# Quantum Gravity Seminar

"Quantization

&

Categorification"

Fall 2003

John C. Baez

(Notes by Derek K. Wise)

QUANTUM GRAVITY SEMINAR
Fall 2003

25 Sept. 2003

## Quantization & Categorification

Neither of these are completely systematic processes in general yet, but they're "attempted inverses" of some more systematic process.

Quantization: going from classical mechanics to quantum mechanics.

Mathematically, one way to formalize this is as follows: In classical mechanics, observables are described using a commutative algebra of real-valued functions on the "phase space" – the space of states of the system in question. In quantization, we replace this algebra by a noncommutative algebra, where the failure of commutativity is usually on the order of Planck's constant $\hbar \cong 6.6 \times 10^{-34} \underbrace{\frac{m^2 kg}{s}}_{\text{unit of ACTION}}$.

unit of ACTION $= \text{length} \times \text{momentum}$

$= \text{time} \times \underbrace{\text{energy}}_{\text{mass} \times \frac{\text{length}^2}{\text{time}^2}}$

$= \text{angle} \times \underbrace{\text{angular momentum}}_{\text{mass} \times \frac{\text{length}^2}{\text{time}}}$

ACTION is the product of:

Length & Momentum

or Time & Energy

or Angle & Angular Momentum

"canonically" conjugate quantities"

Canonically conjugate quantities fail to commute.

Simplest example:

$q =$ position of particle (length)

$p =$ momentum of particle $\left(\text{mass} \cdot \frac{\text{length}}{\text{time}}\right)$

For a particle on a line, the phase space is $\mathbb{R}^2$ w. $q, p$ as coordinate fns – in classical mechanics.

Heisenberg asserted that in quantum mechanics, we instead have

$$pq - qp = -i\hbar$$

or

$$[p, q] = -i\hbar \quad \text{for short.}$$

In Heisenberg's MATRIX MECHANICS (invented ~1925) he found certain $\infty \times \infty$ matrices $p$ & $q$ satisfying the canonical commutation relations $[p, q] = -i\hbar$.

This is an amazing idea, which we've been struggling to understand ever since. We'll study examples of this:

(1) the harmonic oscillator 

Classically the position $q$ of the oscillator is a fn

$$q : \mathbb{R} \longrightarrow \mathbb{R}$$
$$t \longmapsto q(t)$$

↖ time

satisfying

$$\frac{d^2 q}{dt^2} = -\alpha q$$

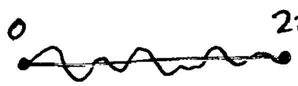
Quantizing the harmonic oscillator forces you to (re)invent:

1) Gaussians
2) Hermite Polynomials
3) Fourier Transform
4) The symplectic group

This is a quantum mechanics problem (as opposed to QFT problem - only finitely many degrees of freedom.)

We can also do quantum field theory - $\infty$ly many degrees of freedom. Eg:

(2) the Wave equation




(e.g. a violin string)

$$\varphi : \mathbb{R} \times [0, 2\pi] \longrightarrow \mathbb{R}$$
$$(t, x) \longmapsto \varphi(t,x)$$

height of string at time $t$, position $x$.

$$\boxed{\frac{\partial^2 \varphi}{\partial t^2} = \frac{\partial^2 \varphi}{\partial x^2}}$$

Or, consider

$$\varphi : \mathbb{R} \times S^1 \longrightarrow \mathbb{R}$$

identify endpts of $[0, 2\pi]$ and use periodic boundary conditions.

or:

$$\varphi : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$$

or:

$$\varphi : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R} \qquad \text{"scalar field"}$$
$$(t, \vec{x}) \longmapsto \varphi(t, \vec{x})$$

$$\frac{\partial^2 \varphi}{\partial t^2} = \nabla^2 \varphi \qquad \text{Wave equation}$$

or:

$$\frac{\partial^2 \varphi}{\partial t^2} = (\nabla^2 - m^2)\varphi \qquad \text{"Klein-Gordon equation"}$$
$$(m \in \mathbb{R} \qquad \text{mass-}m \text{ spin-0 particle}$$

Quantizing these forces you to generalize all the previous stuff to fns of $\infty$ly many variables.
in particular:

i) integration on $\infty$-dim vector spaces (esp. Hilbert spaces)

(Problem: $\nexists$ any "decent" translation invariant countably additive measure on an infinite dimensional space)

## Categorification –

the replacement of sets by categories.

In 1947, Eilenberg & MacLane invented categories, functors, & natural transformations. (in one paper! Their goal was to understand natural transformations / natural isomorphisms not to invent categories)

**Def:** A category $\mathcal{C}$ consists of

1. A collection of objects (if $x$ is an object in $\mathcal{C}$, we write $x \in \mathcal{C}$)
2. given $x, y \in \mathcal{C}$ a collection of morphisms from $x$ to $y$, $\hom(x,y)$.
   (If we have $f \in \hom(x,y)$ we write $f: x \to y$)
3. given $f: x \to y$ & $g: y \to z$, a morphism $fg: x \to z$ called the composite.
4. given $x \in \mathcal{C}$, a morphism $1_x : x \to x$ called the identity of $x$

s.t.

1. $(fg)h = f(gh)$
2. $1_x f = f = f 1_x$

**Examples:** Everything! e.g.

1) $\mathcal{C} = \mathrm{Set}$, where objects are sets, morphisms are functions
2) $\mathcal{C} = \mathrm{FinSet}$, where " finite sets, " " "

## Decategorification:

A morphism $f: x \to y$ in some category is an <u>isomorphism</u> if $\exists\ g: y \to x$ s.t. $fg = 1_x$ & $gf = 1_y$.

<u>Prop:</u> "being isomorphic" is an equivalence relation (we say $x \cong y$ if $\exists$ iso. $f: x \to y$).

Given a category $\mathcal{C}$, its decategorification $\underline{\mathcal{C}}$ is the collection of isomorphism classes of objects. We call the isomorphism class of $x \in \mathcal{C}$ "$\underline{x}$", so $\underline{x} \in \underline{\mathcal{C}}$.

$$
\begin{tikzcd}
\text{categories} \arrow[rr, bend left, "\text{DECATEGORIFICATION}"] & & \text{sets} \\
\text{objects} & & \text{elements} \arrow[ll, dashed, bend left, "\text{CATEGORIFICATION}"]
\end{tikzcd}
$$

<u>Puzzle:</u> FinSet has what famous set as its decategorification?

$x, y \in$ FinSet are isomorphic iff there's a bijection $f: x \to y$ iff $x$ & $y$ have the same cardinality. So we get

$$
\begin{aligned}
\underline{\text{FinSet}} &\xrightarrow{\ \cong\ } \mathbb{N} \\
\underline{x} &\longmapsto |x|
\end{aligned}
$$

cardinality of $x$.

$n \times m$ in $\mathbb{N}$ corresponds to $\underline{n} \times \underline{m}$, the Cartesian product in FinSet:

$$\underline{n \times m} := \underline{n} \times \underline{m}$$

Similarly $n + m$ corresponds to the disjoint union $\underline{n} + \underline{m}$:

$$\underline{n + m} := \underline{n} + \underline{m}$$

---

This quarter we'll study "categorified quantum mechanics" in which the equations of quantum mechanics (usually viewed as numerical equations) are viewed instead as morphisms between objects in some category. Part of the weirdness of QM is actually because of decategorification!!

## 30 Sept 2003

"I'm going to quantize today more than categorify"

### Classical Harmonic Oscillator

$q : \mathbb{R} \to \mathbb{R}$

$t \mapsto q(t)$

time  position

satisfies $F = ma$

where $m > 0$ is the mass

$a(t) = \ddot{q}(t)$ is the acceleration

$F = F(q(t))$ is the force when particle is at position $q(t)$

For the oscillator:

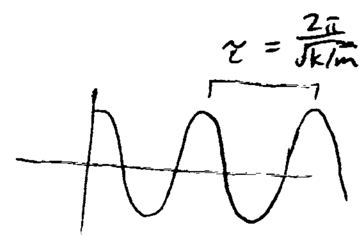
$$F(x) = -kx$$

where $k$ is the spring constant. So $F = ma$ gives

$$\ddot{q}(t) = -\frac{k}{m} q(t)$$

This has solutions

$$q(t) = \alpha \sin \sqrt{\frac{k}{m}}\, t + \beta \cos \sqrt{\frac{k}{m}}\, t$$




The period is $\frac{2\pi}{\sqrt{k/m}} = \frac{2\pi}{\omega}$ where $\omega = \sqrt{\frac{k}{m}}$ is the (angular) frequency.

Soon we will choose units of time s.t. $\omega = 1$ & units of mass s.t. $m = 1$, & thus $k = 1$.

### Hamiltonian approach:

$$E(q, \dot{q}) = \underbrace{K(\dot{q})}_{\text{kinetic}} + \underbrace{V(q)}_{\text{potential}}$$

$$K(\dot{q}) = \frac{1}{2} m \dot{q}^2$$

$V(x)$ has: $F(x) = -V'(x)$ so for oscillator

$$-kx = -V'(x)$$

$$V(x) = \frac{1}{2} k x^2 \quad (+\text{ constant})$$

So, for the harmonic oscillator:

$$E(q,\dot{q}) = \frac{1}{2} m \dot{q}^2 + \frac{1}{2} k q^2$$

- so $E$ is a quadratic form on the vector space ~~of the~~ with basis $\{q, \dot{q}\}$. The point of energy is that it's conserved:

$$\frac{d}{dt} E(q(t), \dot{q}(t)) = \frac{\partial E}{\partial q} \frac{dq}{dt} + \frac{\partial E}{\partial \dot{q}} \frac{d\dot{q}}{dt}$$

$$= \underbrace{V'(q(t))}_{-\text{force}} \frac{dq}{dt} + \underbrace{m \frac{d\dot{q}}{dt}}_{+\text{force}} \dot{q}(t) = 0 \quad \text{by } F = ma$$

Energy Conservation

We can introduce momentum

$$p = m \frac{dq}{dt} = \frac{\partial E}{\partial \dot{q}}$$

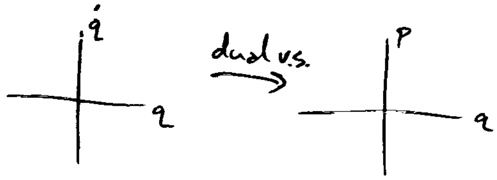
This is important because $F = ma = m\ddot{q}$ says $F = \dot{p}$

In terms of momentum & position (as opposed to velocity and position) energy gets called the "Hamiltonian":

$$H(q,p) = \frac{p^2}{2m} + V(q)$$

or specifically for our oscillator:

$$H(q,p) = \frac{p^2}{2m} + \frac{k}{2} q^2$$




a quadratic form (nondegenerate) lets us switch between these

Now let $m, k, w = 1$ !   $(p = \dot{q})$

$$E(q, \dot{q}) = \frac{1}{2}(q^2 + \dot{q}^2)$$

$$H(q, p) = \frac{1}{2}(q^2 + p^2)$$

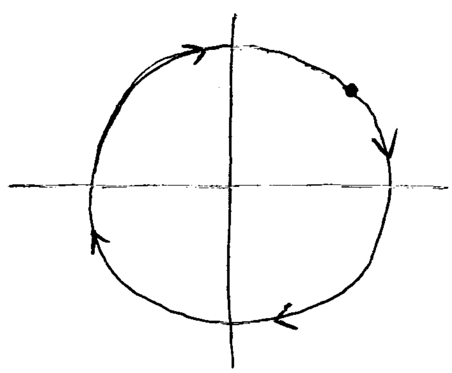
$\frac{dp}{dt} = F = -kq = -q$

$$\frac{dq}{dt} = p \qquad \frac{dp}{dt} = -q$$

This has general solution

$$q = q_0 \cos t + p_0 \sin t \qquad q(0) = q_0$$
$$p = -q_0 \sin t + p_0 \cos t \qquad p(0) = p_0$$

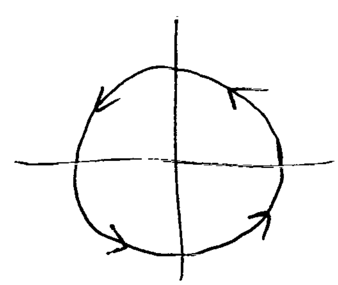
$(q(t), p(t))$ traces out a circle in the phase space $\mathbb{R}^2 \ni (q, p)$



(note the direction! (Due to the bad arbitrary choice of putting $i$ here instead of here))

<u>OR</u> use the opposite order $(p(t), q(t))$ so time evolution goes counterclockwise, so that if we let

$$z = p + iq$$



then

$$\frac{dz}{dt} = -q + ip = iz$$

so
$$z(t) = e^{it} z_0$$
$$\& \ H = \frac{1}{2}(p^2 + q^2) = \frac{1}{2}|z|^2$$

(this may seem like a small point, but if we use $(q, p)$ and have cw circles then we are forced to talk about <u>anti</u>holomorphic fns. on phase space)

Recall that if $F = ma$ & $F = -V'$ we have:

$$\frac{dq}{dt} = \frac{1}{m} p = \frac{\partial H}{\partial p} \qquad \longleftarrow \quad H(q,p) = \frac{1}{2m} p^2 + V(q)$$

$$\frac{dp}{dt} = F(q) = -\frac{\partial H}{\partial q}$$

Hamilton's equations:

$$\boxed{\frac{dq}{dt} = \frac{\partial H}{\partial p} \quad \Big| \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}}$$

reminiscent of
$i(a+ib) = (-b+ia)$

An <u>observable</u> for classical harmonic oscillator is any function $O \in C^\infty(\underset{(p,q)}{\overset{}{\mathbb{R}^2}})$

$$\begin{aligned}\frac{d}{dt} O(p(t), q(t)) &= \frac{\partial O}{\partial q}\frac{dq}{dt} + \frac{\partial O}{\partial p}\frac{dp}{dt} \\ &= \frac{\partial O}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial O}{\partial p}\frac{\partial H}{\partial q} \\ &= \{H, O\}\end{aligned}$$

where the Poisson Bracket of $F, G \in C^\infty(\mathbb{R}^2)$ is

$$\{F, G\} = \frac{\partial F}{\partial p}\frac{\partial G}{\partial q} - \frac{\partial G}{\partial p}\frac{\partial F}{\partial q}$$

In fact, $C^\infty(\mathbb{R}^2), \{\cdot,\cdot\}$ is a <u>Lie algebra</u>:

1) $\{F, G\} = -\{G, F\}$

2) Jacobi id. $\{F, \{G, H\}\} = \{\{F, G\}, H\} + \{G, \{F, H\}\}$.

3) bilinear.

In fact, $C^\infty(\mathbb{R}^2)$ becomes a <u>Poisson algebra</u>:

1) a commutative algebra w. $+, \cdot$ ← (with usual + & · of smooth fns)
2) a Lie algebra w. $\{\cdot,\cdot\}$ (defined by above formula)
3) for any $F$, $\{F,\cdot\}$ is a derivation of comm. alg.:

$$\{F, GH\} = \{F,G\}H + G\{F,H\}$$

Moral: Observables in classical mechanics form a commutative algebra <u>BUT</u> any observable $H$ gives rise to time evolution via: $\frac{dO}{dt} = \{H, O\}$, and time evolution is an automorphism of comm. alg because $\{H,\cdot\}$ is a derivation. ("a derivation is the derivative of an automorphism")

---

$$\{F, G\} = \frac{\partial F}{\partial p}\frac{\partial G}{\partial q} - \frac{\partial F}{\partial q}\frac{\partial G}{\partial p}$$

so
$$\left.\begin{aligned} \{p,p\} &= 0 \\ \{q,q\} &= 0 \\ \{p,q\} &= 1 \end{aligned}\right\} \text{ canonical commutation relations}$$

$$\{p,q^2\} = \{p,q\}q + q\{p,q\} = 2q$$
$$\begin{aligned}\{p,q^3\} &= \{p,q^2\}q + q^2\{p,q\} \\ &= 2q\cdot q + q^2 \cdot 1 = 3q^2\end{aligned}$$

so Poisson bracket satisfies
$$\left.\begin{aligned} \{p,\cdot\} &= \frac{\partial}{\partial q} \\ \& \ \{q,\cdot\} &= -\frac{\partial}{\partial p} \end{aligned}\right\} \text{ note the analogy to momentum \& position operators!}$$

## Quantum Harmonic Oscillator:

Heisenberg quantized the harmonic oscillator by inventing a noncomm. alg. generated by $p$ & $q$ such that if $[F, G] = FG - GF$, then:

$$[p,p] = 0$$
$$[q,q] = 0 \qquad \text{("Heisenberg Algebra" (?))}$$
$$[p,q] = -i\hbar$$

Then he defined the energy to be an elt. of this algebra

$$H = \frac{1}{2}(p^2 + q^2)$$

& then decreed that time evolution of any observable (= alg. elt.) be

$$i\hbar \frac{dO}{dt} = [H, O]$$

---

2 October 2003

## Matrices

We'll consider $n \times n$ matrices $M_n(R)$ with entries in $R$, an arbitrary rig: a "ring without negatives."

