4 November 2003

Let's summarize what we've done so far:

We have a space with basis $\psi_0, \psi_1, \psi_2, \ldots$ on which various operators act:

creation: $a^* \psi_n = \psi_{n+1}$

annihilation: $a \psi_n = n \psi_{n-1}$

position: $q = \frac{a + a^*}{\sqrt{2}}$

momentum: $p = \frac{a - a^*}{\sqrt{2}\, i}$

Hamiltonian: $H = \frac{1}{2}(p^2 + q^2)$ $\qquad H\psi_n = (n + \frac{1}{2})\psi_n$

renormalized Hamiltonian (subtracting off ground state energy): or "number operator" $N = H - \frac{1}{2} = a^* a$ $\qquad N\psi_n = n\psi_n$

Fourier Transform: $F = (-i)^N = e^{-itN}$ where $t = \frac{\pi}{2}$ - a quarter period. $\qquad F\psi_n = (-i)^n \psi_n$

In terms of the Fock representation we call this vector space cx. polynomials in one variable, i.e. $\mathbb{C}[z]$. We call ψ_n "z^n". Thus we get:

$$a^* = M_z \quad \text{(multiplication by } z\text{)}$$

$$a = \frac{d}{dz}$$

$$N = M_z \frac{d}{dz} \qquad (N z^n = z\, n z^{n-1} = n z^n)$$

To evolve a state $\phi \in \mathbb{C}[z]$ by time t we apply e^{-itN}.

What does this do?

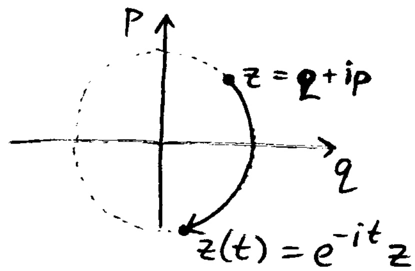
$$\phi(z) = \sum_{n=0}^{\infty} a_n z^n \qquad a_n \in \mathbb{C}$$

$$\begin{aligned}(e^{-itN}\phi)(z) &= \sum_{n=0}^{\infty} a_n e^{-itN} z^n \\ &= \sum_{n=0}^{\infty} a_n e^{-itn} z^n \\ &= \sum_{n=0}^{\infty} a_n (e^{-it} z)^n \\ &= \phi(e^{-it} z)\end{aligned}$$

Note

$$\begin{aligned}e^{-itN} z^n &= \sum \frac{(-itN)^k}{k!} z^n \\ &= \sum \frac{(-itn)^k}{k!} z^n \\ &= e^{-itn} z^n\end{aligned}$$

So: if we think of $z \in \mathbb{C}$ as a point in phase space



we can think of our state as a function of position & momentum and evolve it in time by the rule $z \mapsto e^{-it} z$, same as the classical rule:

$$\begin{aligned} q(t) &= \cos t \, q + \sin t \, p \\ p(t) &= -\sin t \, q + \cos t \, p \end{aligned} \iff z(t) = e^{-it} z$$

Moral: the Fourier transform <u>is</u> a quarter-turn in phase space (in Fock rep.)

So much for the quantum harmonic oscillator ...

...Now : <u>CATEGORIFY IT ALL!</u>

Whenever we see "$n \in \mathbb{N}$" replace it with "n, the n-element set."

First some notes about the article "the" in the above:

- Not all n-element sets are equal (unless $n=0$)
- All n-element sets are isomorphic
- Not all n-elt. sets are isomorphic in a unique way (unless $n=0,1$)

So we must use "the" carefully.

