Given $f: S \to S'$ a bijection

$$Z(f) : Z(S) \to Z(S')$$

is the only possible function; the identity function.

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Structure Types: a <u>structure type</u> is a functor

$$\Phi : \mathrm{FinSet}_0 \longrightarrow \mathrm{Set}$$

where FinSet_0 is the groupoid of finite sets (& bijections) & Set is the category of sets (& functions).

There's a category of structure types: structure types are the objects; what are the morphisms?

$$\begin{tikzcd}
\mathrm{FinSet}_0 \arrow[r, bend left=50, "\Phi"{name=U}] \arrow[r, bend right=50, "\Psi"'{name=D}] & \mathrm{Set}
\arrow[Rightarrow, from=U, to=D, "T"']
\end{tikzcd}$$

They should be <u>natural</u> <u>transformations</u>.

<u>Def</u>: Given categories C & D, functors $\Phi, \Psi : C \to D$, we define a natural transformation $T: \Phi \Rightarrow \Psi$ to be a function assigning to each object $x \in C$ a morphism $T_x : \Phi(x) \longrightarrow \Psi(x)$ in D s.t. for any morphism $f: x \to y$

$$\begin{tikzcd} \Phi(x) \arrow[r, "\Phi(f)"] \arrow[d, "T_x"'] & \Phi(y) \arrow[d, "T_y"] \\ \Psi(x) \arrow[r, "\Psi(f)"] & \Psi(y) \end{tikzcd}$$

the "naturality" square

commutes.

(Note: natural transformations "boost the dimension by 1"
they take objects and send them to morphisms
they take morphisms and send them to commutative diagrams)

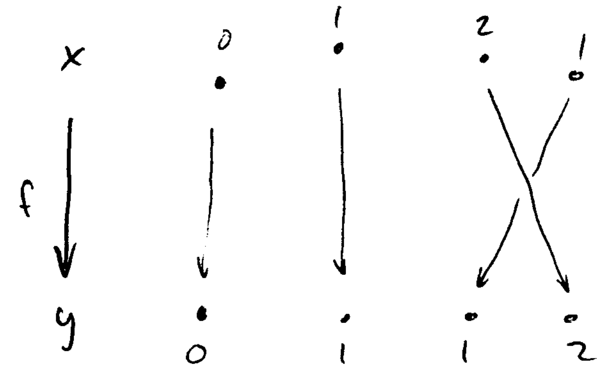
What does this amount to given structure types $\Phi, \Psi : \mathrm{FinSet}_0 \to \mathrm{Set}$? If x is a finite set, $\Phi(x)$ is the set of all Φ-structures on x, $\Psi(x)$ is the set of all Ψ-structures on x, and $T_x : \Phi(x) \to \Psi(x)$ lets us turn any Φ-str. into a Ψ-str.

Let Φ = "3-colorings", so that $\Phi(x) = 3^x$, i.e. the set of maps $k : x \to 3 = \{0,1,2\}$, and given a bijection $f : x \to y$, $\Phi(f) : \Phi(x) \to \Phi(y)$ is given by:

$$\Phi(f)k = k \circ f^{-1} \qquad\qquad k : x \to 3$$

Check that Φ is a functor, i.e. $\Phi(fg) = \Phi(f)\Phi(g)$:

$$\begin{aligned} \Phi(fg)k &= k \circ (fg)^{-1} \\ &= k \circ g^{-1} \circ f^{-1} \\ &= \Phi(f)(k \circ g^{-1}) \\ &= \Phi(f)(\Phi(g)k) = (\Phi(f)\Phi(g))k. \end{aligned}$$



transferring the 3-coloring to y, via the bijection f.