**Def:** A monoid $M$ is a set with an associative binary operation $\circ : M \times M \to M$ & a "unit" $e \in M$ s.t. $e \circ m = m = m \circ e$ $\forall m \in M$.

e.g.: $\mathbb{N}$, the free monoid on one element where $\circ = +$ $e = 0$.

e.g.: $\mathbb{N}$, w. $\circ = \cdot$, $e = 1$. (Note $\mathbb{N}^+ = \mathbb{N} - \{0\}$ w. $\circ = \cdot$, $e = 1$ is the free commutative monoid on $2, 3, 5, 7, 11, 13 \ldots$

**Def:** A rig $M$ is a commutative monoid $(M, +, 0)$ together w. a monoid $(M, \cdot, 1)$ s.t. distributivity holds:

$$a \cdot (b + c) = a \cdot b + a \cdot c$$
$$(b + c) \cdot a = b \cdot a + c \cdot a$$

and $0 \cdot a = 0 = a \cdot 0$.

note we need this hypothesis, since the usual proof involves subtraction:
$(0+0)a = 0a$
$0a + 0a = 0a$

For the elite:

A rig is a monoid object in the monoidal category of comm. monoids (w. usual $\otimes$)

$\cdot : M \otimes M \to M$

---

<u>Prop:</u> If $R$ is a rig, $M_n(R)$ is a rig with usual matrix $+$ & $\cdot$.

<u>Examples of rigs and their matrix rigs:</u>

0.) The set of rigs whose underlying set is $\emptyset$ is $\emptyset$.
i.e there are no rigs with no elements

1.) There is one rig (up to iso.) w. one elt., namely where $0=1$.

2.) There are two rigs with 2 elements:

A) $(\mathbb{Z}_2, +, \cdot, 0, 1)$

B) The Boolean Algebra w. 2 elts.

$$\Omega = (\{F, T\}, \underset{\text{"or"}}{\vee}, \underset{\text{"and"}}{\wedge}, F, T) \quad \text{"truth values"}$$

Using

$$\alpha : \mathbb{Z}_2 \xrightarrow{\sim} \Omega$$
$$0 \longmapsto F$$
$$1 \longmapsto T$$

we can transfer $+, \cdot$ from $\mathbb{Z}_2$ over to $\Omega$

| + | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 0 |

$\longrightarrow$

| | F | T |
|---|---|---|
| F | F | T |
| T | T | F |

So $+$ gets renamed XOR, "exclusive or"

| $\cdot$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 1 |

$\longrightarrow$

| | F | T |
|---|---|---|
| F | F | F |
| T | F | T |

So $\cdot$ is just $\wedge$.

Also we can transfer "or" & "and" from $\Omega$ to $\mathbb{Z}_2$. Then

"a or b" gets renamed $1-(1-a)\cdot(1-b) = a+b+ab$
& "a and b" gets renamed $a \cdot b$.

$$\begin{array}{c|cc} \vee & F & T \\ \hline F & F & T \\ T & T & T \end{array} \longrightarrow \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 1 \end{array}$$

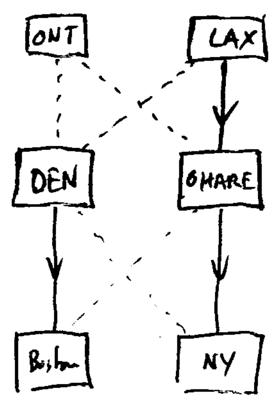
2a) How many 3 elt rigs? JB doesn't know ...

$M_n(\Omega)$ = $n \times n$ matrices with entries T, F

$$\begin{pmatrix} T & F \\ F & T \end{pmatrix}\begin{pmatrix} F & T \\ F & F \end{pmatrix} = \begin{pmatrix} F & T \\ F & F \end{pmatrix}$$

$$\begin{array}{r} \text{den}\to \\ \text{ohare}\to \end{array}\begin{pmatrix} F \\ F \end{pmatrix} = \begin{pmatrix} F & T \\ F & F \end{pmatrix}\begin{pmatrix} T \\ F \end{pmatrix}\begin{array}{l} \leftarrow \text{ont} \\ \leftarrow \text{LAX} \end{array}$$

adjacency matrix of a directed graph




An $n \times m$ matrix of T's and F's is called a relation. Multiplication of these matrices is composition of relations.

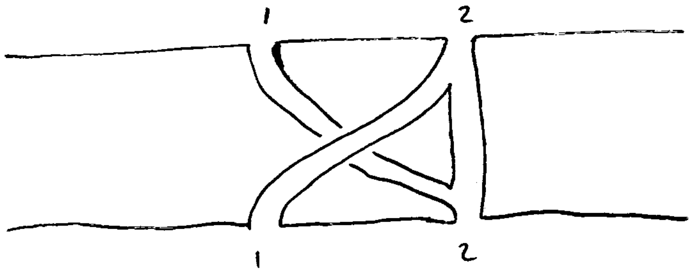
$$\text{If } X_{ij} = \begin{cases} T & \text{if } i \text{ is the father of } j \\ F & \text{otherwise} \end{cases}$$

X is the relation "father of" and $X^2$ is called "grandfather of."

If $X$ & $Y$ are $n \times m$ matrices valued in truth values ($\Omega$), $X+Y$ is the relation "$X$ or $Y$".

---

An $n \times n$ matrix w. entries in $\Omega$ is a description of which input states can go to which output states




a process

$$\begin{pmatrix} F & T \\ T & T \end{pmatrix}$$

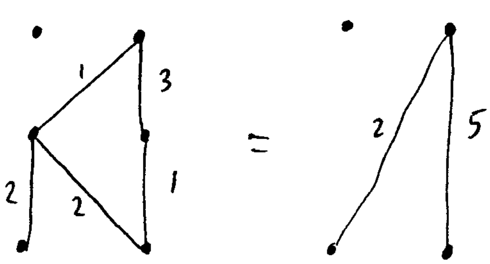
Matrix multiplication corr. to composing processes.
All the strange features of QM come from taking this concept of process and replacing $\Omega$ by $\mathbb{C}$.
One main difference is that in $\mathbb{C}$ we have additive inverses: we can have $x+y=0$ even when $x, y \neq 0$, unlike in $\Omega$. (Note: $T$ has no additive inverse)

---

another rig example:

3) ~~[illegible]~~

$\mathbb{N} = \{\mathbb{N}, +, \cdot, 0, 1\}$




Here we <u>count</u> paths instead of just saying whether there is one (T) or not (F)

$$\begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 5 \end{pmatrix}$$

$$\begin{pmatrix} T & F \\ T & T \end{pmatrix}\begin{pmatrix} F & T \\ F & T \end{pmatrix} = \begin{pmatrix} F & T \\ F & T \end{pmatrix}$$

There's a rig homomorphism

$$f: \mathbb{N} \longrightarrow \Omega$$
$$0 \longmapsto F$$
$$1, 2, 3, \ldots \longmapsto T$$

and this introduces a rig homo. $M_n(\mathbb{N}) \to M_n(\Omega)$

Going from $\mathbb{N}$ to $\mathbb{C}$ is still a big jump!

---

7 October 2003

## Matrix Mechanics

Rigs $R$ and their matrix rigs $M_n(R)$

0.) No rigs w. zero elts.

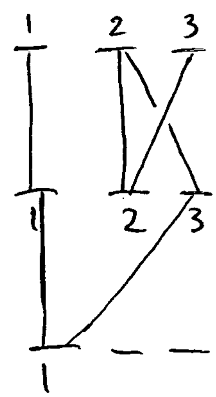
1.) One rig w. one elt. There's at most one rig homo from any rig $R$ to this one, since there's only one map to the one-elt. set, and this indeed is a rig homo, so there's exactly one rig homo from any rig to this one! So we say this is the <u>terminal</u> rig, and call this rig $1$.

2.) Two rigs w. 2 elts:

$$\mathbb{Z}_2 \ \& \ \Omega = \{\{F, T\}, \vee, \wedge, F, T\}$$

We saw that elts of $M_n(\Omega)$ are <u>binary relations</u> on the set $\{1, \ldots, n\}$

composition of processes (composition of relations) = matrix multiplication.




The entries $A_{ij}$ of $A \in M(\Omega)$ say <u>whether or not</u> the process $A$ can take the state $i$ to the state $j$.

Matrix multiplication is "composition of processes"; addition is "superposition" of process

Note there's no destructive interference since $\Omega$ is a rig but <u>not</u> a ring. in $\mathbb{Z}_2$ there <u>is</u> destructive interference.

3.) $\mathbb{N} = (\mathbb{N}, +, \cdot, 0, 1)$ There's at most one rig homo from $\mathbb{N}$ to any other rig, since $f: \mathbb{N} \to R$ must have $f(0) = 0$, $f(1) = 1$ & thus $f(n) = \underbrace{f(1) + \cdots + f(1)}_{n}$. This also lets us define $f$; check that $f$ really is a rig homo. So we say $\mathbb{N}$ is <u>initial</u>.

The entries of $A_{ij} \in M_n(\mathbb{N})$ say how many ways the process $A$ can take state $j$ to state $i$.

4.) $\mathbb{R}^+ = ([0, \infty), +, \cdot, 0, 1)$.

The entries of $A_{ij} \in M_n(\mathbb{R}^+)$ say what's the relative probability that process takes state $j$ to state $i$. This is nicest if the matrix is <u>stochastic</u>:

$$\sum_i A_{ij} = 1$$

– the prob. that state $j$ goes <u>somewhere</u> is 1.

$A, B$ stochastic $\Longrightarrow$ $AB$ stochastic.

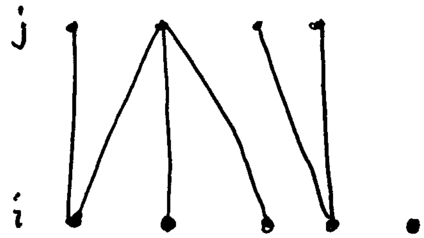
$\psi \in (\mathbb{R}^+)^n$ describes the relative probabilities for the events $i = 1, \ldots, n$ to occur. We say $\psi$ is <u>normalized</u> if

$$\sum \psi_i = 1.$$

In this case the rel. probs. $\psi_i$ are actual probabilities. We can normalize any nonzero $\psi$. Given a normalized $\psi$ and a stochastic $A$, then $A\psi$ is normalized:

$$\sum_i (A\psi)_i = \sum_{i,j} A_{ij} \psi_j = \sum_j \psi_j = 1.$$

If we go back to the rig $\Omega$ of truth values, then the analog of a stochastic matrix is a total relation: one s.t. every $j$ is related to some $i$ (but may be more than one. e.g.



A vector $\psi \in \Omega^n$ is a subset of $n$, & $\bigvee_i \psi_i = T$ says the subset is nonempty.

A silly aside.

Thm: $\sqrt[n]{2}$ is irrational $\forall n > 2$

Pf: by contradiction:

$$\sqrt[n]{2} = \frac{p}{q}$$

$$2 = \frac{p^n}{q^n}$$

$$q^n + q^n = p^n$$

By Fermat's last theorem, this is a contradiction!

For $\mathbb{N}$, a stochastic matrix becomes a function $f : \{1, \ldots, n\} \to \{1, \ldots, n\}$ since $\forall j \in \{1, \ldots, n\}$ $\sum_i A_{ij} = 1$ so $\exists!$ $i$ st. $A_{ij} = 1$, so can $i = f(j)$ for some function $i$. $\psi \in \mathbb{N}^n$ w. $\sum \psi_i = 1$ determines an element of $\{1, \ldots, n\}$.

"Sets & Functions is matrix mechanics over $\mathbb{N}$."

 In QM we usually use:

6) $\mathbb{C} = (\mathbb{C}, +, \cdot, 0, 1)$.

If $A \in M_n(\mathbb{C})$ then $A_{ij}$ says the <u>relative amplitude</u> to get from state $j$ to state $i$ via process $A$.

We'll say $\psi \in \mathbb{C}^n$ is normalized if

$$\sum |\psi_i|^2 = 1$$

- then $|\psi_i|^2$ says the probability for the event $i$ to occur.

We say $A \in M_n(\mathbb{C})$ is <u>unitary</u> or

$$\sum_j |A_{ij}|^2 = 1$$

$\psi$ normalized, $A$ unitary $\Longrightarrow A\psi$ normalized.

In QM, a normalized vector in $\mathbb{C}^n$ (or complex Hilbert space) is called:

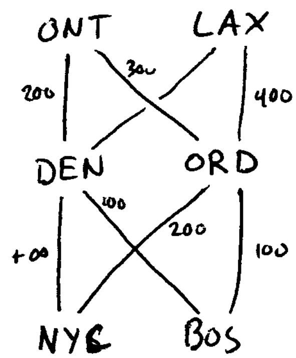
- state vector
- state
- wavefunction.

<u>Note:</u> $\sum |\psi_i|^2 = \sum \psi_i^* \psi_i$ uses the fact that $\mathbb{C}$ is a $*$-rig, i.e. a rig $R$ w. $* : R \to R$ s.t. $(a+b)^* = a^* + b^*$, $(ab)^* = b^* a^*$, $0^* = 0$, $1^* = 1$. Any commutative rig is a $*$-rig with $* = \mathrm{id}$, so we could go back & use $\sum \psi_i^* \psi_i$, $\sum A_{ij} A_{ij}^*$ in previous examples. (This makes $\mathbb{C}$ look less like an exception) (Exercise: do this.)

7) $K = \left( (-\infty, +\infty], \overset{\text{"add"}}{\min}, \overset{\text{"mult"}}{+}, +\infty, 0 \right)$

is {Jim Dolan's / or Bellman's} rig of <u>costs.</u> Given $A \in M_n(K)$, $A_{ij}$ says the <u>cost</u> of going from $j$ to $i$.

Jim said he attributes it to Bellman



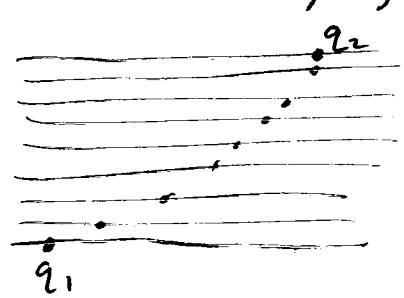

Cost of going from LAX to BOS = (200+100) min. (400+100) – a sum of products in the rig $K$. To compute the cost of a composite process, we do matrix mult. in $M_n(K)$.

In <u>Classical Mechanics</u>, ~~particles travel along~~ (in one dimension) a particle going from position $q_1$ at time $t_1$ to position $q_2$ at $t_2$ usually takes the path

$$q : [t_1, t_2] \longrightarrow \mathbb{R}$$

that minimizes the <u>action</u>

$$S(q) = \int_{t_1}^{t_2} \underbrace{K(\dot{q}(t)) - V(q(t))}_{\text{Lagrangian.}} \, dt$$

We can solve this by discretizing time, multiplying a bunch of $\mathbb{R} \times \mathbb{R}$ matrices valued in $K$, then taking limit as $\Delta t \to 0$.

This lets us turn Lagrangian approach to CM into matrix mechanics over the rig of costs (or rather, actions) "Matrix mechanics over $K$."

There's also a Lagrangian approach to QM which is related to matrix mechanics over $\mathbb{C}$.

---

October 9, 2003

One more "rig":

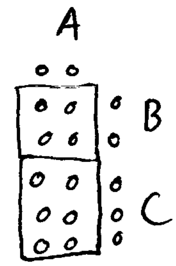
The Category FinSet. It's like a rig with

$$\begin{aligned} + &= \text{disjoint union} \\ \cdot &= \text{Cartesian product} \\ 0 &= \emptyset \\ 1 &= 1\text{-elt set.} \end{aligned}$$

All the rig axioms hold up to canonical isomorphism, e.g.

$$A \times (B + C) \cong A \times B + A \times C$$

example:




recall underline means decategorification

Since $\underline{\text{FinSet}} = \mathbb{N}$, all these operations on FinSet give ops on $\mathbb{N}$ making $\mathbb{N}$ into a rig.

FinSet is a "categorified rig" or "2-rig". Just as $\mathbb{N}$ is the initial rig., FinSet is the "initial 2-rig" (the free 2-rig on nothing (Just as $\mathbb{N}$ is the free rig on nothing)).

Aside:
Example: what's the free rig on 1 generator?
Form all possible elts using no equations but those in the definition of a rig.
e.g. $x \quad (x+x)+x = x+(x+x), \; x^2, \; x^2+x, \ldots$
so it's $\mathbb{N}[x]$.

Plan: (1) Study harmonic oscillator & see that the Hilbert space of states is (a completion of) $\mathbb{C}[x]$.

(2) Notice we can just use $\mathbb{N}$ instead of $\mathbb{C}$ & get "natural numbers oscillator" w. $\mathbb{N}[x]$ as its space of states. (or maybe some completion of $\mathbb{N}[x]$).

(3) $\mathbb{N}[x]$ is the free rig on one generator; why not categorify and use the free 2-rig on one generator, $\mathrm{FinSet}[x]$ — also called the category of <u>species</u>, invented by André Joyal to study combinatorics.

<u>Upshot</u>: "categorifying the (natural numbers) harmonic oscillator gives combinatorics."

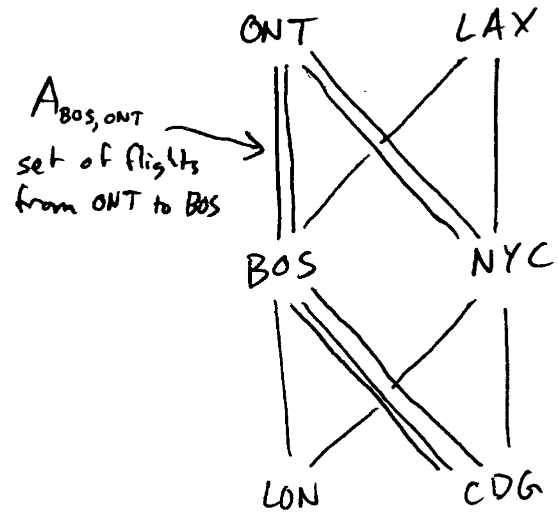
~~Back to Heisenberg's matrix mechanics~~

We can let $M_n(\mathrm{FinSet})$ be the category whose objects are $n \times n$ matrices of finite sets, & whose morphisms are $n\times n$ matrices of functions

$$\begin{pmatrix} f & g \\ h & i \end{pmatrix} : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \longrightarrow \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \quad \text{where} \quad \begin{matrix} f: A \to A' \\ g: B \to B' \\ h: C \to C' \\ i: D \to D' \end{matrix}$$

with obvious composition & identities.