Now let's categorify polynomials, or more generally formal power series, like

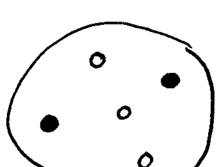
$$\phi(z) = \sum \frac{a_n z^n}{n!} \qquad \text{where } a_n \in \mathbb{N}$$

We'll say a "structure type" or "species" is a type of structure that you can put on a finite set. (Not the real definition yet!) Given a structure type Φ let $\Phi_n \in \mathbb{N}$ be the number of ways you can put this structure on an n-element set. Then define the <u>generating function</u> of Φ, say $|\Phi|$, to be

$$|\Phi|(z) = \sum \frac{\Phi_n z^n}{n!}$$

Examples:

1) Φ = "2-colorings" - ways of coloring each element of a finite set black or white



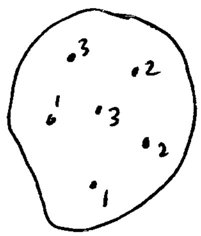
- a 2-coloring of 5.

$$\Phi_n = 2^n$$

So $$|\Phi|(z) = \sum_{n \geq 0} \frac{2^n z^n}{n!} = e^{2z}$$

We think of e^{2z} as the decategorified version of "2-colorings"

2) Φ = "k-colorings" - ways of mapping our finite set to k
= ways of coloring our finite set with the "colors" $\{1, 2, 3, \ldots, k\}$



- a 3-coloring of 6

Φ_n = # ways of k-coloring an n-elt set
$= k^n$

So $$|\Phi|(z) = \sum_{n \geq 0} \frac{k^n}{n!} z^n = e^{kz}$$

If $k = 0$ we get $e^{0z} = 1 = \sum \frac{a_n z^n}{n!}$ where $a_n = 0$ unless $n = 0$, in which case $a_0 = 1$.

If $k = 1$ we get $e^{1z} = \sum \frac{1^n z^n}{n!} = e^z$.

↑ There is 1 way to map $0 \to 0$, 0 ways to map $n \to 0$, $n \geq 1$.

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3) Suppose Φ & Ψ are structure types

Let's invent a structure type $\Phi + \Psi$ such that
$$|\Phi + \Psi| = |\Phi| + |\Psi|.$$

$$|\Phi + \Psi| = \sum \frac{(\Phi + \Psi)_n}{n!} z^n$$

so we need
$$(\Phi + \Psi)_n = \Phi_n + \Psi_n$$

The answer: A $(\Phi+\Psi)$-structure on ~~on~~ n-elt. set is:

a Φ-structure $\underline{\text{XOR}}$ a Ψ-structure

really ~~[illegible]~~ disjoint union, since a Φ-structure might also be a Ψ-structure.

E.g. If Φ = "2-colorings"

Ψ = "3-colorings"

then $\Phi+\Psi$ = "2-colorings xor 3-colorings"

$$|\Phi+\Psi|(z) = e^{2z} + e^{3z} = \sum_{n=0}^{\infty} \frac{2^n+3^n}{n!} z^n$$

---

4) Next puzzle: find $\Phi\Psi$ s.t. $|\Phi\Psi| = |\Phi||\Psi|$

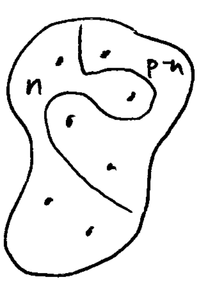
Want

$$|\Phi\Psi|(z) = |\Phi|(z)\,|\Psi|(z)$$

$$= \sum_n \frac{\Phi_n z^n}{n!} \sum_m \frac{\Psi_m z^m}{m!}$$

$$= \sum_{n,m} \frac{\Phi_n \Psi_m}{n!\,m!} z^{\overbrace{n+m}^{p}}$$

$$= \sum_p \sum_{\substack{n,m:\\ n+m=p}} \frac{\Phi_n \Psi_m}{n!\,m!} p! \frac{z^p}{p!}$$



$$= \sum_p \underbrace{\sum_{0\le n\le p} \binom{p}{n} \Phi_n \Psi_{p-n}}_{(\Phi\Psi)_p} \frac{z^p}{p!}$$

$(\Phi\Psi)_p$ = number of ways to chop up p into 2 parts & put a Φ-str. on first part and Ψ-str. on 2nd part.

Next time ... the Catalan Numbers.

6 November 2003

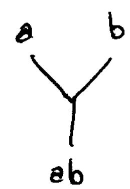
"CATALAN NUMBERS"

Named after Eugene Catalan since discovered by Gauss.

