quantum mechanics in the Fock representation, where states are $\varphi \in \mathbb{C}[[x]]$; now we've categorified & gotten Φ w. $|\Phi| = \varphi$, and they're really categorified formal power series w. Set replacing $\mathbb{C}$.

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2 Dec 2003

Today we categorify differentiation . . .

Now we know str. types are functors

$$\Phi : \mathrm{FinSet}_0 \longrightarrow \mathrm{Set}$$

& we say "n" for the n-elt. set, we can use Φ_n for the set of Φ-structures on n, as opposed to the number of such structures as before. The number of such structures will be $|\Phi_n|$ (i.e. $|\cdot|$ denotes cardinality).

The generating function of Φ is then

$$|\Phi|(z) = \sum_{n=0}^{\infty} \frac{|\Phi_n|}{n!} z^n$$

Any structure type (for which $|\Phi_n| < \infty$) thus gives an elt. $|\Phi| \in \mathbb{C}[[z]]$. In the Fock representation of the Weyl algebra, p & q become operators on $\mathbb{C}[[z]]$ (or $\mathbb{C}[z]$) via:

$$a^* = M_z \qquad\qquad q = \frac{a + a^*}{\sqrt{2}}$$

$$a = \frac{d}{dz} \qquad\qquad p = \frac{a - a^*}{\sqrt{2}\, i}$$

(operators on formal power series)

such that

$$a a^* - a^* a = 1$$

So, let's find "operators" on the category of str. types, A^* & A, which ~~act~~ like a^* & a. Specifically:

$$|A^* \Phi| = a^* |\Phi|$$

(i.e. applying A^* and then decategorifying is the same as applying the ~~same~~ decategorification a^* of A^* to the decategorification $|\Phi|$ of Φ)

$$\text{and} \quad |A \Phi| = a |\Phi|$$

"The bane of categorification is the minus sign" so we want to rewrite $a a^* - a^* a = 1$ and require

$$AA^* \cong A^* A + 1 \quad \leftarrow \text{(naturally isomorphic as functors)}$$

for A & A^*.

A^* is easy to figure out since we know how to multiply structure types

$$A^* \Phi = Z \Phi$$

where Z is the structure type w. $|Z| = z$, i.e. the structure "being the 1-elt. set." So, putting a $Z\Phi$-str. on a set n is writing n as a disjoint union of two sets p & q & putting a Z-str.

on p & $\underline{\Phi}$-str. on q. That is, putting a $Z\underline{\Phi}$ structure on n means choosing an element ~~and~~ $x \in n$ and a $\underline{\Phi}$-structure on $n - \{x\}$. So:

$$\begin{aligned} |A^*\underline{\Phi}| &= |Z\underline{\Phi}| \\ &= |Z||\underline{\Phi}| \\ &= z|\underline{\Phi}| \\ &= a^*|\underline{\Phi}|. \end{aligned}$$

<u>Example:</u> Let $\underline{\Phi}$ be "being an n-element set", i.e.:

$$\underline{\Phi}_m = \begin{cases} 0 & m \not\cong n \\ 1 & m \cong n \end{cases}$$

Here

$$|\underline{\Phi}|(z) = \frac{z^n}{n!}.$$

Now what is $A^*\underline{\Phi}$? To put this structure on a set S, we choose $x \in S$ & put the str. "being an n-elt set" on $S - \{x\}$. I.e. it's "being an n+1 elt. set w. chosen point x." There are n+1 ways to make that choice so:

being an (n+1)-elt. pointed set

$$|A^*\underline{\Phi}|(z) = (n+1)\frac{z^{n+1}}{(n+1)!} = \frac{z^{n+1}}{n!} = z \cdot \frac{z^n}{n!}$$

Another example: let Φ be the str.-type "being a totally ordered n-elt. set."

$$|\Phi_m| = \begin{cases} 0 & n \not\cong m \\ n! & n \cong m \end{cases}$$

So

$$|\Phi|(z) = z^n$$

Now an $A^*\Phi$ str. on S is "picking an elt $x \in S$ and totally ordering $S - \{x\}$" (which must have n elements) But note this is the same as ordering the whole set. I.e. this str.-type is isomorphic to "being a totally ordered n+1-elt. set."

