

## Action as a Functor

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In class I said some mysterious things about how the electromagnetic field is a  $U(1)$  connection on spacetime, while the Feynman path integral involves a  $U(1)$  connection on the cotangent bundle of spacetime. Let's clarify this a little.

Given a smooth manifold  $X$ , let  $\mathcal{P}(X)$  be the **category of paths** in  $X$ . There are lots of different ways to make this precise. I sketched one in class; here's another, where the interval parametrizing the paths can be of arbitrary length. Such paths are called 'Moore paths', but they may actually have been introduced by the topologist R. L. Moore.

The objects of  $\mathcal{P}(X)$  are points of  $X$ . A morphism in  $\mathcal{P}(X)$ , say  $\gamma: x \rightarrow y$ , is a piecewise smooth path  $\gamma: [0, T] \rightarrow X$  with

$$\gamma(0) = x, \quad \gamma(T) = y,$$

where  $T \geq 0$  is arbitrary. The composite of a path  $\gamma_1: [0, T_1] \rightarrow X$  and a path  $\gamma_2: [0, T_2] \rightarrow X$  is a path  $\gamma_1\gamma_2: [0, T_1 + T_2] \rightarrow X$ , defined in the obvious way:

$$(\gamma_1\gamma_2)(t) = \begin{cases} \gamma_1(t) & 0 \leq t \leq T_1 \\ \gamma_2(t - T_1) & T_1 \leq t \leq T_1 + T_2 \end{cases}$$

It's easy to check the associative law and left/right unit laws with this definition, where the identity path  $1_x$  is the path  $\gamma: [0, 0] \rightarrow X$  with  $\gamma(0) = x$ .

Now let us see how a 1-form on  $X$  defines a notion of  $\mathbb{R}$ -valued or  $U(1)$ -valued parallel transport. To do this, we think of the groups  $\mathbb{R}$  and  $U(1)$  as 1-object categories with all morphisms invertible.

1. Suppose that  $A$  is a smooth 1-form on  $X$ . Show that there's a unique functor

$$S: \mathcal{P}(X) \rightarrow \mathbb{R}$$

with

$$S(\gamma) = \int_{\gamma} A$$

for any morphism  $\gamma$  of  $\mathcal{P}(X)$ .

In the last homework you saw that group homomorphisms are actually functors. Thus, we can compose the above functor  $S$  with the homomorphism  $t \mapsto \exp(it)$  from  $\mathbb{R}$  to  $U(1)$  to get a functor from  $\mathcal{P}(X)$  to  $U(1)$ . Let's call this functor

$$e^{iS}: \mathcal{P}(M) \rightarrow U(1).$$

Next, let's show that whenever  $M$  is a smooth manifold, the cotangent bundle  $T^*M$  has a god-given smooth 1-form on it. This is called the **tautologous 1-form** or **symplectic potential**, and denoted by  $\alpha$ .

Recall that  $T^*M$  is the manifold whose points are pairs  $(q, p)$  where  $q \in M$  and  $p \in T_q^*M$  is a **cotangent vector** at  $q$ , that is, a linear functional  $p: T_qM \rightarrow \mathbb{R}$  where  $T_qM$  is the tangent space of  $M$  at  $q$ . The manifold  $T^*M$  becomes a vector bundle over  $M$  with projection

$$\begin{aligned} \pi: T^*M &\rightarrow M \\ (q, p) &\mapsto q \end{aligned}$$

It is then called the **cotangent bundle** of  $M$ .

To define a 1-form  $\alpha$  on  $T^*M$ , all we need is a linear functional  $\alpha_x$  on the tangent space of each point  $x \in T^*M$ . If  $x = (q, p)$  as above, this is given by

$$\begin{aligned} \alpha_x: T_x(T^*M) &\rightarrow \mathbb{R} \\ v &\mapsto p(d\pi(v)) \end{aligned}$$

*Sneaky, huh? You should ponder this carefully until you get it.*

To understand  $\alpha$  better, let's work out a formula for it in terms of local coordinates on  $T^*M$  coming from local coordinates on  $M$ .