Given a matrix $A \in M_n(\mathrm{FinSet})$, the entry $A_{ij}$ describes the finite set of ways of getting from state $j$ to state $i$.




We describe composition of these processes by matrix multiplication in $M_n(\mathrm{FinSet})$:

$$(AB)_{ik} = \underbrace{\sum_j}_{\text{disjoint union}} A_{ij} \underbrace{\times}_{\text{cartesian product}} B_{jk}$$

Now ... Back to Heisenberg's Matrix Mechanics, for the particle on a line, especially the harmonic oscillator.

Recall: classically observables for particle on line form the comm. alg. $C^\infty(\mathbb{R}^2)$ where $(p,q) \in \mathbb{R}^2$, which becomes a Poisson algebra with

$$\{p,q\} = 1 \qquad \begin{array}{l}\{p,p\} = 0 \\ \{q,q\} = 0\end{array}$$

& if the Hamiltonian $H \in C^\infty(\mathbb{R}^2)$ is

$$H(p,q) = \frac{p^2}{2} + V(q) \qquad \begin{array}{c}(m=1) \\ V \in C^\infty(\mathbb{R})\end{array}$$

then any observable $O$ gives a 1-parameter family of observables $O(t)$ s.t.

$$\frac{d}{dt} O(t) = \{H, O(t)\}$$

(this is general, independent of form of Hamiltonian) & in particular for <u>this</u> Hamiltonian:

$$\frac{d}{dt} q(t) = p(t)$$

$$\frac{d}{dt} p(t) = -V'(q).$$

For the harmonic oscillator,

$$V(x) = \frac{1}{2} x^2$$

these reduce to

$$\frac{dq(t)}{dt} = p(t) \qquad \frac{dp(t)}{dt} = -q(t)$$

In Heisenberg's approach to quantum mechanics, observables form a <u>noncomm.</u> algebra containing elts $p$ & $q$ s.t.

$$[p, q] = -i\hbar$$

& if the Hamiltonian is some observable

$$H = \frac{p^2}{2} + V(q)$$

(easy to define if $V$ is algebraic (a polynomial), harder otherwise — use the <u>functional calculus</u>)

then any observable $O$ gives observables $O(t)$ s.t.

$$-i\hbar \frac{d}{dt} O(t) = [H, O(t)]$$

[(Problem: does this d.e. have a solution? In the classical case, the answer has to do with integrability of vector fields (see JB's CM notes))]

Really? Is there a solution?

The solution <u>wants</u> to be:

$$O(t) = e^{itH/\hbar} O e^{-itH/\hbar}$$

because:

$$\frac{d}{dt} O(t) = \frac{d}{dt}\left(e^{itH/\hbar} O e^{-itH/\hbar}\right)$$

$$= \frac{d}{dt}\left(e^{itH/\hbar}\right) O e^{-itH/\hbar} + e^{itH/\hbar} O \frac{d}{dt}\left(e^{-itH/\hbar}\right)$$

$$= \frac{iH}{\hbar} e^{itH/\hbar} O e^{-itH/\hbar} - e^{itH/\hbar} O e^{itH/\hbar} \frac{iH}{\hbar}$$

can pull out $H$ on either side because $H$ commutes with $e^{itH/\hbar} = f(H)$

Functional Calculus!

$$= \frac{i}{\hbar}[H, O(t)]$$

as desired

Note: $\hbar$ has units of action = time × energy, so $\frac{tH}{\hbar}$ is dimensionless so $e^{itH/\hbar}$ makes sense & is dimensionless. Pick units of action so $\hbar = 1$ now! Then the new rules of the game are:

$$[p, q] = -i$$

$$-i \frac{dO(t)}{dt} = [H, O(t)]$$

This should have the solution:

$$O(t) = e^{itH} O e^{-itH}$$

<u>Problem</u>: $e^{itH} = \sum_{n \geq 0} \frac{(itH)^n}{n!}$ doesn't lie in alg. gen by $p$ & $q$ - even if $V$ is a polynomial. So, we need some completion of this algebra.

---

Now let's start calculating

$$[p, q] = -i$$

To do $[p, q^2] = ?$ you use:

$$[A, BC] = [A, B]C + B[A, C]$$

$$ABC - BCA = ABC - BAC + BAC - BCA$$

(so $[A, -]$ is a derivation, just as $\{A, \cdot\}$ was a derivation of the comm. alg. in classical mechanics)

$$[p,q^2] = [p,q]q + q[p,q] = -2iq$$

14 October 2003

JB forgot his notebook today...

The Weyl algebra is the associative alg. over $\mathbb{C}$ generated by $p, q$ satisfying

$$pq - qp = -i$$

We'll do calculations in this algebra (or some larger algebra containing things like $e^{itH}$ where $H$ is an elt. of Weyl algebra). Starting with:

$$[p,q] = -i$$
$$[p,q^2] = [p,q]q + q[p,q] = -2iq$$
$$\vdots$$
$$[p,q^n] = -inq^{n-1}$$

or for any polynomial $F$,

$$[p, F(q)] = -i F'(q)$$

or for short

$$[p, \cdot\,] = -i\frac{\partial}{\partial q}.$$

Next:

$$[q,p^2] = [q,p]p + p[q,p] = 2ip$$
$$[q,p^n] = i n p^{n-1}$$

or for any poly. $F$

$$[q, F(p)] = i F'(p)$$

or:

$$[q, \cdot\,] = i\frac{\partial}{\partial p}.$$

If $H = \frac{1}{2}p^2 + V(q)$ (V a poly. or perhaps an analytic fn. (power-series))

then we have

$$\frac{d}{dt} O(t) = [iH, O(t)]$$

where $O(t) = e^{itH} O e^{-itH}$

We are doing QM in a nonstandard more algebraic way to show that all of elementary QM is really the result of $[p,q] = -i$.

In particular, for the observable $q$,

$$\frac{d}{dt} q(t) = \frac{d}{dt}\left(e^{itH} q e^{-itH}\right)$$
$$= e^{itH}(iHq - qiH)e^{-itH}$$
$$= e^{itH}[iH, q]e^{-itH}$$

← note: this is general.
e.g. $\frac{d}{dt} p(t) = e^{itH}[iH,p]e^{-itH}$.

Now $[iH, q] = i\left[\frac{p^2}{2} + V(q), q\right]$

$[V(q), q] = 0$ since $V$ is a poly. in $q$ and the alg. is associative

$$= \frac{i}{2}[p^2, q]$$
$$= \frac{i}{2} \cdot -2ip$$
$$= p$$

So $$\frac{d}{dt} q(t) = e^{itH} p e^{-itH} = p(t)$$

i.e $$\dot{q}(t) = p(t)$$ ← "velocity = momentum" (when mass = 1)

Next, try

$$[iH, p] = i\left[\frac{p^2}{2} + V(q), p\right]$$
$$= i[V(q), p]$$
$$= i \cdot i V'(q)$$
$$= -V'(q)$$ ← force at time 0.

so

$$\frac{d}{dt} p(t) = e^{itH}(-V'(q))e^{-itH} = -V'(e^{itH} q e^{-itH}) = -V'(q(t))$$

↖ if V is a poly, do this term by term.

In short

$$\dot{p}(t) = -V'(q(t)) \quad \leftarrow \text{"time derivative of momentum is force"}$$

These give

$$\ddot{q}(t) = -V'(q) \qquad \text{"}F = a\text{"}$$

or if we put m back in & define $F(t) = -V'(q(t))$ – the force operator – ~~accept~~ and ~~the~~ $a(t) = \ddot{q}(t)$ – the acceleration operator, we get

$$F(t) = ma(t)$$

(For some reason, nobody talks about where $F = ma$ goes in QM. JB has spent lots of time on spr. trying to convince people this is how it works.)

For the harmonic oscillator:

$$H = \frac{1}{2}(p^2 + q^2)$$

& then

$$\dot{q}(t) = p(t) \qquad \dot{p}(t) = -q(t)$$

$$\& \quad \ddot{q}(t) = -q(t)$$

note: these look just like the eqs we wrote down for the classical case. They are still valid in QM.

These have solutions

$$\left.\begin{aligned} q(t) &= (\cos t)q + (\sin t)p \\ p(t) &= -(\sin t)q + (\cos t)p \end{aligned}\right]$$

same formula as classically! (but it means something different)

(JB is trying to give the impression that QM is "just like" CM except done in a Weyl algebra instead of a Poisson algebra.)

To go further, we want to look at representations of the Weyl algebra as linear operators, i.e. some vector space $V$ & linear operators $p, q : V \to V$ s.t. $pq - qp = -i$.

$$\uparrow_{-i1_V}$$

You could hope for a finite-dim $V$, e.g. $V=\mathbb{C}^n$ so then $p,q\in M_n(\mathbb{C})$. (Matrix mechanics!) In fact there are <u>no</u> fin. dim. reps except for the zero-dim rep $n=0$.

<u>Proof:</u> Suppose $p,q\in M_n(\mathbb{C})$, then

$$0 = tr(pq-qp) = tr(-i\mathbb{1}) = -i\dim(V) = -ni$$

So we conclude $n=0$ (toby's rk. if we kept $\hbar$ in, then the other alternative would be $\hbar=0$). ∎

Next: try to find some normed vector space $V$ s.t. $p,q: V\to V$ are bounded lin. ops. on $V$. In fact, there are no reps like this except the zero dimensional one.

<u>Proof:</u>

$$\begin{aligned} n\|q^{n-1}\| = \|-inq^{n-1}\| = \|[p,q^n]\| &= \|pq^n - q^np\| \\ &\leq \|pq^n\| + \|q^np\| \\ &\leq \|pq\|\cdot\|q^{n-1}\| + \|qp\|\cdot\|q^{n-1}\| \end{aligned}$$

So

$$n \leq \|pq\| + \|qp\| \quad \forall n$$

— a contradiction (since $p$ & $q$ are bded) unless $\|q^{n-1}\|=0$, in which case $\|q^n\| \leq \|q\|\cdot\|q^{n-1}\| = 0$ so $q^n=0$ so $\|[p,q^n]\|=0$ so $\|q^{n-1}\|=0$ so $q^{n-1}=0$ ... so $q=0$. ∎

So $1_V = 0$ so $\dim V = 0$.

So: we can try to describe $p$ & $q$ as <u>unbounded operators</u> on a Hilbert space. The classic example is called the <u>Schrödinger representation</u> of the Weyl algebra.

Idea: to let $p$ & $q$ be operators on some space of fns $\psi : \mathbb{R} \to \mathbb{C}$.

$$(p\psi)(x) = -i\psi'(x) \qquad \text{or} \qquad p = -i\frac{\partial}{\partial x}$$
$$(q\psi)(x) = x\psi(x) \qquad\qquad \underset{\uparrow \text{ mult. by } x}{q = M_x}$$

$$p(q\psi)(x) - q(p\psi)(x) = -i\frac{d}{dx}(x\psi)(x) + ix\frac{d\psi}{dx}(x) =$$
$$-i\psi(x) - ix\frac{d\psi}{dx}(x) + ix\frac{d\psi}{dx}(x) = -i\psi(x)$$

The idea: the fn $\psi$ is a "wavefunction" that describes the probability of finding the particle in some $^{\text{set}}$ $S \subseteq \mathbb{R}$: the probability is

$$\int_S |\psi(x)|^2\,dx$$

(if $\psi$ is normalized so that $\int_{\mathbb{R}} |\psi(x)|^2 = 1$)

(The Plan:)

To write $p$ & $q$ as $\infty \times \infty$ matrices, as Heisenberg did, we'll take Schrödinger rep & pick a basis of functions on the real line — a basis of eigenvectors of

$$H = \frac{1}{2}(p^2 + q^2)$$
$$= \frac{1}{2}\Big(-\frac{d^2}{dx^2} + \underset{\uparrow \text{ the operator "mult. by } x^2\text{"}}{x^2}\Big)$$

What kind of fns on the real line? One choice: Schwartz functions $\mathcal{S}(\mathbb{R}) = \{\psi : \mathbb{R} \to \mathbb{C} : \underset{\forall n,m,\ \exists C \text{ s.t.}}{\nearrow} \left|x^m \frac{d^n}{dx^n}\psi(x)\right| < C\}$

"All derivatives of $\psi$ exist & vanish faster than the reciprocal of any polynomial."

<u>Note</u>: Schwartz functions satisfy both of the nice properties in analysis:

- smoothness
- fast decay

] note: these are <u>dual</u> notions. The Fourier transform takes smooth fns. to ones that decay fast & vise versa.

$$\mathcal{S}(\mathbb{R}) = \{\psi : \quad q^m p^n \psi \text{ is bounded } \forall n,m\}$$
$$= \{\psi : \quad p^n q^m \psi \text{ is bounded } \forall n,m\}.$$

---

We will find linearly independent eigenfns of $H$, say $\psi_i$, s.t. finite lin. combs are dense in the natural topology on $\mathcal{S}(\mathbb{R})$.

(These are in practice Hermite polynomials times gaussians.)

---

16 October 2003

Let the Schwartz space $\mathcal{S}(\mathbb{R}) = \{\psi : \mathbb{R} \to \mathbb{C} : x^m \frac{d^n\psi}{dx^n} \text{ is bounded } \forall n,m\}$

Given a sequence or net $\psi_\alpha \in \mathcal{S}(\mathbb{R})$ we say $\psi_\alpha \to \psi \in \mathcal{S}(\mathbb{R})$ if

$$\sup \left| x^m \frac{d^n}{dx^n}(\psi_\alpha - \psi)\right| \longrightarrow 0 \quad \forall n,m$$

making $\mathcal{S}(\mathbb{R})$ into a topological vector space.

We have cts. lin. ops

$$p, q : \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R})$$

given by
$$(p\psi)(x) = -i\psi'(x)$$
$$(q\psi)(x) = x\psi(x)$$

(Easy Exercise: check that these are continuous)

& also

$$H: \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$$

given by

$$H = \frac{1}{2}(p^2 + q^2)$$

note: $p$ & $q$ have no eigenvectors in this space! They should look like $\delta$-fns. and these clearly aren't in $\mathcal{S}$.

Q: What are the eigenvectors of $H$?

(Note: $p$ & $q$ don't have eigenvectors in $\mathcal{S}(\mathbb{R})$, since $e^{ikx}$ & $\delta(x-a)$ aren't in $\mathcal{S}(\mathbb{R})$.)

Here's <u>one</u> eigenvector of $H$:

$$\psi_0(x) = e^{-x^2/2}$$

Let's check how this works:

$$\frac{d\psi_0}{dx} = -x e^{-x^2/2}$$

$$\frac{d^2\psi_0}{dx^2} = -(1+x^2)e^{-x^2/2}$$

$$p^2\psi_0 = (1-x^2)\psi_0$$

$$q^2\psi_0 = x^2\psi_0$$

$$\Rightarrow H\psi_0 = \frac{1}{2}(p^2+q^2)\psi_0 = \frac{1}{2}\psi_0.$$

It turns out $\frac{1}{2}$ his the lowest eigenvalue of $H$. In general, if

$$H\psi = E\psi$$

we say $\psi$ describes a state where the particle has energy $E$.
So $\psi_0 = e^{-x^2/2}$ is the state of least energy, i.e. the ground state

of the harmonic oscillator. Note that the ground state energy is $\frac{1}{2}$, not (as in the classical harmonic oscillator) 0!

really $\frac{1}{2}\hbar\omega$

Due to the uncertainty principle we can't get both $p$ & $q$ to be 0 in QM. Now how do we get more eigenvectors?

We'll now define cts. lin. ops

$$a, a^* : \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R})$$

s.t. if

$$H\psi = E\psi$$

LADDER OPERATORS
aka
RAISING and LOWERING OPS
aka
CREATION / ANNIHILATION OPS.

then

$$H a^*\psi = (E+1)a^*\psi$$

(so $a^*$ raises the energy by 1)

&

$$H a\psi = (E-1)a\psi$$

($a$ lowers the energy by 1)

We call $a$ the <u>annihilation operator</u> (because $a\psi_0 = 0$ – $a$ annihilates $\psi_0$) or <u>lowering operator</u>; we call $a^*$ the <u>creation op.</u> or <u>raising op.</u>

← note from the def. of $a$, we must have $a\psi_0 = 0$, or else $\psi_0$ couldn't be the ground state.

These ladder operators are

$$a = q + ip$$
$$a^* = q - ip$$

(People usually divide these by $\sqrt{2}$; we'll do that later.)