So if we use Z^n to mean "being a totally ordered n-elt. set" we see

$$A^* Z^n \cong Z^{n+1}$$

& thus

$$|A^* Z^n| = |Z^{n+1}|$$

$$\Downarrow$$

$$a^* z^n = z^{n+1}$$

which is what we want for a "creation operator."

What about A? What can we do to a str. type that has the effect of differentiating (formally) its generating function?

If Φ has gen. function

$$|\Phi|(z) = \sum_{n=0}^{\infty} \frac{|\Phi_n|}{n!} z^n$$

we want a str. type $A\Phi = \frac{D}{DZ}\Phi$ w.

$$\left|\frac{D}{DZ}\Phi\right|(z) = \frac{d}{dz}|\Phi|(z) = \sum_{n=\phi}^{\infty} \frac{|\Phi_n|}{(n-1)!} z^{n-1}$$

$$= \sum_{n=0}^{\infty} \frac{|\Phi_{n+1}|}{n!} z^n$$

So there must be $|\Phi_{n+1}|$ ways to put an $A\Phi$-str. on an n-elt. set. So: say an $A\Phi$-str. on the set n is a Φ-str. on the set $n+1$.

Examples: Let Φ be the str. "being a finite set" — I.e. $\Phi_n = 1$. This is called the "vacuous structure" — every finite set has this in exactly 1 way!

$$|\Phi|(z) = \sum_{n=0}^{\infty} \frac{|\Phi_n|}{n!} z^n \qquad \text{but } |\Phi_n| = 1$$

$$= \sum_{n=0}^{\infty} \frac{Z^n}{n!} = e^Z$$

So we should call this Φ "E^Z"!

We know $\frac{d}{dZ} e^Z = e^Z$, but is $\frac{D}{DZ} E^Z \cong E^Z$?

This would imply $\frac{d}{dz} e^Z = e^Z$ by applying $|\cdot|$ to both sides.

To put a $\frac{D}{DZ} E^Z$ - str. on a finite set S is to put the E^Z structure (structure of being a finite set) on $S+1$. This is the same as putting the str. of being a finite set on S, so

$$\frac{D}{DZ} E^Z = E^Z.$$

So $\frac{d}{dz} e^z = e^z$ means "S is finite iff $S+1$ is."

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4 Dec 2003

Creation and Annihilation operators - Categorified Version

Recall:

$$A^* \Phi = Z \Phi \qquad \text{creation}$$

$$A \Phi = \frac{D}{DZ} \Phi \qquad \text{annihilation}$$

An $A^*\Phi$ - structure on a finite set S is a choice of an element $x \in S$ and a Φ - str. on $S - \{x\}$. An $A\Phi$ - structure on a finite set S is a Φ - structure on $S+1$. Note the irksome contravariance: "creation" is related to removing an element while "annihilation" is related to throwing in an extra element

Examples:

1) $\Phi = \dfrac{Z^n}{n!}$ = "being an n-element set"

$A\Phi = \dfrac{Z^{n-1}}{(n-1)!}$ = "being an (n-1)-element set"

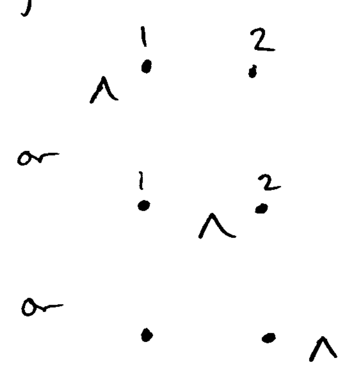
since putting the structure "being an n-elt. set" on $S+1$ is the same as putting the structure "being an (n-1)-elt." set on S. (This shows the irksome contravariance)

2) $\Psi = Z^n$ = "being ~~a~~ totally ordered n-elt set"

Let's do this one without cheating:

$A\Psi$ should be like "being ~~a~~ totally ordered (n-1)-elt. set with a 'mark'"

e.g. if $n=3$



i.e. $A\Psi = nZ^{n-1}$ since putting a total ordering on $S+1$ is same as putting a total ordering on S & equipping it with a "mark" (indicating the position of the extra elt. in $S+1$)

3) $\Phi = \dfrac{Z^n}{n!}$ = "being an n-elt. set"

$$A^*\Phi = Z\frac{Z^n}{n!}$$

$$= (n+1)\frac{Z^{n+1}}{(n+1)!}$$ = "being an (n+1)-elt. pointed set"

since choosing $x \in S$ & putting the structure "being an n-elt. set" on $S-\{x\}$ is the same as putting the structure "being a pointed (n+1)-elt. set" on S.