Let Ψ = "2-colorings", so $\Psi(x) = 2^x$, etc., where $2 = \{0,1\}$. An example of a natural transformation $T : \Phi \Longrightarrow \Psi$ is:

$$T_x : \Phi(x) \longrightarrow \Psi(x) \quad \text{given by} \quad T_x k = r \circ k$$

where r, the "recoloring function", is any function $r : 3 \to 2$.

$$\begin{tikzcd} x \arrow[r] \arrow[dr, "T_x k"'] & 3 \arrow[d, "r"] \\ & 2 \end{tikzcd}$$

Let's check that T makes the naturality square commute: given a bijection $f : x \to y$, we want:

$$\begin{tikzcd} \Phi(x) \arrow[r, "\Phi(f)"] \arrow[d, "T_x"'] & \Phi(y) \arrow[d, "T_y"] \\ \Psi(x) \arrow[r, "\Psi(f)"] & \Psi(y) \end{tikzcd}$$

to commute

i.e. we need to show $T_y \Phi(f) = \Psi(f) T_x$. Given $k : x \to 3$ check $T_y \Phi(f) k = \Psi(f) T_x k$.

$$T_y(k \circ f^{-1}) \overset{?}{=} \psi(f)(f \circ k)$$
$$r \circ (k \circ f^{-1}) = (r \circ k) \circ f^{-1}$$

$$\begin{tikzcd} x \arrow[r] \arrow[dr, "T_x"'] & 3 \arrow[d, "r"] & y \arrow[l] \arrow[dl, "T_y"] \\ & 2 & \end{tikzcd}$$

This only worked because we used the same recoloring function r for defining T_x & T_y, i.e. naturality says we're changing colors in a systematic way for all finite sets $x, y, \ldots$.

Digression:
Note: associativity is a special case of commutativity:

Associativity:

$$(ab)c = a(bc)$$
$$R_c L_a b = L_a R_c b$$
$$R_c L_a = L_a R_c$$

"left & right multiplication commute

Picture Proof:

$$\begin{tikzcd} x \arrow[r, "k"] \arrow[rr, bend right, "T_x k"'] \arrow[d, "f"'] & 3 \arrow[r, "r"] \arrow[d, "1_3"] & 2 \arrow[d, "1_2"] \\ y \arrow[r, "\Phi(f)k"] \arrow[rr, bend right, "\psi(f) T_x k"'] & 3 \arrow[r, "r"] & 2 \end{tikzcd}$$

$$T_x k \circ f^{-1} = \Phi(f) T_x k$$
$$\overset{?}{=} T_y \Phi(f) k$$

Yes, since prism commutes.

The category of species (structure types) deserves the name hom(FinSet$_0$, Set), since hom(C, D) for any categories C, D is the category whose objects are functors $\Phi : C \to D$ & whose morphisms are natural transformations between these. It also deserves the name Set$[\![x]\!]$. We had started doing quantum mechanics in the Fock representation, where states are $\varphi \in \mathbb{C}[\![x]\!]$; now we've categorified & gotten Φ w. $|\Phi| = \varphi$, and they're really categorified formal power series w. Set replacing $\mathbb{C}$.

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2 Dec 2003

Today we categorify differentiation ...

Now we know str. types are functors

$$\Phi : \mathrm{FinSet}_0 \longrightarrow \mathrm{Set}$$

& we say "n" for the n-elt. set, we can use Φ_n for the set of Φ-structures on n, as opposed to the number of such structures as before. The number of such structures will be $|\Phi_n|$ (i.e. $|\cdot|$ denotes cardinality).

The generating function of Φ is then

$$|\Phi|(z) = \sum_{n=0}^{\infty} \frac{|\Phi_n|}{n!} z^n$$

Any structure type (for which $|\Phi_n| < \infty$) thus gives an elt. $|\Phi| \in \mathbb{C}[\![z]\!]$. In the Fock representation of the Weyl algebra, p & q become operators on $\mathbb{C}[\![z]\!]$ (or $\mathbb{C}[z]$) via:

$$a^* = M_z \qquad\qquad q = \frac{a + a^*}{\sqrt{2}}$$

$$a = \frac{d}{dz} \qquad\qquad p = \frac{a - a^*}{\sqrt{2}\, i}$$

(operators on formal power series)