Recall:

$$\frac{dO(t)}{dt} = i\,[H, O(t)]$$

$$[iH, p] = -q \qquad [iH, q] = p$$

so

$$\begin{aligned}[iH, a] &= [iH, q+ip] \\ &= [iH, q] + [iH, ip] \\ &= p - iq \\ &= -ia\end{aligned}$$

$$\Rightarrow \boxed{[H, a] = -a}$$

$$\begin{aligned}[iH, a^*] &= [iH, q-ip] \\ &= p + iq \\ &= ia^*\end{aligned}$$

$$\Rightarrow \boxed{[H, a^*] = a^*}$$

So if $H\psi = E\psi$ then

$$\begin{aligned}Ha^*\psi &= (a^*H + [H, a^*])\psi \\ &= (a^*H + a^*)\psi \\ &= a^*E + a^* \\ &= (E+1)a^*\psi\end{aligned}$$

&

$$\begin{aligned}Ha\psi &= (aH + [H, a])\psi \\ &= (aE - a)\psi \\ &= (E-1)a\psi\end{aligned}$$

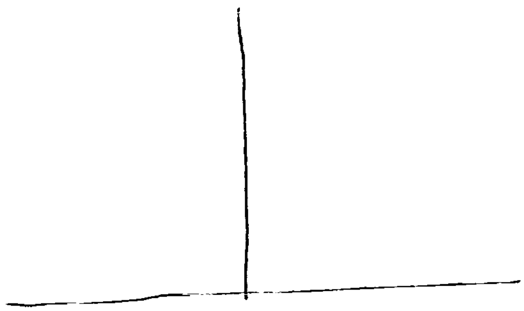
So we can take $\psi_0$ and hit it with $a^*$ $n$ times & get

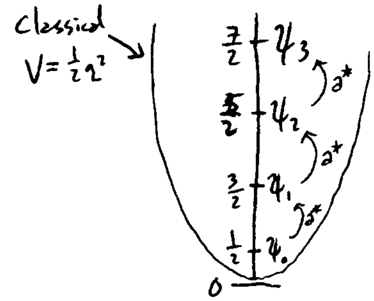
$$\psi_n = a^{*n}\psi_0$$

so

$$H\psi_n = (n + \tfrac{1}{2})\psi_n .$$

These are in fact <u>all</u> of the eigenvectors of $H$, so we have:






What happens if we apply the lowering operator $a$ to $\psi_n$?

$$\begin{aligned} a\psi_0 &= (q+ip)\psi_0 \\ &= \left(x + \frac{d}{dx}\right)e^{-x^2/2} \\ &= 0, \quad \text{as expected.} \end{aligned}$$

If $n \geq 1$,

$$a\psi_n = a\, a^{*n}\psi_0$$

To do this we would like to commute the $a$ through all of the $a^*$'s. We need the commutator:

$$\begin{aligned} [a, a^*] &= [q+ip, q-ip] \\ &= [q,q] + i[p,q] - i[q,p] + [p,p] \\ &= 2i[p,q] \\ &= 2i(-i) \\ &= 2 \end{aligned}$$

this 2 is why people usually divide and $a^*$ by $\sqrt{2}$.

So:

$$\begin{aligned} a\psi_n &= aa^*a^{*n-1}\psi_0 \\ &= (a^*a + [a, a^*])\,a^{*n-1}\psi_0 \\ &= (a^*a + 2)\,a^{*n-1}\psi_0 = \cdots \end{aligned}$$

& etc. pushing $a$ to the right.

An easier way:

$[a, \cdot]$ is a derivation, so

$[a, a^*] = 2$ implies

$$\boxed{[a, a^{*n}] = 2n a^{*n-1}}$$

$[a, \cdot] =$ "$2\frac{d}{da^*}$"

~~$a a^* ...$~~

Then

$$a\psi_n = a\, a^{*n}\psi_0 = (a^{*n} a + [a, a^{*n}])\psi_0$$
$$= (0 + 2n a^{*n-1})\psi_0$$
$$= 2n\, \psi_{n-1}$$

Note: $a$ & $a^*$ are not inverses of each other.
There's a big difference between creating a particle and then annihilating a particle, and the other way round.

Usually, people get rid of these "2's" by setting

$$a = \frac{q + ip}{\sqrt{2}} \qquad a^* = \frac{q - ip}{\sqrt{2}}$$

Then

$$[a, a^*] = 1$$

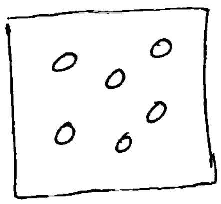
but we still have
$$[H, a] = -a$$
$$[H, a^*] = a^*$$

& now if we set

$$\psi_n = a^{*n}\psi_0$$

we get

$$H\psi_n = (n+\tfrac{1}{2})\psi_n$$
$$a^*\psi_n = \psi_{n+1}$$
$$a\psi_n = n\psi_{n-1}$$



$\psi_n$ is like a box
with $n$ indistinguishable balls

$a^*\psi_n = \psi_{n+1}$ says there's only one way to add a ball to the box.

$a\psi_n = n\psi$ says there are $n$ ways to take a ball out.

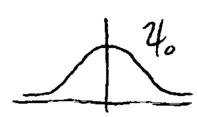
$$[a, a^*] = 1 \quad \text{or}$$
$$aa^* = a^*a + 1$$

"There's one more way to ~~take~~ put in a ball and then take one out than to take one out and then put one in.

---

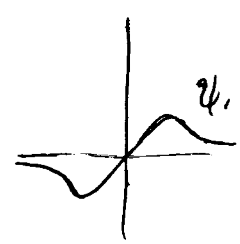
What are the functions $\psi_n$ like?

$$\psi_0 = e^{-x^2/2}$$

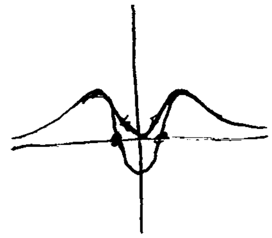



the gaussian of std. deviation 1.

$$\psi_1 = a^*\psi_0 = \frac{q-ip}{\sqrt{2}}\psi_0$$
$$= \frac{1}{\sqrt{2}}\left(x - \frac{d}{dx}\right)e^{-x^2/2}$$
$$= \frac{1}{\sqrt{2}}\, 2x\, e^{-x^2/2}$$



$$\psi_2 = a^* \psi_1$$
$$= \tfrac{1}{\cancel{2}}\left(x - \tfrac{d}{dx}\right) \cancel{2} x e^{-x^2/2}$$
$$= (x^2 - 1 + x^2) e^{-x^2/2}$$
$$= (2x^2 - 1) e^{-x^2/2}$$



Certainly
$$\psi_n(x) = H_n(x) e^{-x^2/2}$$
where $H_n$ is some polynomial of degree $n$ — the $n$th <u>Hermite polynomial</u> (up to fudge factors). It's easy to see $H_n(x)$ has degree $n$, and $H_n$ is also even/odd depending on parity of $n$ (this comes from the fact that mult by $x$ and diff by $x$ both turn odd fns into even fns and even fns into odd fns, so $x - \frac{d}{dx}$ does too). It's also true (but not trivial) that $H_n$ has $n$ real roots.

<u>Thm:</u> The fns. $\psi_n$ form a (topological) basis of $\mathcal{S}(\mathbb{R})$ i.e. they're linearly indep. & the finite lin. combs $\sum_{n=0}^{N} a_n \psi_n$ are dense in $\mathcal{S}(\mathbb{R})$ ← A souped up version of the Stone Weierstrass Thm but noncompactness makes it harder.

<u>"Pf:"</u> They're obviously lin. indep. & finite lin. combs are the same as fns $P(x)e^{-x^2/2}$ where $P$ is any poly. So we need: polynomials times $e^{-x^2/2}$ are dense in $\mathcal{S}(\mathbb{R})$. This follows from the "$L^2$ Stone-Weierstrass Thm" & some extra stuff. ∎

Prop. – $\psi_n$ are orthogonal in $L^2(\mathbb{R})$:

$$\langle \psi_n, \psi_m \rangle = \int_{-\infty}^{\infty} \overline{\psi_n}\, \psi_m \, dx = 0 \quad \text{if } n \neq m$$

Pf: Note

$$\langle p\psi, \varphi \rangle = \langle \psi, p\varphi \rangle$$

$\forall \psi, \varphi \in \mathcal{S}(\mathbb{R})$

$$\langle q\psi, \varphi \rangle = \langle \psi, q\varphi \rangle$$

via integration by parts (note this works rigorously: $\int_{-\infty}^{\infty} \overline{-i\psi'}\,\varphi = \int_{-\infty}^{\infty} \overline{\psi} \cdot -i\varphi'$ + boundary terms which are zero because our functions go to zero fast enough.)

$$\int x \overline{\psi} \varphi = \int \overline{\psi}\, x \varphi$$

So

$$\langle H\psi, \varphi \rangle = \langle \psi, H\varphi \rangle \quad \text{since } H = \tfrac{1}{2}(p^2 + q^2)$$

Then

$$\frac{1}{E_n}\langle H\psi_n, \psi_m \rangle = \langle \psi_n, \psi_m \rangle = \frac{1}{E_m}\langle \psi_n, H\psi_m \rangle$$

$$\parallel$$

$$\frac{1}{E_n}\langle \psi_n, H\psi_m \rangle$$

So either $E_n = E_m$ or $\langle \psi_n, \psi_m \rangle = 0$. ∎

(note: $E_n \neq 0$)

Prop: $\mathcal{S}(\mathbb{R})$ are dense in $L^2(\mathbb{R})$.

Pf: Even $C_0^\infty(\mathbb{R}) \subseteq \mathcal{S}(\mathbb{R})$ are dense in $L^2(\mathbb{R})$. ∎

<u>Thm:</u> If we normalize $\psi_n$ to get

$$|n\rangle = \frac{\psi_n}{\|\psi_n\|}$$

these form an o.n. basis of $L^2(\mathbb{R})$.

<u>Pf:</u> They're orthonormal, since the $\psi_n$ are orthogonal.
They form a basis since the finite lin. combs. of the $\psi_n$ are dense in $\mathcal{S}(\mathbb{R})$ & thus by previous prop. in $L^2(\mathbb{R})$. ∎

---

21 October 2003

The Weyl Algebra: the assoc. alg. gen by $p, q$ s.t.

$$[p,q] = -i.$$

is also generated by

$$a = \frac{q+ip}{\sqrt{2}}$$

$$a^* = \frac{q-ip}{\sqrt{2}}$$

with $[a, a^*] = 1$, since

$$\frac{a+a^*}{\sqrt{2}} = q$$

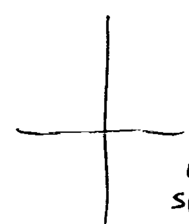
$$\frac{a-a^*}{\sqrt{2}\,i} = p$$

note the similarity with

$z = x+iy$
$\bar{z} = x-iy$

$\frac{z+\bar{z}}{2} = x$
& $\frac{z-\bar{z}}{2i} = y$

$\mathbb{R}^2 \cong \mathbb{C}$



is phase space for classical harm. oscillator

We considered the <u>Schrödinger</u> <u>rep</u> of the Weyl algebra, where

$$p, q : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$$

$$\mathcal{S}(\mathbb{R}) = \left\{ \psi : \mathbb{R} \to \mathbb{C} \ : \ \left| x^n \frac{d^n}{dx^n} \psi \right| \text{ bded} \right\}$$

Now we're doing a quantum version of the same thing: $a, a^*$ are like $z$ & $\bar{z}$

($\to$ noncommutative complex analysis)

and

$$p = -i\frac{d}{dx}$$

$$q = M_x \qquad \text{(mult. by } x\text{)}$$

We considered the eigenfunctions of

$$H = \frac{1}{2}(p^2+q^2) : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$$

& saw they were

$$\psi_0 = e^{-x^2/2} \qquad H\psi_0 = \frac{1}{2}\psi_0$$

and

$$\psi_n = a^{*n}\psi_0 \qquad H\psi_n = (n+\tfrac{1}{2})\psi_n.$$

In fact if we normalize these:

$$|n\rangle = \frac{\psi_n}{\|\psi_n\|} \qquad \|\psi\|^2 = \int_{\mathbb{R}} |\psi|^2\,dx$$

we get an o.n. basis of $L^2(\mathbb{R})$. We saw

$$a^*\psi_n = \psi_{n+1}$$

$$a\,\psi_n = n\,\psi_{n-1}$$

$$H\psi_n = (n+\tfrac{1}{2})\psi_n$$

As operators on $\mathcal{S}(\mathbb{R})$ we thus have

$$H = a^*a + \tfrac{1}{2}I.$$

In the Weyl algebra we indeed have $H = a^*a + \frac{1}{2}$:

$$a^*a = \left(\frac{q-ip}{\sqrt{2}}\right)\left(\frac{q+ip}{\sqrt{2}}\right)$$

$$= \frac{1}{2}(q-ip)(q+ip)$$

$$= \frac{1}{2}(q^2 \underbrace{-ipq+iqp}_{-i[p,q]=-1} + p^2) = \frac{1}{2}(p^2+q^2-1)$$

$$= H - \frac{1}{2} \quad \text{the harmonic oscillator minus } \tfrac{1}{2}.$$

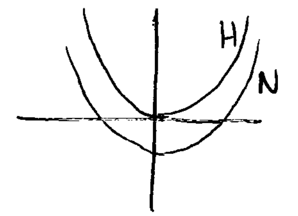
It would be nice to have everything in our eqns. be natural numbers. Why? Because we know how to categorify $\mathbb{N}$ as FinSet. We have:

$$H\psi_n = (n + \tfrac{1}{2})\psi_n$$

the fly in the ointment – an unnatural number.

To categorify we modify the Hamiltonian:

$$N = H - \tfrac{1}{2}$$
$$= a^* a.$$




If we let $N$, the <u>number operator</u> be the "new, improved" Hamiltonian for our oscillator, we're really replacing

$$\tfrac{1}{2}(p^2 + \underbrace{q^2}_{\text{potential}}) \quad \text{by} \quad \tfrac{1}{2}(p^2 + \underbrace{q^2 - 1}_{\text{potential}})$$

This doesn't change any <u>physics</u> since:

1) classically: $F = -V'$ doesn't change if you add a constant to $V$

2) quantumly $\dfrac{dO}{dt} = i[H, O]$ doesn't change if you add a constant to $H$.

So we get a rep. of the Weyl algebra with

$$\begin{cases} a^*\psi_n = \psi_{n+1} \\ a\psi_n = n\psi_{n-1} \end{cases}$$

$$(\text{and } N\psi_n = n\psi_n) \qquad (N = a^* a)$$

We call this the Heisenberg representation of the Weyl algebra if we treat $\psi_n$ as abstract basis vectors, not wavefunctions.

BUT we can also think of the Heisenberg rep this way

$$\psi_n = z^n$$
$$a = \frac{d}{dz}$$
$$a^* = M_z$$

In short, we get a rep on the Weyl algebra on $\mathbb{C}[z]$ by these formulas. This is called the Fock Representation. If we think of $a = \frac{d}{dz}$ & $a^* = M_z$ as operators on polynomials, or analytic functions, or ..., on the complex plane, we call this the Bargmann - Segal representation.

We will categorify the Fock representation & see this is related to combinatorics. But before we get into that ...

We want to talk about:
(1) The Uncertainty Principle
(2) Fourier Transform.

(1). Suppose $A$ is a self-adjoint operator on a Hilbert space $H$, finite-dim. for simplicity. We think of $A$ as an "observable", but how does this work? We think of unit vectors $\psi \in H$ as states: ways our system can be. We say $|\langle \psi, \varphi \rangle|^2$ is the probability of finding system in state $\psi$ given that it is in state $\varphi$. Note this number doesn't change if we multiply $\psi$ by a phase ($c \in \mathbb{C}$ s.t. $|c| = 1$), so really states are equivalence classes of unit vectors mod phase.

If $A$ is s.a. then $H$ has an o.n. basis of eigenvectors $e_i \in H$

$$Ae_i = \lambda_i e_i$$

The numbers $\lambda_i$ are all of the possible outcomes of measuring $A$, & ~~the~~ $e_i$ is the state in which measuring $A$ always yields the value $\lambda_i$. Suppose we measure $A$ in some arbitrary state $\psi$. We can write

$$\psi = \sum \langle e_i, \psi \rangle e_i$$

where $|\langle e_i, \psi\rangle|$ is prob. of system in state $\psi$ to be found in state $e_i$.
So when we measure $A$ in state $\psi$ we get the answer $\lambda_i$ with prob $|\langle e_i, \psi \rangle|^2$. These probabilities sum to 1:

$$\sum |\langle e_i, \psi\rangle|^2 = \|\psi\|^2 = 1$$

The <u>mean</u> or <u>expected</u> value of $A$ in state $\psi$:

conj. linear, linear

$$\begin{aligned}
\sum |\langle e_i, \psi\rangle|^2 \lambda_i &= \sum \langle \psi, e_i\rangle \langle e_i, \psi\rangle \lambda_i \\
&= \sum \langle \psi, Ae_i\rangle \langle e_i, \psi\rangle \\
&= \sum \langle A\psi, e_i\rangle \langle e_i, \psi\rangle \\
&= \langle A\psi, \psi\rangle \\
&= \langle \psi, A\psi\rangle
\end{aligned}$$

23 October 2003

A quantum system has some Hilbert space $H$ s.t.

1) states are unit vectors $\psi \in H$ (really equivalence classes where $\psi \sim \psi'$ if $\psi' = c\psi$ for some $c \in \mathbb{C}$ with $|c| = 1$)

2) observables are self-adjoint operators $A$ on $H$.

If $A$ has an o.n. basis of eigenvectors

$$A e_i = \lambda e_i$$

then

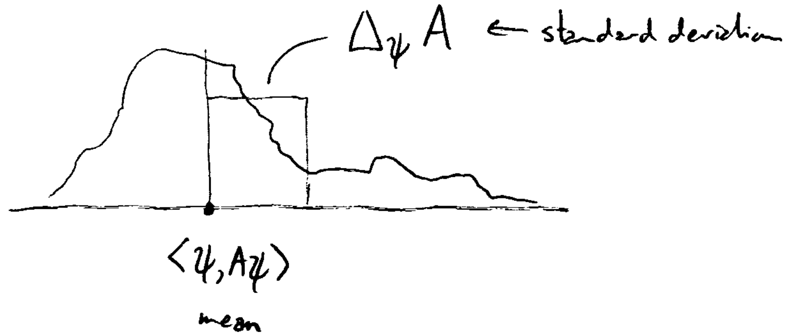
i) when you measure $A$ in state $e_i$ you always get $\lambda_i$.

ii) when you measure $A$ in some state $\psi$ you get $\lambda_i$ with prob. $|\langle e_i, \psi\rangle|^2$.

