset{(p,q)}{\overset{}{\mathbb{R}^2}})$

$$\frac{d}{dt} O(p(t), q(t)) = \frac{\partial O}{\partial q}\frac{dq}{dt} + \frac{\partial O}{\partial p}\frac{dp}{dt}$$

$$= \frac{\partial O}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial O}{\partial p}\frac{\partial H}{\partial q}$$

$$= \{H, O\}$$

where the Poisson Bracket of $F, G \in C^\infty(\mathbb{R}^2)$ is

$$\{F, G\} = \frac{\partial F}{\partial p}\frac{\partial G}{\partial q} - \frac{\partial G}{\partial p}\frac{\partial F}{\partial q}$$

In fact, $C^\infty(\mathbb{R}^2), \{\cdot,\cdot\}$ is a <u>Lie algebra</u>:

1) $\{F, G\} = -\{G, F\}$
2) Jacobi id. $\{F, \{G, H\}\} = \{\{F, G\}, H\} + \{G, \{F, H\}\}$.
3) bilinear.

In fact, $C^\infty(\mathbb{R}^2)$ becomes a <u>Poisson algebra</u>:

1) a commutative algebra w. $+, \cdot$ ← (with usual + & · of smooth fns)
2) a Lie algebra w. $\{\cdot,\cdot\}$ (defined by above formula)
3) for any F, $\{F,\cdot\}$ is a derivation of comm. alg.:

$$\{F, GH\} = \{F,G\}H + G\{F,H\}$$

Moral: Observables in classical mechanics form a commutative algebra <u>BUT</u> any observable H gives rise to time evolution via: $\frac{dO}{dt} = \{H, O\}$, and time evolution is an automorphism of comm. alg because $\{H,\cdot\}$ is a derivation. ("a derivation is the derivative of an automorphism")

$$\{F, G\} = \frac{\partial F}{\partial p}\frac{\partial G}{\partial q} - \frac{\partial F}{\partial q}\frac{\partial G}{\partial p}$$

so
$$\left.\begin{aligned} \{p,p\} &= 0 \\ \{q,q\} &= 0 \\ \{p,q\} &= 1 \end{aligned}\right\} \text{ canonical commutation relations}$$

$$\{p,q^2\} = \{p,q\}q + q\{p,q\} = 2q$$
$$\begin{aligned}\{p,q^3\} &= \{p,q^2\}q + q^2\{p,q\} \\ &= 2q\cdot q + q^2\cdot 1 = 3q^2\end{aligned}$$

so Poisson bracket satisfies $\{p,\cdot\} = \frac{\partial}{\partial q}$ & $\{q,\cdot\} = -\frac{\partial}{\partial p}$ } note the analogy to momentum & position operators!

Quantum Harmonic Oscillator:

Heisenberg quantized the harmonic oscillator by inventing a noncomm. alg. generated by p & q such that if
$[F,G] = FG - GF$, then:

$$[p,p] = 0$$
$$[q,q] = 0 \qquad \text{("Heisenberg Algebra" (?))}$$
$$[p,q] = -i\hbar$$

Then he defined the energy to be an elt. of this algebra

$$H = \frac{1}{2}(p^2 + q^2)$$

& then decreed that time evolution of any observable (= alg. elt.) be

$$i\hbar \frac{dO}{dt} = [H, O]$$

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2 October 2003

Matrices

We'll consider $n \times n$ matrices $M_n(R)$ with entries in R, an arbitrary rig: a "ring without negatives."