Suppose  $U \subseteq M$  is an open set in  $M$  equipped with coordinate functions  $x_i: U \rightarrow \mathbb{R}$ . Then the open set  $T^*U \subseteq T^*M$  gets coordinates  $q_i, p^i: T^*U \rightarrow \mathbb{R}$  where for any point  $(q, p) \in T^*M$ ,

$$\begin{aligned} q_i(q, p) &= x_i(q) \\ p^i(q, p) &= p\left(\frac{\partial}{\partial x_i}\right). \end{aligned}$$

Here  $\frac{\partial}{\partial x_i}$  is the tangent vector at  $q \in M$  pointing in the  $x_i$  direction.

2. In terms of the above coordinates, show that on  $T^*U$  we have

$$\alpha = \sum_i p^i dq_i$$

If we use the **Einstein summation convention**, which says we always sum over indices that appear twice, once as a superscript and once as a subscript, we can abbreviate the above formula as:

$$\alpha = p^i dq_i.$$

*In case you were wondering, this is why we write the index on  $p^i$  as a superscript.*

As a consequence of 1 and 2, we see there is a god-given functor

$$S: \mathcal{P}(T^*M) \rightarrow \mathbb{R}$$

given by

$$S(\gamma) = \int_\gamma \alpha$$

for any piecewise smooth path in  $T^*M$ . We call  $S(\gamma)$  the **action** of the path  $\gamma$ . We also get a functor

$$e^{iS}: \mathcal{P}(T^*M) \rightarrow \text{U}(1).$$

In physics we can apply these ideas by letting  $M$  be the **configuration space** of a classical system, that is, the space of possible positions of the system. Then  $T^*M$  becomes the **phase space**, that is, the space of possible **states** of the system. For a state  $(q, p) \in T^*M$ , we call  $q \in M$  the **position** and  $p \in T_q^*M$  the **momentum**.

The functor  $S$  then assigns to any path in phase space a real number called its **action**. The functor  $e^{iS}$  assigns to any path a phase; this becomes important in the path-integral approach to quantum mechanics.

The phase space  $T^*M$  also has a god-given 2-form on it, given by

$$\omega = d\alpha$$

This is called the **symplectic structure**.

3. Using the Einstein summation convention, show that on  $T^*U$  we have

$$\omega = dp^i \wedge dq_i$$

in terms of the previously described coordinates  $p^i, q_i$ .

A path is called a **loop** if it starts where it ends. The action of a loop in phase space is the integral of the symplectic structure over any disk having that loop as its boundary:

4. Suppose that  $\gamma$  is a loop in  $T^*M$ . Suppose  $D$  is a disk in  $T^*M$  whose boundary is  $\gamma$ . Show that

$$S(\gamma) = \int_D \omega.$$

In an early stage of quantum mechanics Bohr and Sommerfeld thought that the ‘energy eigenstates’ of a quantum system corresponded to the periodic motions of the corresponding classical system in which it traced out a loop  $\gamma$  in phase space for which  $S(\gamma)$  was a multiple of  $2\pi$ , and thus  $e^{iS(\gamma)} = 1$ . This is a bit oversimplified, but it’s still a useful idea.

Another application of these ideas to physics arises when we let  $M$  be spacetime. If a particle traces out a path in spacetime, say  $\gamma: [0, T] \rightarrow M$ , the laws governing this particle let us ‘lift’ this path to  $T^*M$ . In other words, they give us a formula for a path  $\tilde{\gamma}: [0, T] \rightarrow T^*M$  such that  $\pi(\tilde{\gamma}(t)) = \gamma(t)$ . To be kind, I will omit this formula. The point, however, is that we can then define the action of the path  $\gamma$  to be

$$S(\gamma) = \int_{\tilde{\gamma}} \alpha.$$

If the particle has electric charge  $c$  and there is an electromagnetic field on spacetime, the formula for its action gets a bit fancier. An electromagnetic field is described by a 1-form  $A$  on the spacetime  $M$ . If the particle traces out a path  $\gamma: [0, T] \rightarrow M$  in spacetime, its action is then defined to be

$$S(\gamma) = \int_{\tilde{\gamma}} \alpha + c \int_{\gamma} A.$$

Here we see the electromagnetic field on spacetime and the symplectic potential on the cotangent bundle of spacetime showing up in the same formula!