Thus the mean value (expected value or expectation value) of $A$ in state $\psi$

$$\sum_i |\langle e_i, \psi\rangle|^2 \lambda_i = \langle \psi, A\psi\rangle$$

(Note: when we measure $A$, the state is not then in state "$A\psi$". $A$ doesn't take a state to a new state – that's what unitary operators do in QM.)



The standard deviation of some quantity $x$ is the ~~mean of~~ square root of the mean of

$$(x - \text{mean of } x)^2$$

note: can't use just mean of $(x-\bar{x})$ that's zero!

(The RMS)

The mean of $(x - \text{mean of } x)^2$ is the <u>variance</u> $(\Delta_\psi \cancel{x})^2$; standard deviation is $\Delta x$.

So in quantum mechanics the variance of $A$ in state $\psi$ is

$$(\Delta_\psi A)^2 = \langle \psi, (A - \langle \psi, A\psi \rangle)^2 \psi \rangle$$

1) <u>Thm:</u> (Uncertainty Principle)

Suppose $A, B$ are self-adjoint operators on a Hilbert space $H$, and $\psi \in H$ is a unit vector. Then

$$\Delta_\psi A \, \Delta_\psi B \geq \frac{1}{2} \left| \langle \psi, [A,B] \psi \rangle \right|$$

<u>Proof:</u> First we'll modify $A$ & $B$ a little. Let

$$A' = A - \langle \psi, A\psi \rangle I$$
$$B' = B - \langle \psi, B\psi \rangle I$$

(i.e. subtract off the means)

so they have mean zero:

$$\langle \psi, A'\psi \rangle = \langle \psi, (A - \langle \psi, A\psi \rangle)\psi \rangle = \langle \psi, A\psi \rangle - \langle \psi, A\psi \rangle \langle \psi, \psi \rangle = 0$$

& $\langle \psi, B'\psi \rangle = 0$

Note $[A', B'] = [A, B]$ and also the standard deviations are unaffected:

$$\Delta_\psi A' = \Delta_\psi A$$
$$\Delta_\psi B' = \Delta_\psi B$$

since for any constant $C$

$$\Delta_\psi (A + C) = \langle \psi, (A + C - \langle \psi, (A+C)\psi \rangle)^2 \psi \rangle$$
$$= \langle \psi, (A - \langle \psi, A\psi \rangle)^2 \psi \rangle$$
$$= \Delta_\psi A.$$

Note: since mean of $A'$ is zero,

$$(\Delta_\psi A')^2 = \langle \psi, A'^2 \psi \rangle$$

and same for $B$. It suffices to show the uncertainty principle for $A'$ & $B'$, i.e. we need to show $\sqrt{\langle \psi, A'^2 \psi \rangle} \sqrt{\langle \psi, B'^2 \psi \rangle} \geq \frac{1}{2} |\langle \psi, [A', B'] \psi \rangle|$

$$\langle \psi, A'^2 \psi \rangle \langle \psi, B'^2 \psi \rangle \geq \frac{1}{4} |\langle \psi, [A', B'] \psi \rangle|^2$$

$$\|A'\psi\|^2 \cdot \|B'\psi\| \geq \frac{1}{4} |\langle \psi, A'B'\psi \rangle - \langle \psi, B'A'\psi \rangle|^2$$

$$\frac{1}{4} |\langle A'\psi, B'\psi \rangle - \langle B'\psi, A'\psi \rangle|^2$$

So it suffices to show

$$\frac{1}{4} |\langle A'\psi, B'\psi \rangle - \langle B'\psi, A'\psi \rangle|^2 \leq \frac{1}{4} |2 \langle A'\psi, B'\psi \rangle|^2$$
$$= |\langle A'\psi, B'\psi \rangle|^2$$
$$\leq \|A'\psi\|^2 \cdot \|B'\psi\|^2 \quad \blacksquare$$

complex conjugates

Note: Cauchy Schwarz $\iff$ Uncertainty Principle.

Note: this is all fine if $A, B$ bded or if $\psi$ is in the domain of $A'$, $B'$, $A'B'$ and $B'A'$.

Cor: If $\psi \in \mathcal{S}(\mathbb{R})$ then

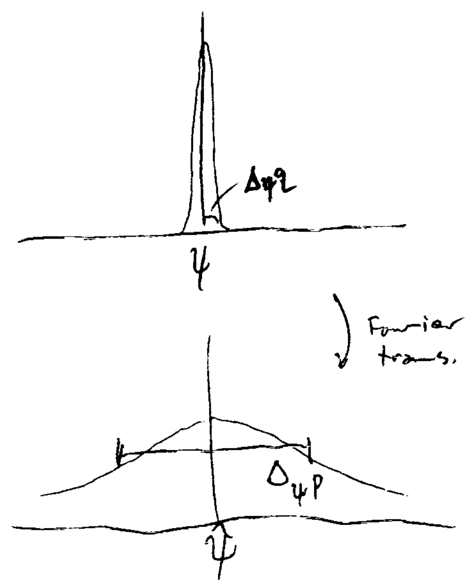
$$\Delta_\psi p \, \Delta_\psi q \geq \frac{1}{2}$$

Pf: in the previous thm, the R.H.S. was

$$\frac{1}{2} \left| \langle \psi, \underbrace{[p,q]}_{=-i} \psi \rangle \right|^2 = \frac{1}{2} \qquad \blacksquare$$

Or, restoring $\hbar$:

$$\boxed{\Delta_\psi p \, \Delta_\psi q \geq \frac{\hbar}{2}}$$




1) This inequality is sharp! (can't make it better and have it still be true.) (i.e. we can get equality)

2) There are lots of $\psi$ that get equality. You can write all of them down. The simplest is the g.s. of the harmonic oscillator: $\psi = \dfrac{e^{-x^2/2}}{\|e^{-x^2/2}\|_{L^2}}$

Thm: If $\psi$ is an eigenvector of the harmonic oscillator hamiltonian

$$H = \frac{1}{2}(p^2+q^2)$$

$$H\psi = E\psi$$

then

$$E \geq \frac{1}{2}.$$

(Note: for $\psi = \frac{e^{-x^2/2}}{\|e^{-x^2/2}\|}$ we do have $H\psi = \frac{1}{2}\psi$)

Proof: By the proof of the uncertainty principle,

$$\langle\psi, p^2\psi\rangle\langle\psi, q^2\psi\rangle \geq \underbrace{\frac{1}{4}|\langle\psi, \overbrace{[p,q]}^{-i}\psi\rangle|^2}_{\frac{1}{4}}$$

(really just Cauchy-Schwarz for p,q s.a.)

But if $H\psi = E\psi$,

$$\langle\psi, \tfrac{1}{2}(p^2+q^2)\psi\rangle = E$$

so we need

$$\langle\psi, p^2\psi\rangle + \langle\psi, q^2\psi\rangle \geq 1.$$

Lemma: If $A, B \geq 0$ and $AB \geq \frac{1}{4}$ then $A+B \geq 1$ if $A, B \geq 0$.

Pf: $AB \geq \frac{1}{4}$ so $B \geq \frac{1}{4A}$ so $A+B \geq A + \frac{1}{4A}$

Why is $A + \frac{1}{4A} \geq 1$? Find min $A = \frac{1}{2}$ & check $A + \frac{1}{4A} = 1$ at minimum. ∎

## 2) Fourier Transform

The Fourier transform:

$$F : \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R})$$

is given by:

$$F\psi = \hat{\psi}$$

where

$$\hat{\psi}(k) = \frac{1}{\sqrt{2\pi}} \int e^{-ikx} \psi(x) \, dx$$

This relates the momentum and position operators nicely:

Example:

$$\begin{aligned}
\widehat{(p\psi)}(k) &= \frac{1}{\sqrt{2\pi}} \int e^{-ikx} \cdot -i\psi'(x) \, dx \\
&= \frac{1}{\sqrt{2\pi}} \int i(-ike^{-ikx})\psi(x) \, dx \\
&= \frac{1}{\sqrt{2\pi}} k \int e^{-ikx} \psi(x) \, dx \\
&= k\hat{\psi}(k) \\
&= (q\hat{\psi})(k)
\end{aligned}$$

integration by parts. No boundary terms because Schwartz fns. are so nice

So

$$\boxed{Fp = qF}$$

$$\widehat{q\psi}(k) = \frac{1}{\sqrt{2\pi}} \int e^{-ikx}\, x \cdot \psi(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\, i \int \frac{d}{dk}(e^{-ikx})\, \psi(x)\, dx$$

by niceness of Schwartz functions

$$= \frac{1}{\sqrt{2\pi}} \cdot i \frac{d}{dk} \int e^{-ikx}\, \psi(x)\, dx$$

$$= \left(i \frac{d}{dk} \hat{\psi}\right)(k)$$

$$= -(p\hat{\psi})(k)$$

So

$$Fq = -pF$$

---

<u>28 October 2003</u>

We have

$$p, q, F : \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R})$$

Fourier transform

where

$$\mathcal{S}(\mathbb{R}) = \{\psi : \mathbb{R} \to \mathbb{C} \quad : \quad |x^m \tfrac{d^n}{dx^n} \psi| \text{ bounded}\}$$

& $(p\psi)(x) = -i \frac{d\psi}{dx}(x) \qquad q(\psi)(x) = x\,\psi(x)$

$$(F\psi)(k) = \frac{1}{\sqrt{2\pi}} \int e^{-ikx}\, \psi(x)\, dx$$

We saw:

$$Fp = qF$$
$$Fq = -pF$$
$$[p, q] = -i$$

Let's see how this works with the annihilation / creation ops.

Recall
$$a = \frac{q+ip}{\sqrt{2}} \qquad a^* = \frac{q-ip}{\sqrt{2}}$$

Then

$$\begin{aligned} Fa &= \frac{1}{\sqrt{2}} F(q+ip) \\ &= \frac{1}{\sqrt{2}}(Fq + iFp) \\ &= \frac{1}{\sqrt{2}}(-pF + iqF) \\ &= \frac{i}{\sqrt{2}}(qF + ipF) \\ &= \frac{i}{\sqrt{2}}(q+ip)F \\ &= iaF \end{aligned}$$

$$\boxed{Fa = iaF}$$

& similarly:

$$\begin{aligned} Fa^* &= \frac{1}{\sqrt{2}} F(q-ip) \\ &= \frac{1}{\sqrt{2}}(-p-iq)F \\ &= -ia^*F \end{aligned}$$

$$\boxed{Fa^* = -ia^*F}$$

---

Recall:
$$\psi_0 = e^{-x^2/2}$$
$$\psi_n = a^{*n}\psi_0$$

These have nice simple Fourier transforms; they are eigenvectors

Thm: $F\psi_n = (-i)^n \psi_n$

(In particular, the Gaussian is its own Fourier transform – that's one reason $e^{-x^2/2}$ is the "best" Gaussian)

Proof: We'll show $F\psi_0 = \psi_0$. From this we get

$$
\begin{aligned}
F\psi_n &= F a^{*n}\psi_0 \\
&= (-i)^n a^{*n} F\psi_0 \\
&= (-i)^n a^{*n}\psi_0 \\
&= (-i)^n \psi_n \qquad \text{since } Fa^* = -ia^*F
\end{aligned}
$$

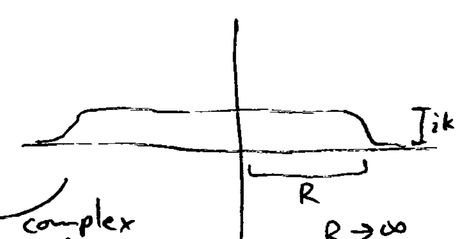
So we only need to check $\psi_0$.

$$
\begin{aligned}
(F\psi_0)(x) &= \frac{1}{\sqrt{2\pi}} \int e^{-ikx} e^{-x^2/2}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int e^{-\frac{1}{2}(x^2 + 2ikx + (ik)^2)} e^{\frac{1}{2}(ik)^2}\, dx \\
&= \frac{1}{\sqrt{2\pi}} e^{-k^2/2} \int e^{-\frac{1}{2}(x+ik)^2}\, dx
\end{aligned}
$$

We want to make a change of variables $y = x + ik$. Technically this is illegal because the new variable doesn't run over the real line. We can use a contour integral in $\mathbb{C}$ to show it works rigorously

$$
\begin{aligned}
&= \frac{1}{\sqrt{2\pi}} e^{-k^2/2} \int_{\text{"}-\infty + ik\text{"}}^{\text{"}\infty + ik\text{"}} e^{-\frac{1}{2}y^2}\, dy \\
&= \frac{1}{\sqrt{2\pi}} e^{-k^2/2} \underbrace{\int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2}\, dy}_{\sqrt{2\pi}}
\end{aligned}
$$

complex analysis




$$
\begin{aligned}
&= e^{-k^2/2} \\
&= \psi_0(k)
\end{aligned}
$$

Thm: $F: \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ extends uniquely to a unitary operator on $L^2(\mathbb{R})$.

Pf: We use the fact that $\psi_n$ form an orthonormal basis of $L^2(\mathbb{R})$, so

$$|n\rangle = \frac{\psi_n}{\|\psi_n\|}$$

is an o.n. basis of $L^2(\mathbb{R})$ with

$$F|n\rangle = (-i)^n |n\rangle$$

BLT Lemma: Suppose $V$ & $W$ are normed vector spaces & $W$ is complete. Suppose $T_0 : V_0 \to W$ is a bounded linear transformation where $V_0 \subseteq V$ is dense. Then $T_0$ extends uniquely to a bounded lin. op. $T: V \to W$.

Pf: $T_0$ bounded means

$$\|T_0 \psi\| < K \|\psi\| \qquad \forall \psi \in V_0$$

$T$ is unique since if $\psi \in V$ $\exists \psi_i \to \psi$ for some $\psi_i \in V_0$, so $\|\psi_i - \psi\| \to 0$ so $\|T(\psi_i - \psi)\| < K\|\psi_i - \psi\| \to 0$ so $\|T\psi_i - T\psi\| \to 0$ so $T\psi_i \to T\psi$.

So:
$$T\psi = \lim_i T\psi_i \qquad \text{for any } \psi_i \to \psi \quad \psi_i \in V_0$$

As for existence, we need to check if $\psi_i \to \psi$ & $\psi_i' \to \psi$ then $\lim T\psi_i = \lim T\psi_i'$, so $T\psi$ is well-defined. Also need to show the limit exists:

$\psi_i \to \psi \implies \psi_i$ is Cauchy seq.

$\implies \|T\psi_i - T\psi_j\| \to 0$ since $T$ bdd.

$\Rightarrow T\psi_i$ converges, since $W$ is complete.

So $\lim_{i\to\infty} T\psi_i$ exists

Then we need to check:

① $\lim_{i\to\infty} T\psi_i$ is independent of our choice of $\psi_i \to \psi$.

② $T\psi = \lim_{i\to\infty} T_0\psi_i$ is linear in $\psi$

③ $T$ is bded.

We won't do this. ∎ back to pf of our thm.

---

We apply the BLT lemma to $\mathcal{S}(\mathbb{R}) \subseteq L^2(\mathbb{R})$ — they're dense and extend $F : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R}) \subseteq L^2(\mathbb{R})$ to $F : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ in a unique way s.t. $F$ is bded.

<u>Note</u>: Why did we need to do this by extending from $\mathcal{S}(\mathbb{R})$? Why not just use the formula?

$$\psi(k) = \frac{1}{\sqrt{2\pi}} \int \underbrace{e^{-ikx}\psi(x)}_{\in L^2} dx$$

The reason is that the integral may not exist; we cannot in general integrate an $L^2$ function. In fact, it turns out that the integral exists for almost all $k$ (w.r.t. Lebesgue measure)

Check that $F : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is unitary. Use the fact that $|n\rangle$ are an o.n. basis to write

$$\psi = \sum_{n=0}^{\infty} a_n |n\rangle \qquad \sum |a_n|^2 < \infty$$

$$\varphi = \sum_{n=0}^{\infty} b_n |n\rangle \qquad \|\psi\|^2$$

To check $T$ is unitary we need to check:

1) $\langle F\psi, F\varphi\rangle = \langle \psi, \varphi\rangle$
2) $F$ must be onto

Why we need this in $\infty$ dimensions.

$T|n\rangle = |n+1\rangle$

$|n\rangle$ are an o.n. basis of countable dimension v. space

$T$, the "right shift operator" satisfies 1 but not 2

Note: this is the "Hilbert Hotel Trick"

1) $\langle F\psi, F\varphi\rangle = \left\langle \sum (-i)^n a_n |n\rangle, \sum (-i)^m b_m |m\rangle \right\rangle$

$$= \sum_n \overline{(-i)^n}\,(-i)^n\, \bar{a}_n b_n$$

$$= \langle \psi, \varphi\rangle$$

2) Given $\psi = \sum a_n |n\rangle \in L^2(\mathbb{R})$, want $\varphi \in L^2(\mathbb{R})$ s.t. $F\varphi = \psi$.

Let

$$\varphi = \sum i^n a_n |n\rangle$$

$$F\varphi = \sum (-i)^n i^n a_n |n\rangle$$

$$= \sum a_n |n\rangle = \psi$$

<u>Cor:</u> $F^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ has

$$F^{-1}\psi_n = (i)^n \psi_n$$

---

In fact:

$$(F^{-1}\psi)(x) = \frac{1}{\sqrt{2\pi}} \int e^{ikx}\psi(k)\,dk.$$

To check this in our approach we just need to check that this formula gives this one

Note: in most books, the proof that $F^{-1}$ has this explicit formula is terribly technical and complicated. In our case, the technical part was all compacted into proving that $\psi_n$ form a basis of $L^2$.