A <u>magma</u> is a set M equipped w. a binary operation $\cdot : M \times M \to M$. Consider the free magma on one element x. The elements include:

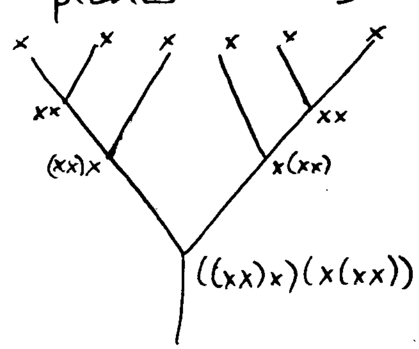
$c_1 = 1$ $\quad x$
$c_2 = 1$ $\quad xx$
$c_3 = 2$ $\quad (xx)x \quad x(xx)$
$c_4 = 5$ $\quad (xx)(xx) \quad x((xx)x) \quad ((xx)x)x \quad x(x(xx)) \quad (x(xx))x$
$\vdots$

We define c_n to be the number of elts. built from n x's in the free magma on x. (Normally people use some different convention - but this seems more basic)



This is a picture of $M \times M \to M$
(multiplication in our magma)

This lets us draw elts of the free magma on x as binary planar trees:

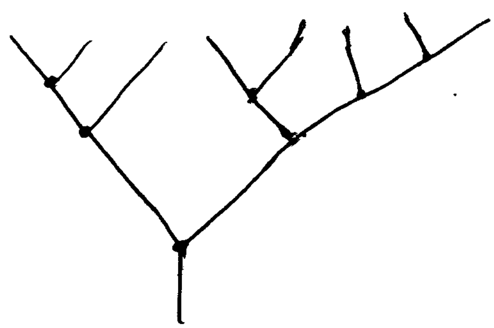


"a binary planar tree with 6 leaves"

And conversely:

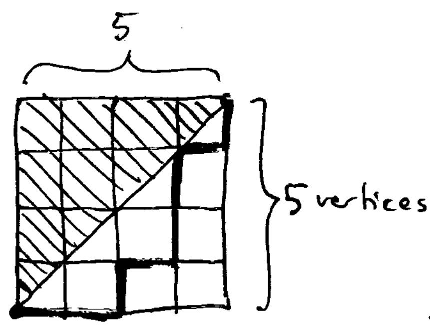
$$((xx)x)((xx)(x(xx)))$$

$$\downarrow$$



So: c_n is the number of binary planar trees with n leaves.

There are other things the Catalan #s are good for



Let's count paths on an $n \times n$ vertex grid from SW corner to NE corner that only go N or E & never enter the shaded NW region

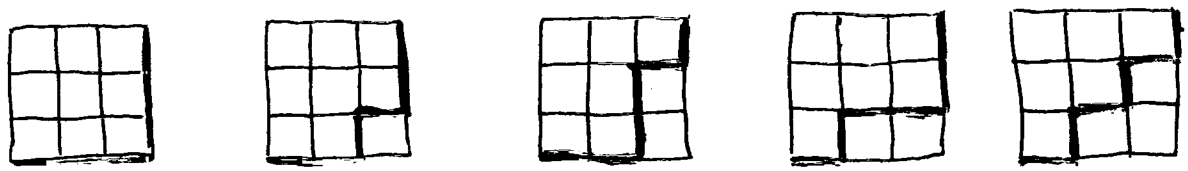
1 x

1 

$xx\cdot = (xx)$

2 

$xxx\cdot\cdot = (x(xx))$ $\quad$ $xx\cdot x\cdot = ((xx)x)$

5 

$xxxx\cdot\cdot\cdot$ $\quad$ $xxx\cdot x\cdot\cdot$ $\quad$ $xxx\cdot\cdot x\cdot$ $\quad$ $xx\cdot xx\cdot\cdot$ $\quad$ $xx\cdot x\cdot x\cdot$

Claim: the number of these paths is c_n.

An element of the free magma on x with n x's is secretly just a reverse Polish notation expression with n x's and $(n-1)$ "dots", which is secretly just a path of the above sort.