4) $\Psi = Z^n$ = "being a totally ordered n-elt. set."

$A^*\Psi = Z^{n+1}$ = "being a totally ordered (n+1)-elt set"

since choosing $x \in S$ & putting ~~a total ordering~~ the structure "total ordering on an n-elt set" on $S-\{x\}$ is the same (isomorphic) as putting the structure "total ordering on an (n+1)-elt. set" on S.

And now the punchline (of the whole course)

• • •

Newton thought

$$pq = qp$$

(presumably, if anyone had bothered to ask him)

Heisenberg said: not quite!

$$pq = qp - i\hbar$$

For us, p & q come from a & a^* via

$q = \frac{a+a^*}{\sqrt{2}}$, $p = \frac{a-a^*}{\sqrt{2}i}$ & noncommutativity of p & q comes from

$$aa^* = a^*a + 1 \qquad (\hbar = 1)$$

But why is this true?

Think of the energy levels of the harmonic oscillator as saying how many "quanta of energy" it has. These mysterious "quanta" are, for us, the elements of these <u>finite sets</u> we're discussing now. This makes $aa^* = a^*a + 1$ <u>obvious</u>, since it follows from the categorified version:

$$AA^* \cong A^*A + 1$$

which is even more obvious.

Huh? ... We'll show

$$AA^*\Phi \cong A^*A\Phi + \Phi$$

for any structure type.

- An $A^*A\Phi$-str. on S is:
  - a choice of $x \in S$ and an $A\Phi$-str. on $S-\{x\}$, which is:
    - a choice of $x \in S$ and ~~a~~ Φ-str. on $S-\{x\}+1$.

- An $AA^*\Phi$-str. on S is:
  - an $A^*\Phi$-str. on $S+1$, which is:
    - a choice of $x \in S+1$ and a Φ-str. on $(S+1)-\{x\}$.

      i.e.: <u>either</u> a choice of $x \in S$ and a Φ-str. on $(S+1)-\{x\} = (S-\{x\})+1$

      <u>xor</u> a Φ structure on $(S+1)-\{x\} = S$ (in the case where the element removed is the same as the one just added)

      i.e: <u>either</u> an $A^*A\Phi$-str. on S

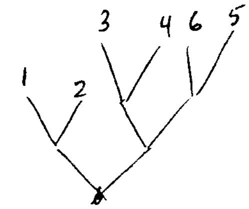
      <u>xor</u> a Φ-str. on S

      i.e: $A^*A\Phi + \Phi$ str. on S

So $\boxed{AA^*\Phi \cong A^*A\Phi + \Phi}$

Moral: Noncommutativity comes from the fact that "putting a ball in a box w. n balls in it, then taking one out" can be done in one more way than: ~~"putting~~ "taking a ball out of a box w. n balls in it and then putting one in".

The str-type "binary trees," T has:



A T-str. on $\{1, \ldots, 6\}$

$$T \cong Z + T^2$$

either the 1-elt. set or two trees

So
$$|T| = |Z + T^2|$$
$$= z + |T|^2$$

$$|T|^2 - |T| + z = 0$$

$$|T|(z) = \frac{1 - \sqrt{1-4z}}{2} = z + z^2 + 2z^3 + 5z^4 + 14z^5 + 42z^6 + \cdots$$

Suppose $z = 1$. We get
$$e^{-\frac{i\pi}{3}} = \frac{1 - \sqrt{3}\, i}{2} = 1 + 1 + 2 + 5 + 14 + 42 + \cdots$$

So: The right hand side is the sum of all Catalan Numbers, i.e. the "cardinality" of the set B

of all (planar) binary trees!

$$|B| = e^{-\frac{i\pi}{3}} \quad \text{in some sense.}$$

So $$|B|^6 = 1$$

Alas, $B^6 \not\cong 1$. BUT $B^7 \cong B$ in a very nice way!! There's a natural isomorphism between trees and 7-tuples of trees. Using $B \cong 1 + B^2$ we get $B \cong B^7$!