Note: In fact the argument that $F^{-1}\psi$ has this integral formula is the same argument as we used for $F$, just replacing $i$ by $-i$ (Galois theory!)

$$F\psi_n = (-i)^n \psi_n$$

$$F^2\psi_n = (-1)^n \psi_n \implies (F^2\psi_n)(x) = \psi_n(-x)$$

$$\implies (F^2\psi)(x) = \psi(-x) \quad \forall \psi \in L^2(\mathbb{R})$$

---

30 October 2003

The Fourier transform, we saw is a unitary operator:

$$F : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$$

with

$$F\psi_n = (-i)^n \psi_n$$

where $\psi_n$ are an (unnormalized) basis of eigenvectors of

$$H = \frac{1}{2}(p^2 + q^2)$$

with

$$H\psi_n = (n + \tfrac{1}{2})\psi_n$$

or, these are also eigenvectors of

$$N := H - \tfrac{1}{2} = \tfrac{1}{2}(p^2+q^2-1) = a^*a$$

(This amounts to picking a new origin for energy)

with

$$N\psi_n = n\psi_n$$

## Time Evolution:

According to Heisenberg, observables evolve in time: for any observable (s.a. operator on Hilbert space) $O$, and any time $t \in \mathbb{R}$, there is an observable $O(t)$ which is "measuring $O$ after waiting time $t$."
$O(t)$ satisfies Heisenberg's equation:

$$\frac{dO(t)}{dt} = i[H, O(t)]$$

where $H$ is the energy or Hamiltonian. The solution is

$$O(t) = e^{itH} O e^{-itH}$$

There are also states $\psi$ (unit vectors in the Hilbert space) — but Heisenberg didn't describe how these change in time. The Heisenberg picture of a state is that it describes the system for all time — that way we only need to worry about $O(t)$, not "$\psi(t)$." The expectation value of $O(t)$ in state $\psi$ is

$$\langle \psi, O(t)\psi \rangle$$

According to Schrödinger, states (that is, the vectors) evolve in time, while observables stay fixed. States evolve according to Schrödinger's equation: Given a state $\psi$ at time zero, we get a state $\psi(t)$ at time $t$ satisfying

$$\frac{d\psi(t)}{dt} = -iH\psi(t)$$

The solution is:

$$\psi(t) = e^{-iHt}\psi$$

Schrödinger says the expectation value of an observable $O$ in state $\psi(t)$ is

$$\langle \psi(t), O\,\psi(t) \rangle$$

In fact, Heiseberg & Schröding pictures agree:

$$\begin{aligned}
\langle \psi(t), O\psi(t) \rangle &= \langle e^{-itH}\psi, Oe^{-itH}\psi \rangle \\
&= \langle \psi, (e^{-itH})^* O\, e^{-itH}\psi \rangle \\
&= \langle \psi, e^{itH} O e^{-itH}\psi \rangle \\
&= \langle \psi, O(t)\psi \rangle .
\end{aligned}$$

Now: what's $e^{-itH}$ like for the harmonic oscillator?
Or: what's $e^{-itN}$ like?

$$e^{-itN}\psi_n = \sum_{k=0}^{\infty} \frac{(-itN)^k}{k!}\psi_n$$

and using $N\psi_n = n\psi_n$,

$$e^{-itN}\psi_n = \sum_{k=0}^{\infty} \frac{(-itn)^k}{k!}\psi_n$$
$$= e^{-itn}\psi_n.$$

Recall:

$$F\psi_n = (-i)^n\psi_n$$

These are the same when $e^{-it} = -i$, e.g. $t = \frac{\pi}{2}$!

(Upshot: "we can take Fourier transforms just by waiting around.")

So

$$F = e^{-i\frac{\pi}{2}N}$$

In short:

"To take the Fourier transform of a function, get a particle whose $\psi$ is that function & wait a quarter period of the harmonic oscillator $(\frac{1}{4}2\pi)$" (with ground state energy subtracted off)

This is the time for "momentum energy" to turn into "position energy" (kinetic into potential)

Or:

$$F = e^{-i\frac{\pi}{2}N} = (-i)^N .$$

---

Note: $F^2 \psi_n = (-1)^n \psi_n$

$\psi_n$ with $n$ even form a basis of even $L^2$ functions
$\psi_n$ with $n$ odd form a basis of odd $L^2$ functions

So: for any $\psi \in L^2(\mathbb{R})$

$$(F^2\psi)(x) = \psi(-x)$$
$$= (P\psi)(x)$$

for the parity operator $P$

So "The Fourier Transform is a square root of Parity."

Next, note:

$$F^4 \psi_n = \psi_n$$

so

$$F^4 = \mathbb{1}$$

What's going on? Why is the Fourier transform like a quarter turn?

Answer:

$$\begin{aligned} q(t) &= \cos t\, q + \sin t\, p \\ p(t) &= -\sin t\, q + \cos t\, p \end{aligned} \qquad (*)$$

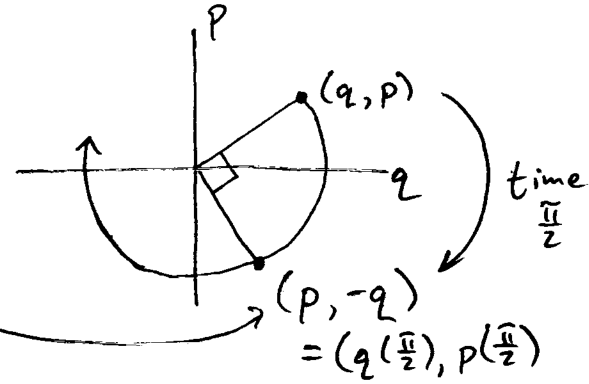
We saw this before with

$$q(t) = e^{+itH} q e^{-itH} = e^{it(H-\frac{1}{2})} q e^{-it(H-\frac{1}{2})} = e^{itN} q e^{-itN}$$

and similarly

$$p(t) = e^{itH} p e^{-itH} = e^{itN} p e^{-itN}$$

Think of this in the classical picture




got this by using (*) on previous page

time evolution by $\frac{\pi}{2}$ is a quarter turn in phase space.

In the quantum picture: we can plug $\frac{\pi}{2}$ in to our eqns. for $q(t)$, $p(t)$ above to get

$$p = q\left(\tfrac{\pi}{2}\right) = e^{i\frac{\pi}{2}N} q e^{-i\frac{\pi}{2}N} = F^{-1} q F \implies \boxed{Fp = qF}$$

$$-q = p\left(\tfrac{\pi}{2}\right) = e^{i\frac{\pi}{2}N} q e^{-i\frac{\pi}{2}N} = F^{-1} p F \implies \boxed{Fq = -pF}$$

as before

---

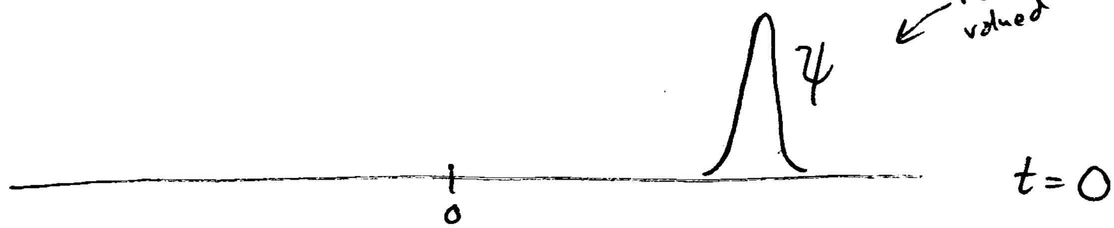
$\delta_a(x)$ – position is perfectly known:

$$q\,\delta_a(x) = a\,\delta_a(x)$$

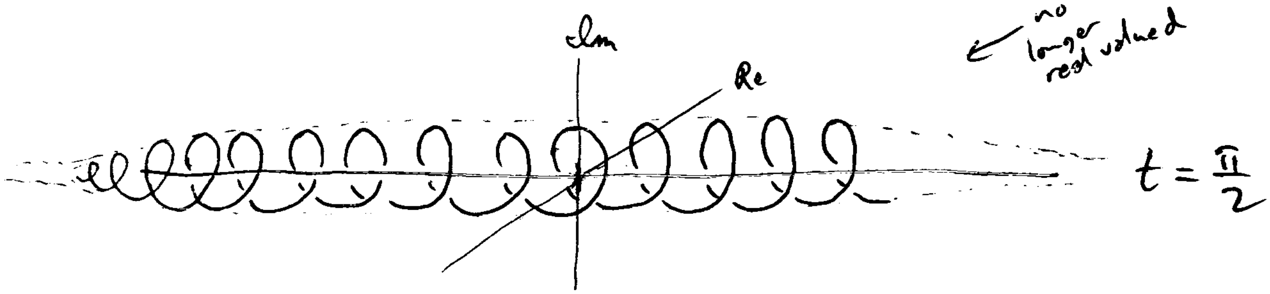
$$F\delta_a(x) = \frac{e^{-iax}}{\sqrt{2\pi}}$$

$$p\,\frac{e^{-iax}}{\sqrt{2\pi}} = a\,\frac{e^{-iax}}{\sqrt{2\pi}}$$ – momentum is perfectly well known

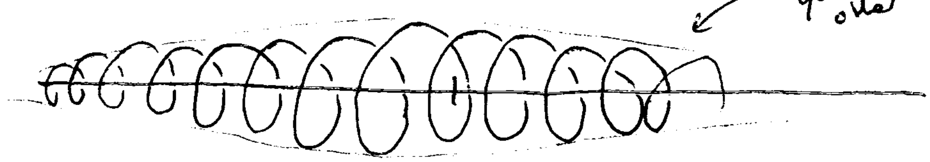
# The quantum harmonic oscillator




real valued

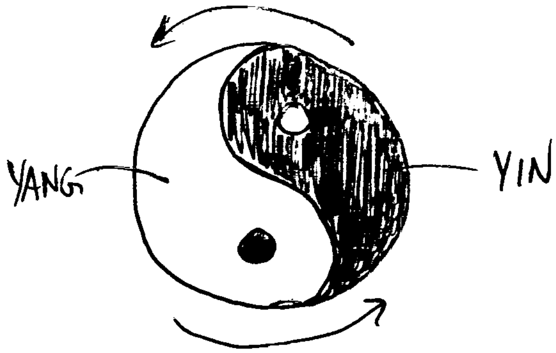



no longer real valued

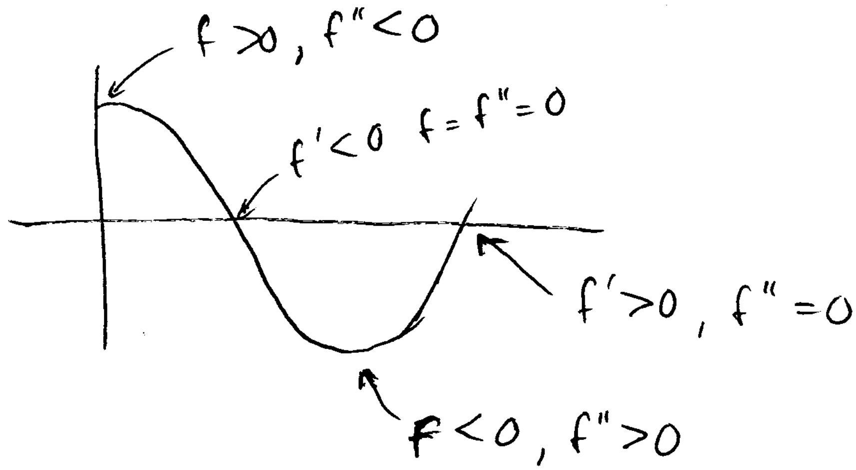
corkscrew goes the other way

# The Mathematics of Tao

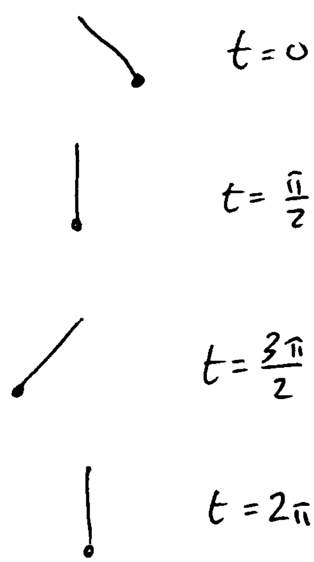



YANG = + = HOT = SUNNY

YIN = − = COLD = DARK




4 most interesting points on a cyclic process



4 November 2003

Let's summarize what we've done so far:

We have a space with basis $\psi_0, \psi_1, \psi_2, \dots$ on which various operators act:

creation: $a^* \psi_n = \psi_{n+1}$

annihilation: $a \psi_n = n \psi_{n-1}$

position: $q = \frac{a + a^*}{\sqrt{2}}$

momentum: $p = \frac{a - a^*}{\sqrt{2}\, i}$

Hamiltonian: $H = \frac{1}{2}(p^2 + q^2)$ $\qquad H\psi_n = (n + \frac{1}{2})\psi_n$

renormalized Hamiltonian (subtracting off ground state energy): or "number operator" $N = H - \frac{1}{2} = a^* a$ $\qquad N\psi_n = n\psi_n$

Fourier Transform: $F = (-i)^N = e^{-itN}$ where $t = \frac{\pi}{2}$ - a quarter period. $\qquad F\psi_n = (-i)^n \psi_n$

In terms of the Fock representation we call this vector space cx. polynomials in one variable, i.e. $\mathbb{C}[z]$. We call $\psi_n$ "$z^n$". Thus we get:

$$a^* = M_z \quad \text{(multiplication by } z\text{)}$$

$$a = \frac{d}{dz}$$

$$N = M_z \frac{d}{dz} \qquad (N z^n = z\, n z^{n-1} = n z^n)$$

To evolve a state $\phi \in \mathbb{C}[z]$ by time $t$ we apply $e^{-itN}$.

What does this do?

$$\phi(z) = \sum_{n=0}^{\infty} a_n z^n \qquad a_n \in \mathbb{C}$$

$$(e^{-itN}\phi)(z) = \sum_{n=0}^{\infty} a_n e^{-itN} z^n$$

$$= \sum_{n=0}^{\infty} a_n e^{-itn} z^n$$

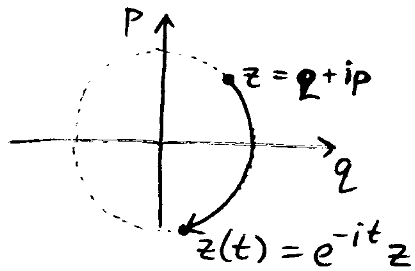
$$= \sum_{n=0}^{\infty} a_n (e^{-it} z)^n$$

$$= \phi(e^{-it} z)$$

Note

$$e^{-itN} z^n = \sum \frac{(-itN)^k}{k!} z^n = \sum \frac{(-itn)^k}{k!} z^n = e^{-itn} z^n$$

So: if we think of $z \in \mathbb{C}$ as a point in phase space




we can think of our state as a function of position & momentum and evolve it in time by the rule $z \mapsto e^{-it} z$, same as the classical rule:

$$\begin{aligned} q(t) &= \cos t \, q + \sin t \, p \\ p(t) &= -\sin t \, q + \cos t \, p \end{aligned} \iff z(t) = e^{-it} z$$

Moral: the Fourier transform <u>is</u> a quarter-turn in phase space (in Fock rep.)

So much for the quantum harmonic oscillator ...

# ...Now: CATEGORIFY IT ALL!

Whenever we see "$n \in \mathbb{N}$" replace it with "$n$, the $n$-element set."

First some notes about the article "the" in the above:

- Not all $n$-element sets are equal (unless $n=0$)
- All $n$-element sets are isomorphic
- Not all $n$-elt. sets are isomorphic in a unique way (unless $n=0,1$)

So we must use "the" carefully.

Now let's categorify polynomials, or more generally formal power series, like

$$\phi(z) = \sum \frac{a_n z^n}{n!} \qquad \text{where } a_n \in \mathbb{N}$$

We'll say a "structure type" or "species" is a type of structure that you can put on a finite set. (Not the real definition yet!)

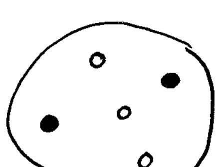
Given a structure type $\Phi$ let $\Phi_n \in \mathbb{N}$ be the number of ways you can put this structure on an $n$-element set.

Then define the generating function of $\Phi$, say $|\Phi|$, to be

$$|\Phi|(z) = \sum \frac{\Phi_n z^n}{n!}$$

Examples:

1) $\Phi$ = "2-colorings" - ways of coloring each element of a finite set black or white



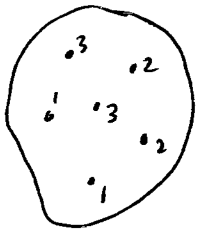
- a 2-coloring of 5.

$$\Phi_n = 2^n$$

So $$|\Phi|(z) = \sum_{n\geq 0} \frac{2^n z^n}{n!} = e^{2z}$$

We think of $e^{2z}$ as the decategorified version of "2-colorings"

2) $\Phi$ = "k-colorings" - ways of mapping our finite set to $k$
= ways of coloring our finite set with the "colors" $\{1, 2, 3, \ldots, k\}$




- a 3-coloring of 6

$$\Phi_n = \#\text{ ways of } k\text{-coloring an } n\text{-elt set} = k^n$$

So $$|\Phi|(z) = \sum_{n\geq 0} \frac{k^n}{n!} z^n = e^{kz}$$

If $k=0$ we get $e^{0z} = 1 = \sum \frac{a_n z^n}{n!}$ where $a_n = 0$ unless $n=0$, in which case $a_0 = 1$.

If $k=1$ we get $e^{1z} = \sum \frac{1^n z^n}{n!} = e^z$.

There is 1 way to map $0 \to 0$, 0 ways to map $n \to 0$, $n \geq 1$.