Def: A monoid M is a set with an associative binary operation $\circ : M \times M \to M$ & a "unit" $e \in M$ s.t. $e \circ m = m = m \circ e$ $\forall m \in M$.

e.g.: $\mathbb{N}$, the free monoid on one element where $\circ = +$ $e = 0$.

e.g.: $\mathbb{N}$, w. $\circ = \cdot$, $e = 1$. (Note $\mathbb{N}^+ = \mathbb{N} - \{0\}$ w. $\circ = \cdot$, $e = 1$ is the free commutative monoid on $2, 3, 5, 7, 11, 13 \ldots$

Def: A rig M is a commutative monoid $(M, +, 0)$ together w. a monoid $(M, \cdot, 1)$ s.t. distributivity holds:

$$a \cdot (b+c) = a \cdot b + a \cdot c$$
$$(b+c) \cdot a = b \cdot a + c \cdot a$$

and $0 \cdot a = 0 = a \cdot 0$.

note we need this hypothesis, since the usual proof involves subtraction:
$(0+0)a = 0a$
$0a + 0a = 0a$

For the elite:

A rig is a monoid object in the monoidal category of comm. monoids (w. usual $\otimes$)

$\cdot : M \otimes M \to M$

<u>Prop:</u> If R is a rig, $M_n(R)$ is a rig with usual matrix $+$ & $\cdot$.

<u>Examples of rigs and their matrix rigs:</u>

0.) The set of rigs whose underlying set is $\emptyset$ is $\emptyset$.
i.e there are no rigs with no elements

1.) There is one rig (up to iso.) w. one elt., namely where $0=1$.

2.) There are two rigs with 2 elements:

A) $(\mathbb{Z}_2, +, \cdot, 0, 1)$

B) The Boolean Algebra w. 2 elts.

$$\Omega = (\{F, T\}, \underset{\text{"or"}}{\vee}, \underset{\text{"and"}}{\wedge}, F, T) \qquad \text{"truth values"}$$

Using

$$\alpha : \mathbb{Z}_2 \xrightarrow{\sim} \Omega$$
$$0 \longmapsto F$$
$$1 \longmapsto T$$

we can transfer $+, \cdot$ from $\mathbb{Z}_2$ over to Ω

+	0	1
0	0	1
1	1	0

$\longrightarrow$

	F	T
F	F	T
T	T	F

So $+$ gets renamed XOR, "exclusive or"

$\cdot$	0	1
0	0	0
1	0	1

$\longrightarrow$

	F	T
F	F	F
T	F	T

So $\cdot$ is just $\wedge$.

Also we can transfer "or" & "and" from Ω to $\mathbb{Z}_2$. Then

"a or b" gets renamed $1-(1-a)\cdot(1-b) = a+b+ab$
& "a and b" gets renamed $a \cdot b$.

$\vee$	F	T
F	F	T
T	T	T

$\longrightarrow$

	0	1
0	0	1
1	1	1

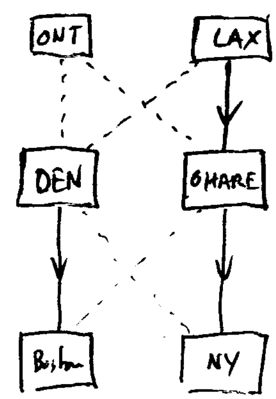
2a) How many 3elt rigs? JB doesn't know ...

$M_n(\Omega)$ = $n \times n$ matrices with entries T, F

$$\begin{pmatrix} T & F \\ F & T \end{pmatrix}\begin{pmatrix} F & T \\ F & F \end{pmatrix} = \begin{pmatrix} F & T \\ F & F \end{pmatrix}$$

$$\begin{array}{r} \text{den}\to \\ \text{ohare}\to \end{array}\begin{pmatrix} F \\ F \end{pmatrix} = \begin{pmatrix} F & T \\ F & F \end{pmatrix}\begin{pmatrix} T \\ F \end{pmatrix}\begin{array}{l} \leftarrow \text{ont} \\ \leftarrow \text{LAX} \end{array}$$

adjacency matrix of a directed graph



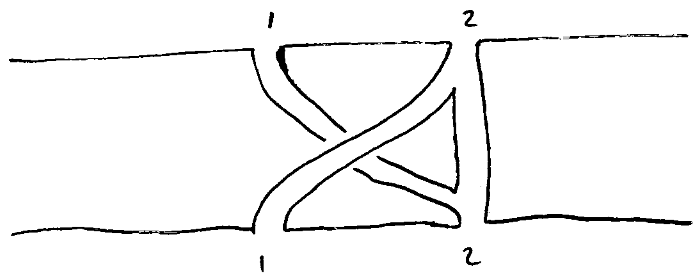
An $n \times m$ matrix of T's and F's is called a <u>relation</u>. Multiplication of these matrices is <u>composition of relations</u>.