Next: Consider chopping a regular polygon into triangles:

1

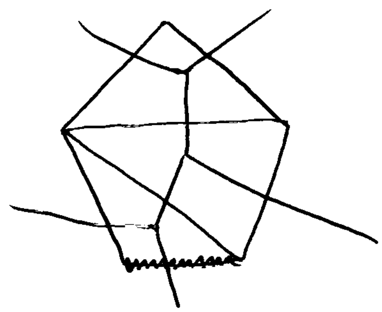
2

5

$c_1 =$
$c_2 = 1$
$c_3 = 2$
$c_4 = 5$
$c_5 = 14$
$c_6 = 42$
$\vdots$

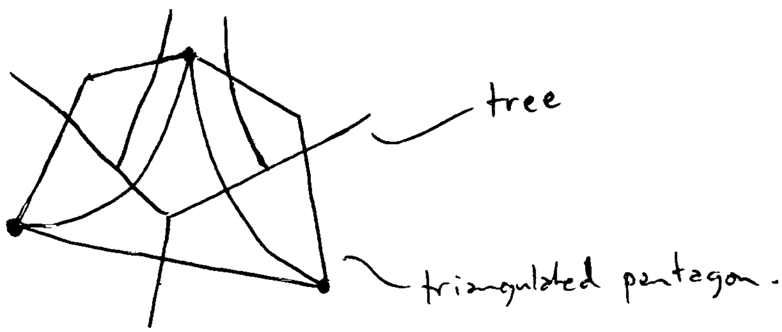
Claim: there are c_n ways to chop a regular $(n+1)$-gon into triangles.

How does this work:



Pick a side of the $(n+1)$-gon to be the "output" side and then draw the tree that is Poincaré dual to the triangulation.

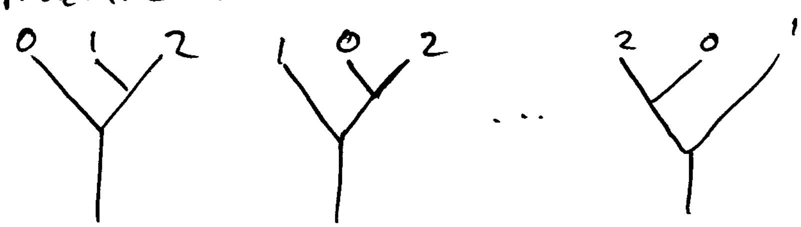
Conversely:



Let's calculate c_n!

Let T be the structure type "planar binary trees": a T-structure on a finite set is a way of making its elts into the leaves of a planar binary tree.

Eg. if our set is $3 = \{0,1,2\}$, a T-structure on it could be



etc.

for a total of $3! \cdot c_3 = 12$ T-structures.

So let $T_n = n!\, c_n$ be the number of T-structures on the set n, & get the generating function:

$$|T|(z) = \sum_{n=0}^{\infty} \frac{T_n z^n}{n!} = \sum_{n=0}^{\infty} c_n z^n$$

$$= x + x^2 + 2x^3 + 5x^4 + \cdots$$

Recall:

1) Given str. types Φ & Ψ, a "$\Phi + \Psi$-str" on a set is a Φ-structure <u>xor</u> Ψ-str. on that set.

$$|\Phi + \Psi| = |\Phi| + |\Psi|$$

2) Given str. types Φ & Ψ, to put a "$\Phi\Psi$-str." on a set is to chop the set into 2 disjoint subsets & put a Φ-str. on the first, a Ψ-str. on the second.

3) There's a str. type Z with $|Z| = z$.

Since the coefficient of z^n in this power series is 0 unless $n=1$, in which case it's 1, there are <u>No</u> ways to put a Z structure on a set unless it has 1 element, in which case there is one way. So we say

$$Z = \text{"being a 1-element set"}$$

More generally, there's a structure type called $\frac{Z^n}{n!}$, or "being an n-element set," & whose ~~the~~ generating function is:

$$\left|\frac{Z^n}{n!}\right| = \frac{z^n}{n!}$$

If $n=0$ we get a str. type $\mathbf{1}$ with

$$|\mathbf{1}| = 1$$

& is "being the empty set."

Use these to calculate $|T|(z)$!