---

3) Suppose $\Phi$ & $\Psi$ are structure types

Let's invent a structure type $\Phi + \Psi$ such that
$$|\Phi + \Psi| = |\Phi| + |\Psi|.$$

$$|\Phi + \Psi| = \sum \frac{(\Phi+\Psi)_n}{n!} z^n$$

so we need
$$(\Phi + \Psi)_n = \Phi_n + \Psi_n$$

The answer: A $(\Phi + \Psi)$-structure on ~~on~~ $n$-elt. set is:

a $\Phi$-structure <u>XOR</u> a $\Psi$-structure

really ~~fake~~ disjoint union,
since a $\Phi$-structure might also be
a $\Psi$-structure.

E.g. If $\Phi$ = "2-colorings"
$\Psi$ = "3-colorings"

then $\Phi + \Psi$ = "2-colorings xor 3-colorings"

$$|\Phi + \Psi|(z) = e^{2z} + e^{3z} = \sum_{n=0}^{\infty} \frac{2^n + 3^n}{n!} z^n$$

---

4) Next puzzle: find $\Phi\Psi$ s.t. $|\Phi\Psi| = |\Phi||\Psi|$

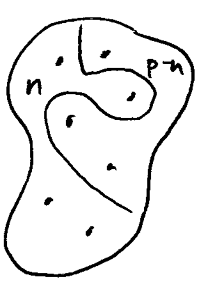
Want

$$|\Phi\Psi|(z) = |\Phi|(z)\,|\Psi|(z)$$

$$= \sum_n \frac{\Phi_n z^n}{n!} \sum_m \frac{\Psi_m z^m}{m!}$$

$$= \sum_{n,m} \frac{\Phi_n \Psi_m}{n!\,m!} z^{\overbrace{n+m}^{p}}$$

$$= \sum_p \sum_{\substack{n,m: \\ n+m=p}} \frac{\Phi_n \Psi_m}{n!\,m!} p! \frac{z^p}{p!}$$




$$= \sum_p \underbrace{\sum_{0 \le n \le p} \binom{p}{n} \Phi_n \Psi_{p-n}}_{} \frac{z^p}{p!}$$

$(\Phi\Psi)_p$ = number of ways to chop up $p$ into 2 parts & put a $\Phi$-str. on first part and $\Psi$-str. on 2nd part.

Next time ... the Catalan Numbers.

6 November 2003

# "CATALAN NUMBERS"

Named after Eugene Catalan since discovered by Gauss.

A magma is a set $M$ equipped w. a binary operation $\cdot : M \times M \to M$. Consider the free magma on one element $x$. The elements include:

$c_1 = 1$ $\quad x$

$c_2 = 1$ $\quad xx$

$c_3 = 2$ $\quad (xx)x \quad x(xx)$

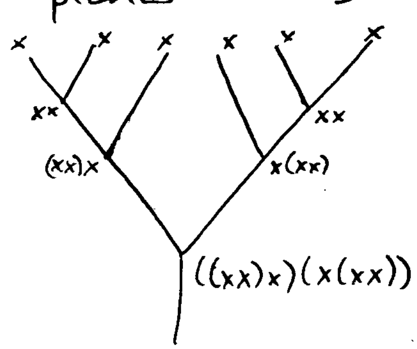
$c_4 = 5$ $\quad (xx)(xx) \quad x((xx)x) \quad ((xx)x)x \quad x(x(xx)) \quad (x(xx))x$

$\vdots$

We define $c_n$ to be the number of elts. built from $n$ $x$'s in the free magma on $x$. (Normally people use some different convention - but this seems more basic)


This is a picture of $M \times M \to M$
(multiplication in our magma)

This lets us draw elts of the free magma on $x$ as binary planar trees:

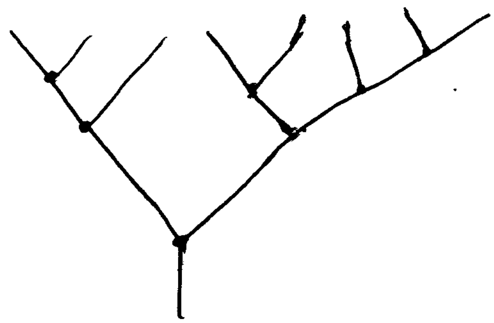



"a binary planar tree with 6 leaves"

And conversely:

$$((xx)x)((xx)(x(xx)))$$

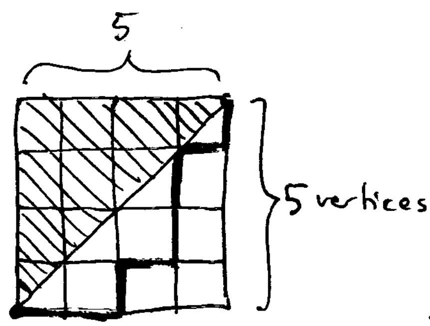
$$\downarrow$$



So: $c_n$ is the number of binary planar trees with $n$ leaves.

---

There are other things the Catalan #s are good for




Let's count paths an on $n \times n$ vertex grid from SW corner to NE corner that only go N or E & never enter the shaded NW region

1 $\dot{x}$

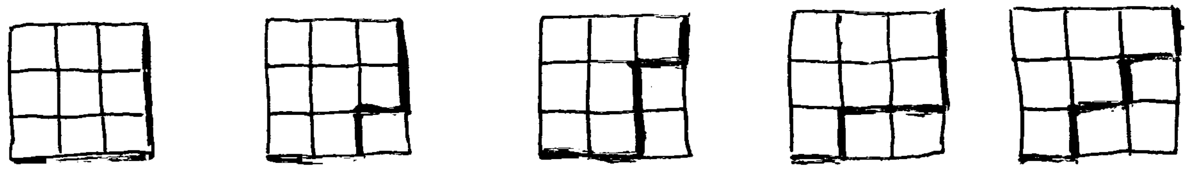
1

$xx\cdot = (xx)$

2

$xxx\cdot\cdot = (x(xx))$ $\quad xx\cdot x\cdot = ((xx)x)$

5



$xxxx\cdot\cdot\cdot \quad xxx\cdot x\cdot\cdot \quad xxx\cdot\cdot x\cdot \quad xx\cdot xx\cdot\cdot \quad xx\cdot x\cdot x\cdot$

Claim: the number of these paths is $c_n$.

An element of the free magma on $x$ with $n$ $x$'s is secretly just a reverse Polish notation expression with $n$ $x$'s an $(n-1)$ "dots", which is secretly just a path of the above sort.

---

Next: Consider chopping a regular polygon into triangles:

$c_1 =$
$c_2 = 1$
$c_3 = 2$
$c_4 = 5$
$c_5 = 14$
$c_6 = 42$
$\vdots$

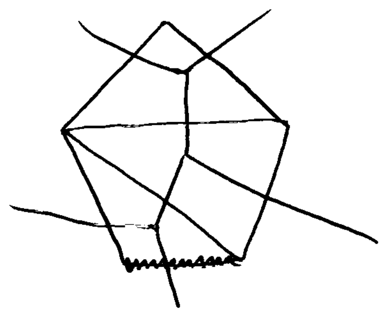
1

2

5

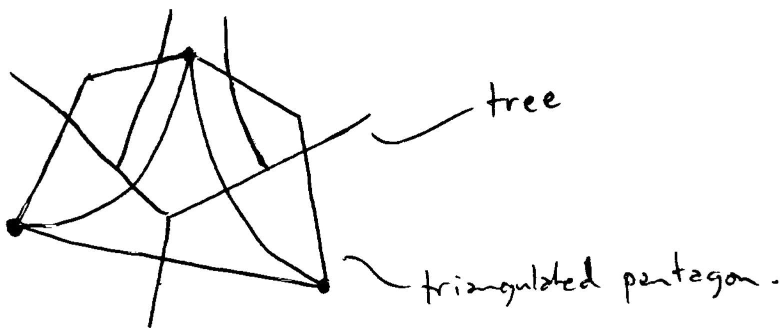
Claim: there are $c_n$ ways to chop a regular $(n+1)$-gon into triangles.

How does this work:



Pick a side of the $(n+1)$-gon to be the "output" side and then draw the tree that is Poincaré dual to the triangulation.

Conversely:

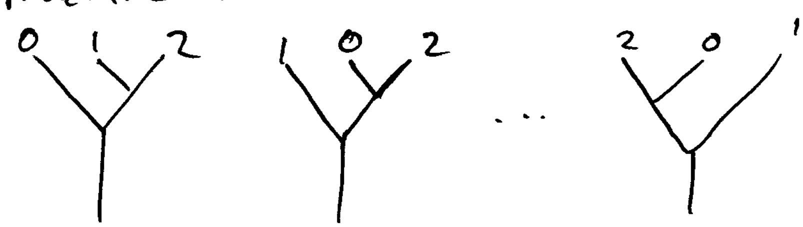



---

Let's calculate $c_n$!

Let $T$ be the structure type "planar binary trees": a $T$-structure on a finite set is a way of making its elts into the leaves of a planar binary tree.

Eg. if our set is $3 = \{0,1,2\}$, a $T$-structure on it could be




etc.

for a total of $3! \cdot c_3 = 12$ $T$-structures.

So let $T_n = n!\, c_n$ be the number of T-structures on the set $n$, & get the generating function:

$$|T|(\cancel{z}) = \sum_{n=0}^{\infty} \frac{T_n z^n}{n!} = \sum_{n=0}^{\infty} c_n z^n$$

$$= x + x^2 + 2x^3 + 5x^4 + \cdots$$

Recall:

1) Given str. types $\Phi$ & $\Psi$, a "$\Phi + \Psi$-str" on a set is a $\Phi$-structure <u>xor</u> $\Psi$-str. on that set.

$$|\Phi + \Psi| = |\Phi| + |\Psi|$$

2) Given str. types $\Phi$ & $\Psi$, to put a "$\Phi\Psi$-str." on a set is to chop the set into 2 disjoint subsets & put a $\Phi$-str. on the first, a $\Psi$-str. on the second.

3) There's a str. type $Z$ with $|Z| = z$.

Since the coefficient of $z^n$ in this power series is 0 unless $n=1$, in which case it's 1, there are <u>No</u> ways to put a $Z$ structure on a set unless it has 1 element, in which case there is one way. So we say

$$Z = \text{"being a 1-element set"}$$

More generally, there's a structure type called $\frac{Z^n}{n!}$, or "being an n-element set," & whose ~~the~~ generating function is:

$$\left|\frac{Z^n}{n!}\right| = \frac{z^n}{n!}$$

If $n=0$ we get a str. type $\mathbf{1}$ with

$$|\mathbf{1}| = 1$$

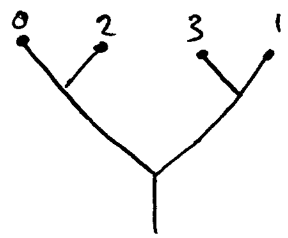
& is "being the empty set."

Use these to calculate $|T|(z)$!

13 Nov 2003

We said a "T-structure" on a finite set $S$ is a way of labeling the leaves of a planar binary tree with elements of $S$, using each elt. exactly once:

If $S = \{0, 1, 2, 3\}$ here's a T-structure on $S$:




There are $n!\,c_n$ T-structures on an $n$-element set, where $c_n$ is the number of $n$-leaved planar binary trees.

$$|T|(z) = \sum_{n=0}^{\infty} (c_n n!) \frac{z^n}{n!}$$
$$= \sum_{n=0}^{\infty} c_n z^n$$

Now what is $c_n$?

A T-structure on a set $S$ is: either $S$ being the 1-elt. set XOR writing $S$ as a disjoint union of $S'$ and $S''$ & putting a T-structure on $S'$ & $S''$.

$$T = Z + T^2$$

Taking generating functions:

$$|T| = |Z + T^2|$$
$$= |Z| + |T|^2$$
$$= z + |T|^2$$

So: Here's one way to solve this quadratic equation

$$|T| = z + |T|^2$$
$$= z + (z + |T|^2)^2$$
$$= z + (z + (z + |T|^2)^2)^2$$
$$= z + (z + z^2 + 2z|T|^2 + |T|^4)^2$$
$$= z + z^2 + 2z^2 + \cdots$$

and we get the beginning of our power series.

Another way to solve a quadratic equation:

$$|T|^2 - |T| + z = 0$$

so

$$|T| = \frac{1 \pm \sqrt{1-4z}}{2}$$

What is this as a power series

$$\sqrt{1+z} = 1 + \frac{1}{2} z + \frac{1}{2} \cdot \frac{-1}{2} \frac{z^2}{2!} + \frac{1}{2} \cdot \frac{-1}{2} \cdot \frac{-3}{2} \frac{z^3}{3!} + \cdots$$

$$\sqrt{1-4z} = 1 - 2z - 2^2(1\cdot 1)\frac{z^2}{2!} - 2^3(1\cdot 1\cdot 3)\frac{z^3}{3!} - 2^4(1\cdot 1\cdot 3\cdot 5)\frac{z^4}{4!} - \cdots$$

double factorial

Since $c_0 = 0$, $|T|(0) = 0$ so must have $|T|(z) = \frac{1-\sqrt{1-4z}}{2}$:

$$\frac{1-\sqrt{1-4z}}{2} = z + 2\cdot 1\cdot \frac{z^2}{2!} + 2^2(1\cdot 3)\frac{z^3}{3!} + 2^3(1\cdot 3\cdot 5)\frac{z^4}{4!} + 2^4(1\cdot 3\cdot 5\cdot 7)\frac{z^5}{5!} + \cdots$$

$$= z + z^2 + 2z^3 + 5z^4 + 14z^5 + \cdots$$

So we've solved a quadratic equation and counted binary trees:

$$c_n = \frac{2^{n-1}(2n-3)!!}{n!}$$

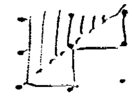
$$= \frac{(2(n-1))!}{n!(n-1)!}$$

$$= \frac{1}{n}\frac{(2(n-1))!}{(n-1)!(n-1)!}$$

$$= \frac{1}{n}\binom{2(n-1)}{n-1}$$

Note, our $c_n$ is most people's $C_{n-1}$, i.e.:

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

e.g. there are $c_3$ of these paths , or $C_2$ of them.

Finally ... What *is* a structure type? It's a certain kind of functor. More precisely it's a functor

the groupoid of finite sets

$$\Phi : \mathrm{FinSet}_0 \longrightarrow \mathrm{Set}$$

Huh? FinSet is the category whose objects are finite sets and whose morphisms are just set functions. Set is the category of *all* set, with functions.

For any category $C$ we get a category $C_0$ with the same objects as $C$ but only the isomorphisms of $C$ as its morphisms. This is a groupoid, since all morphisms are isomorphisms.

E.g.: $\mathrm{FinSet}_0$ has finite sets as objects and bijections as morphisms

$\Phi$ will assign to any finite set $S$ the set $\Phi(S)$ of "$\Phi$-structures on $S$."

---

Given two categories $C$ & $C'$, a functor $F: C \longrightarrow C'$ is a map sending each object $c \in C$ to a object $F(c) \in C'$, and each morphism $f: c_1 \to c_2$ of $C$ to a morphism $F(f): F(c_1) \to F(c_2)$ of $C'$ such that

$$F(fg) = F(f)\,F(g)$$

&

$$F(1_c) = 1_{F(c)}.$$

So, saying $\Phi : \text{FinSet}_0 \longrightarrow \text{Set}$ is a functor says

1) $\Phi$ sends any finite set $S$ to a set $\Phi(S)$: the set of $\Phi$-structures on $S$.

2) Given a bijection $f : S \to S'$ we get a map

$$\Phi(f) : \Phi(S) \longrightarrow \Phi(S')$$

sending $\Phi$-str. on $S$ to $\Phi$-str. on $S'$.

3) $\Phi(fg) = \Phi(f)\,\Phi(g)$ says transferring a $\Phi$-str. along $fg$ is same as transferring it along $g$ & then $f$.

4) $\Phi(1_S) = 1_{\Phi(S)}$ says transferring a $\Phi$-str. along an identity fn. leaves it alone.

What is $Z$, the structure of "being a 1-elt. set"? It should be a functor

$$Z : \text{FinSet}_0 \longrightarrow \text{Set}$$

What is it? Given a finite set $S$, $Z(S) = \begin{cases} \emptyset & \text{if } |S| \neq 1 \\ 1 & \text{if } |S| = 1 \end{cases}$

where $1$ is 'the' one-element set.

Given $f: S \to S'$ a bijection

$$Z(f) : Z(S) \to Z(S')$$

is the only possible function: the identity function.

---

<u>25 Nov 2003</u>

Structure Types: a <u>structure type</u> is a functor

$$\Phi : \mathrm{FinSet}_0 \longrightarrow \mathrm{Set}$$

where $\mathrm{FinSet}_0$ is the groupoid of finite sets (& bijections) & Set is the category of sets (& functions).

There's a category of structure types: structure types are the objects; what are the morphisms?

$$\begin{tikzcd} \mathrm{FinSet}_0 \arrow[r, bend left=50, "\Phi"] \arrow[r, bend right=50, "\Psi"'] \arrow[r, phantom, "T \Downarrow"] & \mathrm{Set} \end{tikzcd}$$

They should be <u>natural transformations</u>.

<u>Def</u>: Given categories $C$ & $D$, functors $\Phi, \Psi : C \to D$, we define a natural transformation $T : \Phi \Rightarrow \Psi$ to be a function assigning to each object $x \in C$ a morphism $T_x : \Phi(x) \longrightarrow \Psi(x)$ in $D$ s.t. for any morphism $f : x \to y$

$$\begin{array}{ccc} \Phi(x) & \xrightarrow{\Phi(f)} & \Phi(y) \\ {\scriptstyle T_x}\downarrow & & \downarrow{\scriptstyle T_y} \\ \Psi(x) & \xrightarrow{\Psi(f)} & \Psi(y) \end{array}$$

the "naturality" square

commutes.