$$\text{If } X_{ij} = \begin{cases} T & \text{if } i \text{ is the father of } j \\ F & \text{otherwise} \end{cases}$$

X is the relation "father of" and X^2 is called "grandfather of."

If X & Y are $n \times m$ matrices valued in truth values (Ω), $X+Y$ is the relation "X or Y".

An $n \times n$ matrix w. entries in Ω is a description of which input states can go to which output states



a process

$$\begin{pmatrix} F & T \\ T & T \end{pmatrix}$$

Matrix multiplication corr. to composing processes.
All the strange features of QM come from taking this concept of process and replacing Ω by $\mathbb{C}$.
One main difference is that in $\mathbb{C}$ we have additive inverses: we can have $x+y=0$ even when $x, y \neq 0$, unlike in Ω. (Note: T has no additive inverse)

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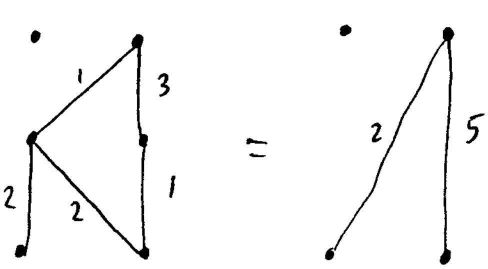
another rig example:

3) $\mathbb{N} = \{\mathbb{N}, +, \cdot, 0, 1\}$



Here we count paths instead of just saying whether there is one (T) or not (F)

$$\begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 5 \end{pmatrix}$$

$$\begin{pmatrix} T & F \\ T & T \end{pmatrix}\begin{pmatrix} F & T \\ F & T \end{pmatrix} = \begin{pmatrix} F & T \\ F & T \end{pmatrix}$$

There's a rig homomorphism

$$\begin{aligned} f: \mathbb{N} &\longrightarrow \Omega \\ 0 &\longmapsto F \\ 1,2,3,\ldots &\longmapsto T \end{aligned}$$

and this introduces a rig homo. $M_n(\mathbb{N}) \to M_n(\Omega)$

Going from $\mathbb{N}$ to $\mathbb{C}$ is still a big jump!

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7 October 2003

Matrix Mechanics

Rigs R and their matrix rigs $M_n(R)$

0.) No rigs w. zero elts.

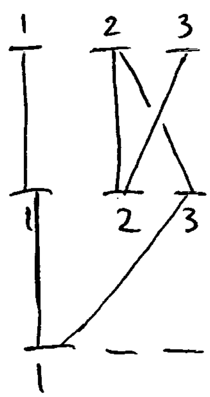
1.) One rig w. one elt. There's at most one rig homo from any rig R to this one, since there's only one map to the one-elt. set, and this indeed is a rig homo, so there's exactly one rig homo from any rig to this one! So we say this is the <u>terminal</u> rig, and call this rig 1.

2.) Two rigs w. 2 elts:

$$\mathbb{Z}_2 \ \& \ \Omega = \{\{F,T\}, \vee, \wedge, F, T\}$$

We saw that elts of $M_n(\Omega)$ are <u>binary relations</u> on the set $\{1,\ldots,n\}$

composition of processes (composition of relations) = matrix multiplication.



The entries A_{ij} of $A \in M(\Omega)$ say <u>whether or not</u> the process A can take the state i to the state j.

Matrix multiplication is "composition of processes"; addition is "superposition" of process

Note there's no destructive interference since Ω is a rig but <u>not</u> a ring. in $\mathbb{Z}_2$ there <u>is</u> destructive interference.