(Note: natural transformations "boost the dimension by 1"
they take objects and send them to morphisms
they take morphisms and send them to commutative diagrams)

---

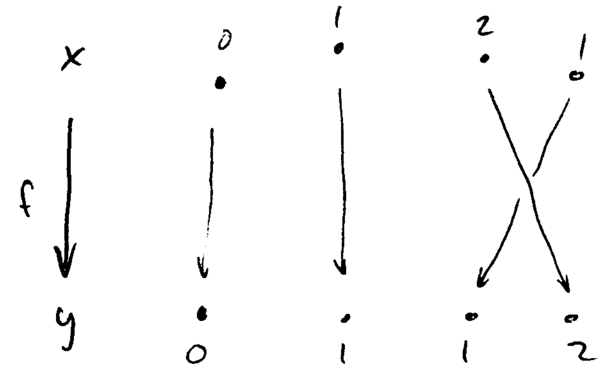
What does this amount to given structure types $\Phi, \Psi : \mathrm{FinSet}_0 \to \mathrm{Set}$? If $x$ is a finite set, $\Phi(x)$ is the set of all $\Phi$-structures on $x$, $\Psi(x)$ is the set of all $\Psi$-structures on $x$, and $T_x : \Phi(x) \to \Psi(x)$ lets us turn any $\Phi$-str. into a $\Psi$-str.

Let $\Phi$ = "3-colorings", so that $\Phi(x) = 3^x$, i.e. the set of maps $k : x \to 3 = \{0,1,2\}$, and given a bijection $f : x \to y$, $\Phi(f) : \Phi(x) \to \Phi(y)$ is given by:

$$\Phi(f)k = k \circ f^{-1} \qquad\qquad k : x \to 3$$

Check that $\Phi$ is a functor, i.e. $\Phi(fg) = \Phi(f)\Phi(g)$:

$$\begin{aligned} \Phi(fg)k &= k \circ (fg)^{-1} \\ &= k \circ g^{-1} \circ f^{-1} \\ &= \Phi(f)(k \circ g^{-1}) \\ &= \Phi(f)(\Phi(g)k) = (\Phi(f)\Phi(g))k. \end{aligned}$$




transferring the 3-coloring to $y$, via the bijection $f$.

Let $\Psi$ = "2-colorings", so $\Psi(x) = 2^x$, etc., where $2 = \{0,1\}$. An example of a natural transformation $T : \Phi \Longrightarrow \Psi$ is:

$$T_x : \Phi(x) \longrightarrow \Psi(x) \quad \text{given by} \quad T_x k = r \circ k$$

where $r$, the "recoloring function", is any function $r : 3 \to 2$.

$$\begin{tikzcd} x \arrow[r] \arrow[dr, "T_x k"'] & 3 \arrow[d, "r"] \\ & 2 \end{tikzcd}$$

Let's check that $T$ makes the naturality square commute: given a bijection $f : x \to y$, we want:

$$\begin{tikzcd} \Phi(x) \arrow[r, "\Phi(f)"] \arrow[d, "T_x"'] & \Phi(y) \arrow[d, "T_y"] \\ \Psi(x) \arrow[r, "\Psi(f)"] & \Psi(y) \end{tikzcd}$$

to commute

i.e. we need to show $T_y \Phi(f) = \Psi(f) T_x$. Given $k : x \to 3$ check $T_y \Phi(f) k = \Psi(f) T_x k$.

$$T_y(k\circ f^{-1}) \overset{?}{=} \psi(f)(f\circ k)$$
$$r\circ(k\circ f^{-1}) = (r\circ k)\circ f^{-1}$$

$$\begin{tikzcd}
x \arrow[r] \arrow[dr, "T_x"'] & 3 \arrow[d, "r"] & y \arrow[l] \arrow[dl, "T_y"] \\
& 2 &
\end{tikzcd}$$

This only worked because we used the same recoloring function $r$ for defining $T_x$ & $T_y$, i.e. naturality says we're changing colors in a systematic way for all finite sets $x, y, \ldots$.

Digression:
Note: associativity is a special case of commutativity:

Associativity:
$(ab)c = a(bc)$
$R_c L_a b = L_a R_c b$
$R_c L_a = L_a R_c$
"left & right multiplication commute

Picture Proof:

$$\begin{tikzcd}
& 3 \arrow[dr, "r"] \arrow[dd, "1_3"] & \\
x \arrow[ur, "k"] \arrow[rr, "T_x k"'] \arrow[dd, "f"'] & & 2 \arrow[dd, "1_2"] \\
& 3 \arrow[dr, "r"] & \\
y \arrow[ur, "\Phi(f)k"] \arrow[rr, "\psi(f)T_x k"'] & & 2
\end{tikzcd}$$

$$T_x k\circ f^{-1} = \Phi(f)\, T_x k$$
$$\overset{?}{=} T_y\, \Phi(f)\, k$$

Yes, since prism commutes.

---

The category of species (structure types) deserves the name hom(FinSet$_0$, Set), since hom$(C, D)$ for any categories $C, D$ is the category whose objects are functors $\Phi: C \to D$ & whose morphisms are natural transformations between these. It also deserves the name Set$[\![x]\!]$. We had started doing quantum mechanics in the Fock representation, where states are $\varphi \in \mathbb{C}[\![x]\!]$; now we've categorified & gotten $\Phi$ w. $|\Phi| = \varphi$, and they're really categorified formal power series w. Set replacing $\mathbb{C}$.

---

2 Dec 2003

Today we categorify differentiation . . .

Now we know str. types are functors

$$\Phi : \mathrm{FinSet}_0 \longrightarrow \mathrm{Set}$$

& we say "n" for the n-elt. set, we can use $\Phi_n$ for the set of $\Phi$-structures on n, as opposed to the number of such structures as before. The number of such structures will be $|\Phi_n|$ (i.e. $|\cdot|$ denotes cardinality).

The generating function of $\Phi$ is then

$$|\Phi|(z) = \sum_{n=0}^{\infty} \frac{|\Phi_n|}{n!} z^n$$

Any structure type (for which $|\Phi_n| < \infty$) thus gives an elt. $|\Phi| \in \mathbb{C}[\![z]\!]$. In the Fock representation of the Weyl algebra, $p$ & $q$ become operators on $\mathbb{C}[\![z]\!]$ (or $\mathbb{C}[z]$) via:

$$a^* = M_z \qquad\qquad q = \frac{a + a^*}{\sqrt{2}}$$

$$a = \frac{d}{dz} \qquad\qquad p = \frac{a - a^*}{\sqrt{2}\, i}$$

(operators on formal power series)

such that

$$a a^* - a^* a = 1$$

So, let's find "operators" on the category of str. types, $A^*$ & $A$, which ~~act~~ like $a^*$ & $a$. Specifically:

$$|A^* \Phi| = a^* |\Phi|$$

(i.e. applying $A^*$ and then decategorifying is the same as applying the ~~...~~ decategorification $a^*$ of $A^*$ to the decategorification $|\Phi|$ of $\Phi$ )

and $$|A \Phi| = a |\Phi|$$

"The bane of categorification is the minus sign" so we want to rewrite $a a^* - a^* a = 1$ and require

$$A A^* \cong A^* A + 1$$ ← (naturally isomorphic as functors)

for $A$ & $A^*$.

$A^*$ is easy to figure out since we know how to multiply structure types

$$A^* \Phi = Z \Phi$$

where $Z$ is the structure type w. $|Z| = z$, i.e. the structure "being the 1-elt. set." So, putting a $Z\Phi$-str. on a set $n$ is writing $n$ as a disjoint union of two sets $p$ & $q$ & putting a $Z$-str.

on $p$ & $\underline{\Phi}$-str. on $q$. That is, putting a $Z\underline{\Phi}$ structure on $n$ means choosing an element ~~and~~ $x \in n$ and a $\underline{\Phi}$-structure on $n - \{x\}$. So:

$$\begin{aligned} |A^*\underline{\Phi}| &= |Z\underline{\Phi}| \\ &= |Z||\underline{\Phi}| \\ &= z|\underline{\Phi}| \\ &= a^*|\underline{\Phi}|. \end{aligned}$$

<u>Example:</u> Let $\underline{\Phi}$ be "being an $n$-element set", i.e.:

$$\underline{\Phi}_m = \begin{cases} 0 & m \not\cong n \\ 1 & m \cong n \end{cases}$$

Here

$$|\underline{\Phi}|(z) = \frac{z^n}{n!}.$$

Now what is $A^*\underline{\Phi}$? To put this structure on a set $S$, we choose $x \in S$ & put the str. "being an $n$-elt set" on $S - \{x\}$. I.e. it's "being an $n+1$ elt. set w. chosen point $x$." There are $n+1$ ways to make that choice so:

being an $(n+1)$-elt. pointed set

$$|A^*\underline{\Phi}|(z) = (n+1)\frac{z^{n+1}}{(n+1)!} = \frac{z^{n+1}}{n!} = z \cdot \frac{z^n}{n!}$$

Another example: let $\Phi$ be the str.-type "being a totally ordered n-elt. set."

$$|\Phi_m| = \begin{cases} 0 & n \ncong m \\ n! & n \cong m \end{cases}$$

So

$$|\Phi|(z) = z^n$$

Now an $A^*\Phi$ str. on $S$ is "picking an elt $x \in S$ and totally ordering $S - \{x\}$" (which must have n elements) But note this is the same as ordering the whole set.
I.e. this str.-type is isomorphic to "being a totally ordered n+1-elt. set."
So if we use $Z^n$ to mean "being a totally ordered n-elt. set" we see

$$A^* Z^n \cong Z^{n+1}$$

& thus

$$|A^* Z^n| = |Z^{n+1}|$$
$$\Downarrow$$
$$a^* z^n = z^{n+1}$$

which is what we want for a "creation operator."

What about $A$? What can we do to a str. type that has the effect of differentiating (formally) its generating function?

If $\Phi$ has gen. function

$$|\Phi|(z) = \sum_{n=0}^{\infty} \frac{|\Phi_n|}{n!} z^n$$

we want a str. type $A\Phi = \frac{D}{DZ}\Phi$ w.

$$\left|\frac{D}{DZ}\Phi\right|(z) = \frac{d}{dz}|\Phi|(z) = \sum_{n=\emptyset}^{\infty} \frac{|\Phi_n|}{(n-1)!} z^{n-1}$$

$$= \sum_{n=0}^{\infty} \frac{|\Phi_{n+1}|}{n!} z^n$$

So there must be $|\Phi_{n+1}|$ ways to put an $A\Phi$-str. on an n-elt. set. So: say an $A\Phi$-str. on the set $n$ is a $\Phi$-str. on the set $n+1$.

Examples: Let $\Phi$ be the str. "being a finite set" — I.e. $\Phi_n = 1$. This is called the "vacuous structure" — every finite set has this in exactly 1 way!

$$|\Phi|(z) = \sum_{n=0}^{\infty} \frac{|\Phi_n|}{n!} z^n \qquad \text{but } |\Phi_n| = 1$$

$$= \sum_{n=0}^{\infty} \frac{Z^n}{n!} = e^Z$$

So we should call this $\Phi$ "$E^Z$"!

We know $\frac{d}{dZ} e^Z = e^Z$, but is $\frac{D}{DZ} E^Z \cong E^Z$?

This would imply $\frac{d}{dz} e^Z = e^Z$ by applying $|\cdot|$ to both sides.

To put a $\frac{D}{DZ} E^Z$ - str. on a finite set $S$ is to put the $E^Z$ structure (structure of being a finite set) on $S+1$. This is the same as putting the str. of being a finite set on $S$, so

$$\frac{D}{DZ} E^Z = E^Z.$$

So $\frac{d}{dz} e^z = e^z$ means "$S$ is finite iff $S+1$ is."

---

4 Dec 2003

Creation and Annihilation operators - Categorified Version

Recall:

$$A^* \Phi = Z \Phi \qquad \text{creation}$$

$$A \Phi = \frac{D}{DZ} \Phi \qquad \text{annihilation}$$

An $A^*\Phi$ - structure on a finite set $S$ is a choice of an element $x \in S$ and a $\Phi$ - str. on $S - \{x\}$. An $A\Phi$ - structure on a finite set $S$ is a $\Phi$ - structure on $S+1$. Note the irksome contravariance: "creation" is related to removing an element while "annihilation" is related to throwing in an extra element

Examples:

1) $\Phi = \dfrac{Z^n}{n!}$ = "being an n-element set"

$A\Phi = \dfrac{Z^{n-1}}{(n-1)!}$ = "being an (n-1)-element set"

since putting the structure "being an n-elt. set" on $S+1$ is the same as putting the structure "being an (n-1)-elt." set on $S$. (This shows the irksome contravariance)

2) $\Psi = Z^n$ = "being ~~a~~ totally ordered n-elt set"

Let's do this one without cheating:

$A\Psi$ should be like "being ~~a~~ totally ordered (n-1)-elt. set with a <u>'mark'</u>"

e.g. if $n=3$

$\wedge\, \overset{1}{\bullet} \quad \overset{2}{\bullet}$

or

$\overset{1}{\bullet} \;\wedge\; \overset{2}{\bullet}$

or

$\bullet \quad \bullet\, \wedge$

I.e. $A\Psi = nZ^{n-1}$ since putting a total ordering on $S+1$ is same as putting a total ordering on $S$ & equipping it with a "mark" (indicating the position of the extra elt. in $S+1$)

3) $\Phi = \frac{Z^n}{n!}$ = "being an n-elt. set"

$$A^* \Phi = Z \frac{Z^n}{n!}$$

$$= (n+1)\frac{Z^{n+1}}{(n+1)!}$$ = "being an (n+1)-elt. pointed set"

since choosing $x \in S$ & putting the structure "being an n-elt. set" on $S - \{x\}$ is the same as putting the structure "being a pointed (n+1)-elt. set" on $S$.

4) $\Psi = Z^n$ = "being a totally ordered n-elt. set."

$A^* \Psi = Z^{n+1}$ = "being a totally ordered (n+1)-elt set"

since choosing $x \in S$ & putting ~~a total ordering~~ the structure "total ordering on an n-elt set" on $S - \{x\}$ is the same (isomorphic) as putting the structure "total ordering on an (n+1)-elt. set" on $S$.

And now the punchline (of the whole course)

• • •

Newton thought

$$pq = qp$$

(presumably, if anyone had bothered to ask him)

Heisenberg said: not quite!

$$pq = qp - i\hbar$$

For us, $p$ & $q$ come from $a$ & $a^*$ via

$q = \frac{a+a^*}{\sqrt{2}}$, $p = \frac{a-a^*}{\sqrt{2}i}$ & noncommutativity of $p$ & $q$ comes from

$$aa^* = a^*a + 1 \qquad (\hbar = 1)$$

But why is this true?

Think of the energy levels of the harmonic oscillator as saying how many "quanta of energy" it has. These mysterious "quanta" are, for us, the elements of these <u>finite sets</u> we're discussing now. This makes $aa^* = a^*a + 1$ <u>obvious</u>, since it follows from the categorified version:

$$AA^* \cong A^*A + 1$$

which is even more obvious.

Huh? ... We'll show

$$AA^*\Phi \cong A^*A\Phi + \Phi$$

for any structure type.

- An $A^*A\Phi$-str. on $S$ is:
  - a choice of $x \in S$ and an $A\Phi$-str. on $S - \{x\}$, which is:
    - a choice of $x \in S$ and $\Phi$-str. on $S - \{x\} + 1$.

- An $AA^*\Phi$-str. on $S$ is:
  - an $A^*\Phi$-str. on $S+1$, which is:
    - a choice of $x \in S+1$ and a $\Phi$-str. on $(S+1) - \{x\}$.

      i.e.: <u>either</u> a choice of $x \in S$ and a $\Phi$-str. on $(S+1) - \{x\} = (S - \{x\}) + 1$

      <u>xor</u> a $\Phi$ structure on $(S+1) - \{x\} = S$ (in the case where the element removed is the same as the one just added)

      i.e: <u>either</u> an $A^*A\Phi$-str. on $S$

      <u>xor</u> a $\Phi$-str. on $S$

      i.e: $A^*A\Phi + \Phi$ str. on $S$

So $\boxed{AA^*\Phi \cong A^*A\Phi + \Phi}$

Moral: Noncommutativity comes from the fact that "putting a ball in a box w. n balls in it, then taking one out" can be done in one more way than: ~~"putting~~

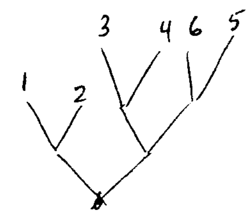
"taking a ball out of a box w. n balls in it and then putting one in".

---

The str-type "binary trees," $T$ has:

$$T \cong Z + T^2$$

either the 1-elt. set or two trees




A T-str. on $\{1, \dots, 6\}$


So $|T| = |Z + T^2|$

$$= z + |T|^2$$

$$|T|^2 - |T| + z = 0$$

$$|T|(z) = \frac{1 - \sqrt{1-4z}}{2} = z + z^2 + 2z^3 + 5z^4 + 14z^5 + 42z^6 + \cdots$$

Suppose $z = 1$. We get

$$e^{-\frac{i\pi}{3}} = \frac{1 - \sqrt{3}\, i}{2} = 1 + 1 + 2 + 5 + 14 + 42 + \cdots$$

So: The right hand side is the sum of all Catalan Numbers, i.e. the "cardinality" of the set $B$

of all (planar) binary trees!

$$|B| = e^{-\frac{i\pi}{3}} \quad \text{in some sense.}$$

So $\quad |B|^6 = 1$

Alas, $B^6 \not\cong 1$. BUT $B^7 \cong B$ in a very nice way!! There's a natural isomorphism between trees and 7-tuples of trees. Using $B \cong 1 + B^2$ we get $B \cong B^7$!